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On Some *n*-Involution and *k*-Potent Operators on Hilbert Spaces

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Abstract: In this paper, we survey various results concerning n-involution operators and k-potent operators in Hilbert spaces. We gain insight by studying the operator equation $T^n = I$, with $T^k \neq I, k \leq n-1$ where $n, k \in \mathbb{N}$. We study the structure of such operators and attempt to gain information about the structure of closely related operators, associated operators and the attendant spectral geometry. The paper concludes with some applications in integral equations.

Keywords: *n*-Involution, Idempotent, Spectral Radius, Twist, Invection, *Q*-Equivalence

1. Introduction

Let H denote a Hilbert space and $B(H)$ denote the Banach algebra of bounded linear operators. If $T \in B(H)$, then T^* denotes the adjoint of T, while $Ker(T), Ran(T), \overline{M}$ and $M^\perp$ stands for the kernel of T, range of T, closure of M and orthogonal complement of a closed subspace M of H, respectively. We denote by $\sigma(T), \|T\|, r(T)$ and $W(T)$ the spectrum, norm, spectral radius of T and numerical range of T, respectively. Recall that an operator $T \in B(H)$ is normal if $T^*T = TT^*$, self-adjoint (or Hermitian) if $T^* = T$, skew-adjoint if $T^* = -T$, unitary if $T^*T = TT^* = I$, a projection(or idempotent) if $T^2 = T$, an orthogonal projection if $T^2 = T$ and $T^* = T$, an involution if $T^2 = I$, a symmetry or a reflection if $T = T^* = T^{-1}$. That is, T is self-adjoint unitary, isometric if $T^*T = I$, n-normal if $T^nT^* = T^*T^n$, an n-th root of identity if $T^n = I, n$ a positive integer, an n-involution if $T^n = I, n \geq 2$ a positive integer and normaloid if $r(T) = \|T\|$, a contraction if $\|Tx\| \leq \|x\|$, for all $x \in H$.

Two operators $A \in B(H)$ and $B \in B(K)$ are said to be *similar* if there exists an invertible operator $N \in B(H, K)$ such that $NA = BN$ or equivalently $A = N^{-1}BN$, and are unitarily equivalent if there exists a unitary operator $U \in B_+(H, K)$ (Banach algebra of all invertible operators in $B(H)$) such that $UA = BU$ (i.e. $A = U^*BU$ equivalently, $A = U^{-1}BU$). Two operators $A \in B(H)$ and $B \in B(K)$ are said to be metrically equivalent if $\|Ax\| = \|Bx\|$, (equivalently, $|\langle Ax, Ax\rangle|^{\frac{1}{2}} = |\langle Bx, Bx\rangle|^{\frac{1}{2}}$ for all $x \in H$) (see [5] for more exposition). Two operators S and T are said to be nearly equivalent if there exists an invertible operator V such that $S^*S = V^{-1}T^*TV$. Clearly similarity, unitary equivalence, near-equivalence and metric equivalence are equivalence relations on $B(H)$. An operator T is said to be *nearly normal* if $TT^* = V^{-1}T^*TV$, for some invertible operator V.

The spectral radius of an operator T denoted by $r(T)$ is defined as $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$.

It is well known that $r(T)$ is equal to the actual radius of the spectrum, that is, $r(T)=\sup\{|\lambda|:\lambda\in\sigma(T)\}$. The numerical range of $T\in B(H)$ is defined by $W(T)=\{\langle Tx,x\rangle:x\in H,\|x\|=1\}$. Behaviour of the powers T^n of a linear operator T on a Hilbert space H has been studied by some researchers, leading to important applications. It is well-known that linear operators and their powers may behave quite differently.

2. Main Results

Theorem 2.1 ([5], Proposition 1.10). $T\in B(H)$ is similar to a unitary operator if and only if T and T^* are both similar to isometries.

An operator $T\in B(H)$ is said to be algebraic if there exists a nonzero polynomial $p(t)\in F[t]$ such that $p(T)=0$. An algebraic operator $T\in B(H)$ is said to be of order n if $p(T)=0$ for $p(t)\in F[t]$ such that the degree of p is n and $q(T)\neq 0$ for any polynomial of degree less than n. Algebraic operators with characteristic polynomial $p(t)=\lambda^n-1,\ n\geq 2$ are said to be generalized involutive of order n or n-involutions and their characteristic roots are nth roots of unity.

Operators such that $T^n=I$ are similar to unitary operators. Indeed, if T^n is a contraction then T is similar to a contraction C, which implies that $C^n=I$ which in turn implies that C is unitary (by use of the Nagy-Foias-Langer Decomposition for contractions ([5], § 5.1)). Operators of the form $T=e^{\frac{2\pi ki}{n}}I$ provide the simplest example of n-involution operators in Hilbert spaces. It has been shown in ([3], Theorem 1) that all n-involutions are of this form. Clearly an n-involution need not be unitary although it is norm-preserving and invertible. Note that if T is an n-involution then T^{-1} and T^* are n-involutions (see [1], [6], [10]). We may have operators T such that $T^n=I$, with $\sigma(T)\subseteq\partial D$ but $\|T\|\geq 1$.

Example 1. Let $A=\begin{pmatrix}-1 & 1\\ 0 & 1\end{pmatrix}$. Clearly $A^2=I$. A simple computation shows that A is not unitary, although it is similar to a unitary operator. Note that every operator similar to a unitary operator is invertible.

Theorem 2.2 If S and T are similar then S^n and T^n are similar.

Theorem 2.3 Every n-involution T is invertible.

Proof. The fact that $T^2=I$ implies that T is left-invertible and right-invertible, and hence invertible.

Clearly if T is n-involution then $T^{n-1}=T^{-1}$ (see [6]) and all the previous results hold. We note that an n-involution need not be unitary. The following result gives a condition under which an n-involution is unitary.

Corollary 2.4 ([2], Corollary 6, § 3.7.3) If T is normaloid and $T^k=I$, then T is unitary.

Proof. Recall that for any operator T, we have that $\|T^k\|\leq\|T\|^k$. Since T is normaloid, we have that $\|T^k\|=\|T\|^k=\|T\|$. Since $T^k=I$ it is true that $1=\|T^k\|=\|T\|^k=\|T\|$, whence $\|T\|\leq 1$.

Thus T is an invertible isometry and is therefore a unitary operator.

Define the set $\eta_T=\{T\in B(H):T^n=I,n\in\mathbb{N}\}$. Clearly, η_T is a self-adjoint sub-algebra of n-involutions on H. It is clear that if $T\in\eta_T$, then $T^*\in\eta_T$.

Theorem 2.5 *Let S and T be unitarily equivalent operators in a Hilbert space H. Then S is normaloid if and only if T is.*

Proof. Suppose U is a unitary operator. Then $S=U^*TU$ implies that $S^n=U^*T^nU$, so that $\|S^n\|=\|T^n\|$ for every integer $n\geq 1$. Thus S is normaloid if and only if T is.

Theorem 2.6 *If $T\in\eta_T$, then T is n-normal.*

Note that $T\in B(H)$ is *n-normal* if and only if T^n is a normal operator. In other words, the *n*-normality of T is equivalent to the statement that $T^{n*}T^n=T^nT^{n*}$.

Theorem 2.7 Let $S,T\in B(H)$ be such that $S\in\eta_S$ and S and T are unitarily equivalent, n then $T\in\eta_T$.

Proof. Using $T^n=I$ and $S=UTU^*$, where U is unitary, we have that $S^n=UT^nU^*$. Rewriting, we have $T^n=US^nU^*=I$. This proves the claim.

Note that Theorem 2.7 is also true if unitary equivalence is replaced by similarity.

The next result shows that quasisimilarity leaves the n-involutory property invariant.

Proposition 2.8 If $S\in\eta_S$ and S and T are quasisimilar, then $T^n=I$.

Proof. Suppose $S^n=I$ and suppose S and T are quasisimilar with quasi-affinities X and Y such that XS = TX and SY = Y T. A simple calculation shows that $XS^n=T^nX$ $S^nY=YT^n$ (i.e. S^n and T^n are quasisimilar). Using the fact that $S^n=I$, we have $T^nX=X$ and $YT^n=Y$. This says that (T^n-I) X = 0 and Y (T^n-I) = 0. This implies that

Ran(X)=Ker(T^n-I) and Ran(T^n-I) = Ker(Y), respectively.

Using the fact that X and Y are quasi-affinities, we have that *Ker*(T^n-I) = H and *Ran*(T^n-I) $=\{0\}$. These two statements imply that T^n-I = 0, which is equivalent to $T^n=I$. This proves the claim.

The following two results follow from definitions.

Proposition 2.9 Let S be a self-adjoint involution which is metrically equivalent to T. If T is self-adjoint then it is an involution.

Proposition 2.10 Let $S\in\eta_S$ and suppose that S and T are nearly equivalent. Then $T \in \eta_T$ if S and T are self-adjoint.

Theorem 2.11 If $T^n = I$ and $S^n = I$, then TS is an n-involution if and only if TS = ST.

Proof. Using TS = ST we have $(TS)^n = T^n S^n = I^2 = I$. The converse is trivial.

Theorem 2.12 If T is an n-involution, then r(T) = 1.

Proof. Recall that $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$. Thus if $T^n = I$, then

$$1 = r(T^n) = \lim_{n\to\infty} \|T^n\|^{1/n} = \lim_{n\to\infty} (\|T^{nk}\|^{\frac{1}{nk}})^k$$

$$= \lim_{n\to\infty} (\|T^{nk}\|^{\frac{1}{nk}})^n = (r(T))^n$$

Thus $r(T) = 1$.

Remark. Note that if $\|T^n\| = 1$, then $r(T) = 1$. However, there are operators T such that $\|T\| = 1$ does not imply that $\|T^n\| = 1$. For instance, the unilateral shift S on $\ell^2(\mathbb{N})$ is such that $\|S\| = 1$ but $\|S^n\| = 0$ as $n \to \infty$. Note also that $\|T^n\| = 1$ need not imply that $\|T\| = 1$. To see this, consider the operator represented by the matrix $T = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. A simple calculation shows that $T^2 = I$ but $\|T\| = 1 + \sqrt{2} > 1$.

The result below gives a condition when $\|T^n\| = 1$ implies that $\|T\| = 1$.

Note that if T is normal then $\|T^n\| = \|T\|^n$. Consequently, if T is a normal n-involution then $\|T\| = 1$. This follows from the fact that $1 = \|T^n\| = \|T\|^n$.

Theorem 2.13 *If T is a normal n-involution then* $\|T\| = 1$.

Proof. This follows from the fact that $1 = \|T^n\| = \|T\|^n$.

Theorem 2.13 also holds when normal is replaced by normaloid operator.

Theorem 2.14 If T is an n-involution, then T is n-normal, for some $n \in \mathbb{N}$.

Proof. This follows easily from $T^n T^* = T^* T^n$.

Example 2. The operator with matrix $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is normal and $T^4 = I$ and hence $T \in \eta_T$. A simple computation shows that T is 4-normal.

Definition 2.15 Let C be a simple smooth closed oriented curve and let $\alpha(t)$ be a one-to-one mapping of C onto itself. The function $\alpha(t)$ is called a shift function or simply a shift on C. A shift $\alpha(t)$ is called a Carleman shift if $\alpha[\alpha(t)] \equiv t, t \in C$.

Here $\alpha_n(t) \equiv \alpha[\alpha_{n-1}(t)], t \in C, \alpha_1 = \alpha(t)$, denotes the n-th iteration of $\alpha(t)$, $n = 2, 3, \cdots$.

Let $C = \partial D$ denote the unit circle and $\alpha(t) = e^{\frac{2\pi i}{n}} t$. Evidently α(t) is a generalized Carleman shift of order n, preserving the orientation of C. The function $\alpha(t) = \frac{1}{t}$ is a Carleman shift of order 2 on C, changing the orientation of C.

3. Spectral Properties of *n*-Involutions

In this section we study some spectral properties of n-involutions.

Theorem 3.1 *If T is an n-involution, then* $0 \notin \sigma(T)$.

Proof. Follows from the fact that *T* is invertible.

Recall that a complex number λ is said to be unimodular if $\lambda = e^{2\pi ik}$ for some $k \in \mathbb{R}$.

Proposition 3.2 *If T is an n-involution then every component of the spectrum of T intersects the unit circle.*

Proof. Since $T^n = I$ we have that $1 = r(T^n) \le \|T^n\| \le \|T\|^n \le r(T^n)$. Suppose that $r(T) > 1$. Then there exists a non-zero vector $y \in H$ such that $\|T^n y\| \to \infty$ as $n \to \infty$, a contradiction to $\|T^n\| = 1$. This proves that $r(T) = 1$.

Theorem 3.3 Let H be a finite dimensional complex Hilbert space. An operator $T \in B(H)$ is an n-involution if and only if it is similar to a diagonal matrix operator with unimodular entries.

Proof. From Theorem 3.2 every component of the spectrum of T intersects the unit circle. Since H is finite dimensional, $\sigma_p(T) = \sigma(T) = \{\lambda_1, \lambda_2, \cdots, \lambda_m\}$ with $\lambda_1, \lambda_2, \cdots \lambda_m \in \partial D$. By the Jordan canonical decomposition, the matrix T is similar to a block-diagonal matrix

$$T = \begin{pmatrix} T_1 & 0 & \cdots & 0 \\ 0 & T_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & T_m \end{pmatrix},$$ where $T_i = \lambda_i I_m$ and I_m is the m-dimensional identity matrix.

Corollary 3.4 If T is an n-involution then the only eigenvalues of T are of the form $\lambda_k = e^{\frac{2\pi ik}{n}}, |\lambda_k| = 1, n, k \in \mathbb{N}, 0 \le k \le n-1$.

Theorem 3.4 can be generalized as follows.

Theorem 3.5 *Let T be an n-involution. A scalar* $\lambda \in \sigma(T)$ *if and only if* $|\lambda| = 1$.

An operator T is called an *n-symmetry* if it is a unitary n-involution, that is,

$$I = T^n = (T^{-1})^n = (T^*)^n.$$

An operator T is an *n-reflection* if $T^n = I$ and $\dim(Ran(T-I)) = 1$.

Example 3. The operators $A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ acting on C^2 and C^3, respectively are 2-reflections since a simple calculation shows that $A^2 = I, B^2 = I$ and

$$\dim(Ran(A-I)) = 1 \text{ and } \dim(Ran(B-I)) = 1.$$

The following claims follow easily from definitions.

Theorem 3.6 An operator $T \in B(H)$ is an n-reflection if and only if T is similar to a diagonal matrix $D = diag(1,1,\cdots,-1)$.

Theorem 3.7 An operator $T \in B(H)$ is an n-involution if and only if T is similar to a diagonal matrix $D = diag(\lambda_i)$, $|\lambda_i| = 1$.

Note also that if T is an n-involution then $\sigma(T^{-1}) = \sigma(T^{n-1})$.

Clearly the following operator class inclusion holds

$n-$ Symmetries $\subset$ $n-$ Reflections $\subset$ $n-$ Involutions $\subset$ n-Normal

Example 4. The operator $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by $(Tf)(t) = \overline{f}(-t)$, where the bar denotes complex conjugation and $t \in R$ is an involution. Clearly

$$(T^2 f)(t) = (T\overline{f})(-t) = \overline{\overline{f}}(-(-t)) = f(t).$$

The reflection operator $R : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by $(Rf)(t) = f(-t)$ is an involution which is unitary.

Clearly reflections are self-adjoint involutions.

4. *n*-Involutions, Associated Idempotents, *k*-Potents and Geometry

Involutions have a wide range of applications in geometry. Interestingly, there is a close relationship between n-involutions and idempotent operators.

Proposition 4.1 *(a). If T is an involution then* $P = \frac{1}{2}(I+T)$ *is an idempotent operator.*

(b). If P is an idempotent operator then $T=2P-I$ *is an involution.*

Proof.

(a). $T^2 = I \Rightarrow P^2 = [\frac{1}{2}(I+T)]^2$

$$= \frac{1}{4}(I + 2T + T^2) = \frac{1}{2}(I+T) = P$$

(b). $P^2 = P \Rightarrow T^2 = (2P-I)^2$

$$= 4P^2 - 2P - 2P + I = 4P - 4P + I = I$$

It is clear that the map $\psi : B(H) \to B(H)$ defined by $\psi(T) = \frac{1}{2}(I+T) = P$ is a bijective correspondence between the class of involutions and that of idempotent operators. We shall call the involution T and the idempotent P *associated.*

Proposition 4.2 *If P is a rank one idempotent operator then the operators*

$$\begin{pmatrix} I-2P & P \\ 0 & I \end{pmatrix}, \begin{pmatrix} I-P & P \\ P & I-P \end{pmatrix}, \begin{pmatrix} I-P & I \\ P & P-I \end{pmatrix}$$

are involutions.

Theorem 4.3 $T \in B(H)$ *is an involution if and only if it is the difference of a pair of complementary idempotent operators.*

Proof. Suppose that $T = P - Q, P^2 = P, Q^2 = Q$ and $P + Q = I$. Then $T^2 = (P-Q)^2 = P^2 + Q^2 = P + Q = I$ because $PQ = QP = 0$. This proves that T is an involution. Conversely, if T is an involution, we put $P = \frac{1}{2}(I+T)$ and $Q = \frac{1}{2}(I-T)$. Then $P - Q = T$ and it is easy to see that this decomposition is unique:

$$T = P - Q, P + Q = I \Rightarrow P = \frac{1}{2}(I+T).$$

Example 5. The projection $P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$ has $T = 2P - I = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ as its associated involution.

Theorem 4.4 *Two involutions are similar if and only if their associated idempotent operators are similar.*

Proof. Suppose T_1 and T_2 are involutions acting on a Hilbert space H such that

$T_1 = X^{-1}T_2X$. Then the associated idempotent operators are $P_1 = \frac{1}{2}(I+T_1)$ and $P_2 = \frac{1}{2}(I+T_2)$, respectively. Clearly, $T_1 = 2P_1 - I$ and $T_2 = 2P_2 - I$. Upon substitution, we have $2P_1 - I = X^{-1}(2P_2 - I)X$ and upon simplification we have that $P_1 = X^{-1}P_2X$. Conversely, if $P_1 = X^{-1}P_2X$, then $\frac{1}{2}(I+T_1) = X^{-1}(\frac{1}{2}(I+T_2))X$. A simple calculation shows that $T_1=X^{-1}T_2X$. This proves the claim.

Remark. An idempotent operator need not be self-adjoint. We give a condition under which it is self-adjoint in terms of its associated involution.

Proposition 4.5 *An involution is self-adjoint if and only if its associated idempotent is self-adjoint.*

Proof. Let T be a self-adjoint involution. Then $P^* = \frac{1}{2}(I+T^*) = \frac{1}{2}(I+T) = P$, where P is the associated idempotent. Conversely if $P^*=P$, then $T^*=2P^*-I=2P-I=T$. This proves the assertion.

Remark. It is clear that if T is an involution, then $\pm I, \pm T, \pm T^*$ exhaust the set of involutions. Note also that the sum of involutions need not be an involution: For instance, if T is an involution, then $-T$ is but $(T+-T)^2=0\neq I$. The product of two involutions also need not be an involution unless they commute, as seen in Theorem 2.11.

Theorem 4.6 *If $T\in B(H)$, then the following assertions are equivalent.*

(a). $T=T^*$ and $T^2=I$.

(b). T is normal and $T^2=I$.

(c). There exists an orthogonal projection such that $P=\frac{1}{2}(I+T)$.

Proof. We first note that assertion (a) says that T is a self-adjoint unitary or a symmetry.

(a)$\Rightarrow$ (b): Suppose $T=T^*$. To see that T is normal, note that $T^*T=TT^*=T^2$.

(b) $\Rightarrow$ (c): Let $P=\frac{1}{2}(I+T)$. Then

$$
\begin{aligned}
P^*P &= [\frac{1}{2}(I+T^*)][\frac{1}{2}(I+T)] \\
&= \frac{1}{4}(T^*T+T+T^*+I) \\
&= \frac{1}{4}(TT^*+T+T^*+I) \\
&= [\frac{1}{2}(I+T)][\frac{1}{2}(I+T^*)] \\
&= PP^*
\end{aligned}
$$

and

$$
\begin{aligned}
P^2 &= [\frac{1}{2}(I+T)]^2 \\
&= \frac{1}{4}(T^2+2T+I) \\
&= \frac{1}{4}(2I+2T) \\
&= \frac{1}{2}(I+T) \\
&= P
\end{aligned}
$$

and

$$P^*=P.$$

(c) $\Rightarrow$ (a): Let $T=2P-I$. Then

$$T^*=2P^*-I=2P-I=T$$

and

$$T^2=(2P-I)^2=4P-4P+I=I$$

The following result gives a condition under which an involution is unitary.

Theorem 4.7 *If P is an orthogonal projection (self-adjoint idempotent), then the associated involution $T=2P-I$ is unitary.*

Proof. This follows from

$$
\begin{aligned}
T^*T &= (2P^*-I)(2P-I) \\
&= 4P-4P+I \\
&= I \\
&= (2P-I)(2P^*-I) \\
&= TT^*
\end{aligned}
$$

If P is a projection, then the linear operator $T=I+(\lambda-1)P, \lambda\in\mathbb{R}, \lambda\neq 0$ is called a *dilatation* in the ratio λ. In the special case $\lambda = 1$, the dilatation becomes the identity operator. If $\lambda = 0$, T is a projection and if $\lambda = -1$, then T is an involution (indeed, a unitary involution, if P is an orthogonal projection).

An *invection* T is a linear operator satisfying $T^4=I$. That is, T is a 4-involution.

Proposition 4.8 *Every involution is an invection.*

Remark. We note that the converse of Proposition 4.8 is not true in general. There exist non-involutory 4-involutions. For instance, $A=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ is a 4-involution which is not an involution. Geometrically, the 4-involutions A and $-A$ represent rotations through 90° and -90°, respectively.

The following operators are also 4-involutions:

$$A=\begin{pmatrix}-4 & 7 & -5\\ -3 & 6 & -5\\ -2 & 4 & -3\end{pmatrix}, B=\begin{pmatrix}-1 & 2 & -2\\ -1 & 2 & -1\\ 0 & 0 & -1\end{pmatrix},$$

$$C=\begin{pmatrix}-11 & -9 & -15\\ -4 & 3 & 8\\ 10 & -8 & -15\end{pmatrix}$$

Maple Software shows that, $\sigma(T)=\{-1,i,-i\}, \sigma(B)=\{-1,i,-i\}, \sigma(C)=\{-1,i,-i\}$ and that $\|A\|\approx 13.733, \|B\|\approx 3.606$ and $\|C\|\approx 25.814$.

A simple computation shows that A and B are similar. A closer scrutiny reveals that $tr(A)=tr(B)$.

Theorem 4.9 *Let $A\in B(H)$ and suppose that $A\neq I$. If*

$A^2 = I$, *then A is similar to an operator with matrix* $B = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$.

We note that in Theorem 4.9 similarity cannot be replaced with unitary equivalence.

Let A ∈ B(H). Consider the subspaces M and N defined by $M = \{x \in H : Ax = x\}$ and $N = \{x \in H : Ax = -x\}$. Clearly M and N are A- reducing subspaces and $H = M \oplus N$.

Two *n*-involutions need not be similar. The following result gives a condition when two involutions are similar.

Corollary 4.10 *Let A, B ∈ B(H). If A and B are n-involutions and* $tr(A) = tr(B)$, *then A and B are similar.*

Proof. Follows easily from Theorem 4.9.

Proposition 4.11 *If T is a n-involution, then* $\|T\| \geq 1$.

We now try to characterize the class of *n*-involutions with norm 1.

Theorem 4.12 *Let T be an n-involution. Then* $\|T\| = 1$ *if and only if T is unitary.*

Proof. The converse is trivial. Suppose $T^n = I$ and $\|T\| = 1$. Then T is invertible, both T and T^{-1} are power bounded and hence similar to a unitary operator ([5], Corollary 1.16 and Proposition 3.8). The condition that $\|T\| = 1$ then shows that T is unitarily equivalent to a unitary and hence a unitary.

Remark. Theorem 4.12 is true if *T* is unitarily equivalent to the operator. Note that unitary equivalence in this case collapses to equality to $\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$. We note also that $\|A\| = \|B\|$ is a necessary but not a sufficient condition for the unitary equivalence or even similarity of any two operators *A* and *B*.

Corollary 4.13 *If* $T \in B(H)$ *is unitary and* $\sigma(T) \subseteq \{-1\}$ *and* $\dim(H) < \infty$, *then T is a self-adjoint involution.*

Proof. T unitary and $\sigma(T) \subseteq \{-1\}$ implies that $T^* = T^{-1}$ and $\sigma(T^*) = \sigma(T^{-1}) \subseteq \{-1\}$.. This is true if T is self-adjoint. Thus $T^2 = T^*T = I$. This proves the claim.

Example 6. The following operators are 4-involutions with norm 1:

$A = \pm\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $B = \pm\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. A simple calculation shows that they are unitary.

Theorem 4.14 *Every* 4-*involution can be decomposed as a product of two involutions.*

Theorem 4.15 If T is a 4-involution then $P = \frac{1}{4}(I + T^2 + T^3)$ is the associated idempotent operator.

Proof. Indeed, using Theorem 4.14, we have

$$P^2 = \frac{1}{16}[I + T^2 + 3T^2 + 4T^3 + 3T^4 + 2T^5 + T^6]$$

$$= \frac{1}{16}[I + T^2 + T^4 + T^6 + 2T + 2T^2 + 4T^3 + 2T^4 + 2T^5]$$

$$= \frac{1}{16}[4I + 4T = 4T^2 + 4T^3]$$

$$= \frac{1}{4}[I + T + T^2 + T^3]$$

$$= P$$

A linear operator T satisfying $T^3 = -T$ is called a *twist*, *tripotent* if $T^3 = T$ and *k-potent* if $T^k = T$, where $k \geq 2$ is a positive integer. For more recent exposition on tripotent operators (see [8], [10]). The following result relates twists and 4-involutions.

Theorem 4.16 *Let* $T \in B(H)$.

(a). If T *is a twist then* $A = I + T + T^2$ *and* $A^{-1} = I - T + T^2 = A^3$ *are* 4-*involutions.*

(b). If A *is a* 4-*involution then* $T = \frac{1}{2}(A - A^3) = \frac{1}{2}(A - A^{-1})$ *is a twist.*

(c). If T *is a k-potent for* $k \geq 2$, *then Ran(T) is closed and* T^{k-1} *is idempotent and that*

$$Ran(T) = Ran(T^2) = \cdots = Ran(T^k),$$

and

$$Ker(T) = Ker(T^2) = \cdots = Ker(T^k).$$

With respect to the direct sum decomposition H = Ran(T) ⊕ Ker(T), the operator T has a 2 × 2 *operator matrix of the form* $T = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$, *where A: Ran(T) → Ran(T) is invertible and* $A^{k-1} = I$. *That is A is a* $k-1$-*involution.*

Remark. Twists and 4-involutions have important applications in geometry. For instance, if $A : \mathbb{R}^2 \to \mathbb{R}^2$ defined by *A*(*x*, *y*) = (−*y*, *x*), then $A^2 = -I, A^3 = -A$ (i.e. *A* is a twist) and $A^4 = I$ (i.e. *A* is a 4-involution), so that

$$A^{2n} = (-1)^n I, A^{2n+1} = -A.$$

Therefore

$$e^{tA} = \cos tA + \sin tA$$

which is a rotation by the angle *t*:

$$e^{tA}(x, y) = (\cos tx - \sin ty, \cos t + \sin tx),$$

and so *A* is a generator of rotation.

Proposition 4.17 *If T is an n-involution and an m-involution, with n ≥ m + 2, then T is an (n − m)-involution.*

Proof. The proof follows from the fact that $I = T^n = T^m T^{(n-m)} = T^{(n-m)}$.

Note that the converse to Proposition 4.17 is not generally true. For instance an involution T cannot be a 3-involution, unless $T = I$.

The following result is trivial.

Proposition 4.18 *If T is an n-involution then* T^n *is an involution.*

Theorem 4.19 *If T is an involution with associated idempotent P, then Ran(T) = Ker(I − P) and Ran(P) = Ker(I − T).*

Proof. $x \in Ran(T) \Leftrightarrow x = Tx$

$$\begin{aligned} &\Leftrightarrow x = (2P - I)x \\ &\Leftrightarrow (I - (2P - I))x = 0 \\ &\Leftrightarrow (I - P)x = 0 \\ &\Leftrightarrow x \in Ker(I - P). \end{aligned}$$

The second part follows by substituting $P = \frac{1}{2}(I + T)$.

$$x \in Ran(P) \Leftrightarrow x = Px$$

$$\begin{aligned} &\Leftrightarrow x = \frac{1}{2}(I + T)x \\ &\Leftrightarrow (I - T)x = 0 \\ &\Leftrightarrow x \in Ker(I - T). \end{aligned}$$

Theorem 4.20 *If T = AB, where A and B are invertible operators, then T is similar to BA.*

Theorem 4.21 *If T is a product of two involutions then T and* T^{-1} *are similar.*

Proof. If $T = AB$, where A and B are involutions, then using Theorem 4.20, we have $T^{-1} = BA = A(AB)A = ATA^{-1}$. This establishes the claim.

Corollary 4.22 *If T is a* 4*-involution, then T and* T^{-1} *are similar.*

Proof. From Theorem 4.14, every 4-involution is decomposable as a product of two involutions. Invoking Theorem 4.21, the claim follows.

Theorem 4.23 *Every idempotent operator T is k-potent, for every integer* $k \geq 2$.

Theorem 4.23 asserts that every projection operator is k-potent, for every integer $k \geq 2$. But not every k-potent, for every integer $k > 2$ is necessarily a projection.

The following results show that if two idempotent operators have equal range then they are similar.

Theorem 4.24 *If H is a Hilbert space and J is an idempotent on H with range M and* P_M *is the orthogonal projection of H onto M, then J and* P_M *are similar operators.*

Corollary 4.25 *If A, B* $\in B(H)$ *and* $A^2 = A$ *and* $B^2 = B$ *with Ran(A) = Ran(B), then A is similar to B.*

Theorem 4.27 *Let T be an n-involution. Then T is unitary if and only if* $\|T\| = \dfrac{1}{\|(T^{n-1})^{-1}\|}$.

Proof. The fact that T is a unitary involution implies that $T^*T = TT^* = I$ and $T^n = I$. Thus $T = (T^{n-1})^{-1}$. Using the substitution $T = (T^{n-1})^{-1}$, we have $T^*(T^{n-1})^{-1} = I$. Taking norms both sides, we have that $\|T^*\| = \|T\| = \dfrac{1}{\|(T^{n-1})^{-1}\|}$, which proves the claim.

Recall that $T \in B(H)$ is a 2-isometry if $T^{*2}T^2 - 2T^*T + I = 0$.

Proposition 4.28 *An operator T is an n-involution if and only if* T^* *is an n-involution.*

Theorem 4.29 *Let T be a* 2*-isometry. If T is an involution, then T is unitary.*

Proof. Using Proposition 4.28 and a simple computation gives $2(I - T^*T) = 0$ or $T^*T = I$. Thus T is an invertible isometry, which must be unitary. This proves the claim.

Theorem 4.30 *Let T be a self-adjoint operator. Then T is an involution if and only if T is unitary.*

Proof. Suppose $T = T^*$ and $T^2 = I$. Then $T^*T = TT^* = I$. Thus T is unitary. Conversely, suppose $T = T^*$ and $T^*T = TT^* = I$. Then $T^2 = I$, which proves our claim.

Theorem 4.30 says that T is a self-adjoint involution if and only if it is a symmetry. This means that the class of unitary operators and the class of involutions intersect at the class of symmetries. We note that there are operators which are involutions but are not isometric and hence not symmetric. For instance, the operator $T = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ is an involution which is not an isometry (and hence not a symmetry).

Remark. It has been shown by Singh et al [8] that the product of two tripotent operators is a tripotent operator if and only if they commute. The sum of involutions need not be an involution, even if they commute. However, it is clear that every n-involution is a sum of two complementary idempotent operators and that every n-involution is an $n + 1$-potent operator. We aver that the claim by Singh etal [8] is equivalent to saying that the product of two commuting involutions is a tripotent operator.

Operators A and B are said to *quasicommute* if $AB = BA + T$, where T is a compact operator (see also [4], Definition 1.23). Operators A and B are said to *anti-commute* if $AB = -BA$ (or equivalently, $AB+BA = 0$).

Example 7. Consider the operators Q and U defined on $L^2(\partial D)$ by $(Qf)(t) = \overline{f(t)}$ and $(Uf)(t) = if(t)$, for all $f \in L^2(\partial D)$. A simple computation shows that Q is an involution and U is unitary and that $(QU)f = Q(if) = \overline{if} = -i\overline{f}$ and $(UQ)f = U(\overline{f}) = i\overline{f}$.

Clearly $UQ = QU + T$, where $(Tf)(t) = 2i\overline{f}(t)$. Thus Q quasicommutes U. Note however, that $[Q^2, U] = 0$.

Theorem 4.32 *Let Q be an involution and U is unitary operator. Then UQ = QU if and only if Q is a symmetry or U = I.*

Proof. Suppose $UQ = QU$. We pre-multiply by Q and postmultiply by Q and use the fact that $Q^2 = I$. Applying the Fuglede-Putnam Theorem ([5], Theorem 0.15), taking adjoints and then solving the resulting equalities, we have that $Q^* = U^*Q^*U = U^*QU$ or $U = I$. This only holds if the invertible operator Q satisfies $Q^* = Q$ (that is, Q is a symmetry) or $U = I$.

Proposition 4.33 *Let A and B be anti-commuting idempotent operators. Then A + B is idempotent.*

The following results are our adaption from linear topological spaces (see [3]) to real or complex Hilbert spaces.

Theorem 4.34 *([3], Theorem 1) Suppose that* $A \in B(H)$ *is an n-involution where H is a complex Hilbert space. Let* $\varepsilon = e^{\frac{2\pi ik}{n}}$. *Then there exist subspaces* $H_1, H_2, \cdots, H_n \subseteq H$ *such that*

$$H = H_1 \oplus H_2 \oplus \cdots \oplus H_n$$

and

$$A\Big|_{H_k} = \varepsilon^k I\Big|_{H_k}, 0 \le k \le n-1$$

Proof. Consider the operators

$$Q_n = \frac{1}{n}(\sum_{j=0}^{n-1} \varepsilon^{n-k} A)^j, 1 \le k \le n.$$

A simple computation and assuming that $A^0 = I$, we have

$$Q_1 = A^0 = I.$$

$$Q_2 = \frac{1}{2}\Big[(\varepsilon^{2-k}A)^0 + (\varepsilon^{2-k}A)\Big]_{k=1,2}$$

$$= \frac{1}{2}\Big[(\varepsilon^1 A)^0 + (\varepsilon^0 A)^0 + (\varepsilon A) + (\varepsilon^0 A)^1\Big]$$

$$= \frac{1}{2}\big[I + I + \varepsilon A + A\big]$$

$$= I + \frac{1}{2}(1+\varepsilon)A$$

A similar computation gives

$$Q_3 = I + \frac{1}{3}(1+\varepsilon+\varepsilon^2)A + \frac{1}{3}(1+\varepsilon^2+\varepsilon^4)A^2$$

and

$$Q_4 = I + \frac{1}{4}(1+\varepsilon+\varepsilon^2+\varepsilon^3)A + \frac{1}{4}(1+\varepsilon^2+\varepsilon^4+\varepsilon^6)A^2$$

$$+ \frac{1}{4}(1+\varepsilon^3+\varepsilon^6+\varepsilon^9)A^3$$

and so on.

A straightforward verification shows that the Q_k are projections (that is, ${Q_k}^2 = Q_k$). If we set $H_k = Q_k(H)$, then one can easily see that the H_k have the desired properties.

Theorem 4.35 *Let* $T \in B(H)$ *be an n-involution. Then*

(a). $Ker(T^{n+1})$ *and* $Ran(T^{n+1})$ *are T-invariant.*

(b). $Ker(T^{n-1})$ *and* $Ran(T^{n-1})$ *are* T^{-1}*-invariant.*

Proof. (a). Note that $T^n = I$ implies that $T^{n+1} = T$. Thus $Ker(T^{n+1}) = Ker(T)$ and $Ran(T^{n+1}) = Ran(T)$ which are invariant under T.

(b). The result follows from part (a) and the fact that $T^{n-1} = T^{-1}$.

We introduce a new relation of operators.

Let $A, B \in B(H)$. If $A = QB$, where Q is an invertible operator then A and B are said to be *Q-equivalent* (notation $A \overset{Q}{\approx} B$). If Q is bounded but not necessarily invertible, we say that A and B are *Q-partial equivalent*. For instance, if Q is an isometry, we say that A and B are *Q-partial isometric equivalent*. Two operators A and B are said to be *Wilf-equivalent* if $B = VA$, where V is a symmetry (that is, $V = V^* = V^{-1}$).

Note that Q-equivalence preserves invertibility of operators. Note also that metric equivalence implies Q-equivalence of operators. Note that these two relations coincide if Q is a unitary operator.

Theorem 4.36 *Any two invertible operators* $T, S \in B(H)$ *are Q-equivalent.*

Corollary 4.37 *Any two n-involutions T, S* $\in B(H)$ *are Q-equivalent.*

Proof. Since n-involutions are invertible operators, the result follows from Theorem 4.36.

Note that Q-equivalence of A and B is equivalent to the existence of invertible operators Q_1 and Q_2 such that $A = Q_1 B$ and $B = Q_2 A$. We note that Q-equivalence, Q-partial equivalence and Q-partial isometric equivalence are not equivalence relations because they are not symmetric relations.

Theorem 4.38 *If Q is an involution operator then Q-equivalence is an equivalence relation on B(H).*

Proof. Clearly $A = IA$. Thus $A \overset{Q}{\approx} A$. This shows that $\overset{Q}{\approx}$ is reflexive. Now suppose $A \overset{Q}{\approx} B$. Then $A = QB$ and $B = Q^{-1}A = QA$. Thus $B \overset{Q}{\approx} A$. This proves that $\overset{Q}{\approx}$ is symmetric. Finally, suppose that $A \overset{Q}{\approx} B$ and $B \overset{Q}{\approx} C$. We show that $A \overset{Q}{\approx} C$. By hypothesis, there exists invertible operators Q_1 and Q_2 such that $A = Q_1 B$ and $B = Q_2 C$. Thus $A = Q_1 B = Q_1 Q_2 C$, which proves that $A \overset{Q}{\approx} C$. This shows that $\overset{Q}{\approx}$ is transitive. Therefore $\overset{Q}{\approx}$ is an equivalence relation.

Remark. Wilf-equivalence is an equivalence relation stronger than Q-equivalence.

Question. Does Q-equivalence preserve self-adjointness, invertibility, norm, numerical range, etc. of operators? How is it related to other operator equivalence relations?

It is clear that Q-equivalence preserves invertibility but it does not preserve norm, spectrum, and self-adjointness of operators and numerical range of operators. To see this, let $A = Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. A simple calculation shows that $A \overset{Q}{\approx} B$, but $W(A) = [-1,1] \neq \{1\} = W(B)$. This example also reveals that Q-equivalent operators need not have equal spectra, even if Q is unitary. If Q is unitary, then $\|A\| = \|B\|$.

Remark. We note that Q-equivalence of operators is weaker than similarity. To see this, suppose $A = QB$. Then $A = QB = QBQ\,Q^{-1} = Q(BQ)\,Q^{-1}$. It is a well-known result (see [1], Lemma 1) if $A = BC$ for some involutions B and C, then A is similar to CB. We generalize this result below as follows.

Proposition 4.39 *If A and B are Q-equivalent then A is similar to BQ.*

The converse of Theorem 4.39 is also true. This leads to the following strong result.

Corollary 4.40 *Two operators A and B are Q-equivalent if and only if A is similar to BQ.*

Proof. We prove the converse since the other direction has been proved in the remark above. Suppose, without loss of generality that $A = Q(BQ)\,Q^{-1}$. Then a simple computation shows that $A = QB$. This proves the claim.

Proposition 4.41 *Let A and B in B(H) be n-involutions. If A and B are U-equivalent for some unitary operator U, then they are metrically equivalent.*

The converse of Theorem 4.41 is not generally true. The unilateral shift and identity operators on $\ell^2(\mathbb{N})$ are metrically equivalent but not U-equivalent for any unitary operator U. We note that the converse of the above statement holds in finite dimensional Hilbert spaces.

Corollary 4.42 *Two operators A and B on H are U-equivalent for some unitary operator U if and only if they are normal and metrically equivalent.*

Proof. We prove the converse. Part of the other direction follows from Proposition 4.39 with $Q = U$, where U is a unitary operator. Now suppose A and B are metrically equivalent normal operators. Then by ([7], Corollary 2.6), there exists a unitary operator U such that

$$A = UB.$$

Corollary 4.43 *If two n-involutions A and B acting on a Hilbert space H are U-equivalent for some unitary operator U then* $\|A\| = \|B\|$.

Proof. Follows from the proof of Proposition 4.41.

5. Discussion

The notion of an operator T satisfying $T^n = I$ is applicable in solving singular integral equations with a Carleman shift that involve an involutive operator Q such that $Q^2 = I$, or more generally, $Q^n = I$. Such equations can be written in the form

$$K\phi := (A + QB)\phi = f$$

or more generally

$$K\phi := (A_1 + QA_2 + Q^2 A_3 + \cdots + Q^{n-1} A_n)\phi = f$$

where $A_1, A_2, \cdots, A_n$ are bounded linear operators in a Banach space under consideration (see [4]). Projection operators (associated with involutions) are useful in vast areas of physics-in quantum theory, many-body physics, applications in group theory, projective geometry, statistical mechanics of irreversibility, to mention but a few. Quadratic forms with idempotent and tripotent operator matrices are extensively used in the theory of statistics, especially in the area of multivariate normal distributions (see [9]).

6. Conclusion

In this paper, the structure of some n-involution and k-potent operators and their relationships has been shown. It has been shown that any normaloid n-involution is unitary. It has been shown that unitary equivalence, similarity and quasisimilarity preserve the n-involutory property of operators and that metric equivalence preserves this property for self-adjoint operators. Several conditions under which an n-involution has norm one has been proved. The notion of Q-equivalence is introduced and it is shown that if two n-involutions are U-equivalent for some unitary operator then they have the same norm.

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