New fixed point theorems of condensing mappings satisfying the interior condition in Banach spaces

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Abstract: In this paper, based on a basic result on condensing mappings satisfying the interior condition, some new fixed point theorems of the condensing mappings of this kind are obtained. As a result, the famous Petryshyn's theorem and some results of the Reference [8] are extended to the condensing mappings satisfying the interior condition.

Keywords: Condensing Mappings, Interior Condition, Fixed Point, Banach Space

1. Introduction and Preliminaries

As is well known, the condensing mappings are a class of important nonlinear operators, existing extensively in the nonlinear differential and integral equations. So, the fixed point theorems of the condensing mappings play a central role in the study of existence of the solutions of these equations (see [1-8]).

Let X be a real Banach space and D an open subset of X with $\theta \in D$ where θ denotes the zero element of X . Denote by $\bar{D}$ and ∂D the closure and boundary of D respectively. It is well known that if D is bounded and if $A: \bar{D} \rightarrow X$ is a condensing mapping, then we have had some well-known theorems as follows (see [1-5, 8]).

Theorem 1.1 Suppose that one of the following conditions is satisfied:

- (i) (Leray – Schauder) $Ax \neq \lambda x, \forall x \in \partial D$;
- (ii) (Roth) $\|A(x)\| \leq \|x\|, \forall x \in \partial \Omega$;
- (iii) (Petryshyn) $\|A(x)\| \leq \|A(x) - x\|, \forall x \in \partial \Omega$;
- (iv) (Altman) $\|A(x) - x\|^2 \geq \|A(x)\|^2 - \|x\|^2, \forall x \in \partial \Omega$.

Then A has at least one fixed point in $\bar{D}$.

Recently, Antonio and Morales [7] introduce a new condition which resembles the Leray-Schauder boundary condition mentioned above. It is called the Interior Condition defined as follows.

Definition 1.1 [7] We say that a mapping $A: \bar{D} \rightarrow X$ satisfies the Interior Condition if there exists $\delta > 0$ such that

$$Ax \neq \lambda x, \forall x \in D^*, \lambda > 1, Ax \notin \bar{D},$$

where

$$D^* = \{x \in D : \text{dist}(x, \partial D) < \delta\}. \quad (1.1)$$

Recall that the set-measure of non-compactness of a bounded subset E of X is defined as (see [2-4])

$\gamma[E] = \inf \{d > 0 : E \text{ can be covered by finitely many sets of diameter } \leq d\}$;

A mapping $A: D \subset X \rightarrow X$ is called a condensing mapping (see [2 - 4]), if A is continuous and $\gamma[A(E)] < \gamma[E]$ for all bounded subset E of D where $\gamma[E] > 0$.

Definition 1.2 [7] Let X be a real Banach space and D an bounded open subset of X with $\theta \in D$. We say that D is strictly star-shaped with respect to the origin if the following condition is satisfied:

$$\{tx : t > 0\} \cap \partial D = \{x\}, \forall x \in \partial D. \quad (1.2)$$

From now on, we will assume without loss of generality,

that the star-shaped assumption will always be considered with respect to the origin, unless the contrary is mentioned.

In this paper, based on a basic result on condensing mappings satisfying the interior condition, some new fixed point theorems of the condensing mappings of this kind are obtained. As a result, the famous Petryshyn's theorem and some results of the Reference [8] are extended to the condensing mappings satisfying the interior condition.

2. Main Results

Lemma 2.1^[7] Let X be a real Banach space and let D be a bounded open and strictly star-shaped subset of X with $\theta \in D$. If $A: \overline{D} \rightarrow X$ is a condensing mapping satisfying the Condition (I-C): there exists $\delta > 0$ such that

$$Ax \neq \lambda x, \forall x \in D^*, \lambda > 1, Ax \notin \overline{D}, \quad (\text{I-C})$$

where

$$D^* = \{x \in D : \text{dist}(x, \partial G) < \delta\}$$

for some $\delta > 0$, then A has at least one fixed point in $\overline{D}$.

We are now in the position to apply the above basic result to derive some new fixed point theorems for the condensing mappings satisfying the Condition (I-C) which extend many well-known results to the case of the mappings satisfying the interior condition.

Theorem 2.1 Let X be a real Banach space and let D be a bounded open and strictly star-shaped subset of X with $\theta \in D$. Let $A: \overline{D} \rightarrow X$ be a condensing mapping. Suppose also that there exist $1 \leq \alpha < \beta$ or $1 < \alpha \leq \beta$ such that

$$\|Ax - x\|^\alpha \|x\|^\beta \geq \|Ax\|^\alpha \|Ax + x\|^\beta - \|Ax\|^\alpha \|x\|^\beta; \quad \forall x \in D^* \text{ and } Ax \notin \overline{D}, \quad (2.1)$$

where

$$D^* = \{x \in D : \text{dist}(x, \partial G) < \delta\}$$

for some $\delta > 0$, then A has at least one fixed point in $\overline{D}$.

Proof If the operator A has a fixed point on ∂D , then A has at least one fixed point on D . Now suppose that A has no fixed points on ∂D . Next we shall prove that the condition (I-C) is satisfied.

Suppose it is not true. Then there exist $x_0 \in \partial D$ and $\mu_0 \geq 1$ such that $Ax_0 = \mu_0 x_0$. It is easy to see that $\mu_0 > 1$. Now, consider the function defined by

$$f(t) = (t-1)^\alpha - t^\alpha (t+1)^\beta + t^\alpha, \forall t \geq 1.$$

Since

$$f'(t) = \alpha(t-1)^{\alpha-1} - \alpha t^{\alpha-1} (t+1)^\beta + \alpha t^{\alpha-1} - \beta t^\alpha (t+1)^{\beta-1}$$

< 0 by formal differentiation, $f(t)$ is a strictly decreasing function in $[1, \infty)$. And so $f(t) < f(1)$ for $t > 1$, i.e.,

$$(t-1)^\alpha < t^\alpha (t+1)^\beta - t^\alpha, \text{ for any } t > 1. \text{ Consequently,}$$

noting that $\|x_0\| \neq 0, \mu_0 > 1$, we have

$$\begin{aligned} & \|Ax_0 - x_0\|^\alpha \|x_0\|^\beta \\ &= \|\mu_0 x_0 - x_0\|^\alpha \|x_0\|^\beta \\ &= (\mu_0 - 1)^\alpha \|x_0\|^{\alpha+\beta} \\ &< [\mu_0^\alpha (\mu_0 + 1)^\beta - \mu_0^\alpha] \|x_0\|^{\alpha+\beta} \\ &= \|Ax_0\|^\alpha \|Ax_0 + x_0\|^\beta - \|Ax_0\|^\alpha \|x_0\|^\beta, \end{aligned}$$

which is a contradiction to (2.1), and so the condition (I-C) is satisfied. Therefore, it follows from Lemma 2.1 that the conclusion of Theorem 2.1 holds.

From Theorem 2.1, we can easily get the subsequent four corollaries.

Corollary 2.1 Let X be a real Banach space and let D be a bounded open and strictly star-shaped subset of X with $\theta \in D$. Let $A: \overline{D} \rightarrow X$ be a condensing mapping. Suppose also that there exists $\alpha > 1$, such that

$$\begin{aligned} & (\|Ax - x\| \|x\|)^\alpha \geq (\|Ax\| \|Ax + x\|)^\alpha - (\|Ax\| \|x\|)^\alpha, \\ & \forall x \in D^* \text{ and } Ax \notin \overline{D}, \end{aligned}$$

where

$$D^* = \{x \in D : \text{dist}(x, \partial G) < \delta\}$$

for some $\delta > 0$, then A has at least one fixed point in $\overline{D}$.

Proof Take $\alpha = \beta$. It follows from Theorem 2.1 that the conclusion of Corollary 2.1 holds true.

Corollary 2.2 Let X be a real Banach space and let D be a bounded open and strictly star-shaped subset of X with $\theta \in D$. Let $A: \overline{D} \rightarrow X$ be a condensing mapping. Suppose also that A satisfies the following condition

$$\begin{aligned} & \sqrt{(\|Ax - x\| \|x\|)^3} \geq \sqrt{(\|Ax\| \|Ax + x\|)^3} - \sqrt{(\|Ax\| \|x\|)^3}; \\ & \forall x \in D^* \text{ and } Ax \notin \overline{D}, \end{aligned}$$

where

$$D^* = \{x \in D : \text{dist}(x, \partial G) < \delta\}$$

for some $\delta > 0$, then A has at least one fixed point in $\overline{D}$.

Proof Take $\alpha = \beta = \frac{3}{2}$. It follows from Theorem 2.1 that

the conclusion of Corollary 2.2 holds true.

Corollary 2.3 Let X be a real Banach space and let D be a bounded open and strictly star-shaped subset of X with $\theta \in D$. Let $A: \overline{D} \rightarrow X$ be a condensing mapping. Suppose also that A satisfies the following condition

$$(\|Ax - x\| \|x\|)^2 \geq (\|Ax\| \|Ax + x\|)^2 - (\|Ax\| \|x\|)^2;$$

$$\forall x \in D^* \text{ and } Ax \notin \overline{D};$$

where

$$D^* = \{x \in D : \text{dist}(x, \partial G) < \delta\}$$

for some $\delta > 0$, then A has at least one fixed point in $\overline{D}$.

Proof Take $\alpha = \beta = 2$. It follows from Theorem 2.1 that the conclusion of Corollary 2.3 holds true.

Corollary 2.4 Let X be a real Banach space and let D be a bounded open and strictly star-shaped subset of X with $\theta \in D$. Let $A: \overline{D} \rightarrow X$ be a condensing mapping. Suppose also that there exist $\beta > 1$, such that

$$\|Ax - x\| \|x\|^\beta \geq \|Ax\| \|Ax + x\|^\beta - \|Ax\| \|x\|^\beta,$$

$$\forall x \in D^* \text{ and } Ax \notin \overline{D},$$

where

$$D^* = \{x \in D : \text{dist}(x, \partial G) < \delta\}$$

for some $\delta > 0$, then A has at least one fixed point in $\overline{D}$.

Proof Take $\alpha = 1$. It follows from Theorem 2.1 that the conclusion of Corollary 2.4 holds true.

Theorem 2.2 Let X be a real Banach space and let D be a bounded open and strictly star-shaped subset of X with $\theta \in D$. Let $A: \overline{D} \rightarrow X$ be a condensing mapping. Suppose also that there exist $\alpha \geq 1, \beta \geq 0$, such that

$$\|Ax + x\|^{\alpha+\beta} \leq \|Ax\|^\beta \|Ax - x\|^\alpha + \|x\|^{\alpha+\beta};$$

$$\forall x \in D^* \text{ and } Ax \notin \overline{D}, \quad (2.2)$$

where

$$D^* = \{x \in D : \text{dist}(x, \partial G) < \delta\}$$

for some $\delta > 0$, then A has at least one fixed point in $\overline{D}$.

Proof If the operator A has a fixed point on ∂D , then A has at least one fixed point on D . Now suppose that A has no fixed points on ∂D . Next we shall prove that the condition (I-C) is satisfied.

Suppose it is not true. Then there exist $x_0 \in \partial D$ and $\mu_0 \geq 1$ such that $Ax_0 = \mu_0 x_0$. It is easy to see that

$\mu_0 > 1$. Now, consider the function defined by

$$f(t) = (t+1)^{\alpha+\beta} - t^\beta (t-1)^\alpha - 1, \forall t \geq 1.$$

Since

$f'(t) = \alpha[(t+1)^{\alpha+\beta-1} - t^\beta (t-1)^{\alpha-1}] + \beta[(t+1)^{\alpha+\beta-1} - t^{\beta-1} (t-1)^\alpha]$
 > 0 by formal differentiation, $f(t)$ is a strictly decreasing function in $[1, \infty)$. And so $f(t) > f(1)$ for $t > 1$. Thus, $(t+1)^{\alpha+\beta} > t^\beta (t-1)^\alpha + 1$, for any $t > 1$. Consequently, noting that $\|x_0\| \neq 0, \mu_0 > 1$, we have

$$\begin{aligned} & \|Ax_0 + x_0\|^{\alpha+\beta} \\ &= \|\mu_0 x_0 + x_0\|^{\alpha+\beta} \\ &= (\mu_0 + 1)^{\alpha+\beta} \|x_0\|^{\alpha+\beta} \\ &> [\mu_0^\beta (\mu_0 - 1)^\alpha + 1] \|x_0\|^{\alpha+\beta} \\ &= \|\mu_0 x_0\|^\beta \|\mu_0 x_0 - x_0\|^\alpha + \|x_0\|^{\alpha+\beta} \\ &= \|Ax_0\|^\beta \|Ax_0 - x_0\|^\alpha + \|x_0\|^{\alpha+\beta}, \end{aligned}$$

which is a contradiction to (2.2), and so the condition (I-C) is satisfied. Therefore, it follows from Lemma 2.1 that the conclusion of Theorem 2.2 holds.

From Theorem 2.2, we can easily obtain the subsequent five corollaries.

Corollary 2.5 Let X be a real Banach space and let D be a bounded open and strictly star-shaped subset of X with $\theta \in D$. Let $A: \overline{D} \rightarrow X$ be a condensing mapping. Suppose also that there exists $\alpha > 1$ such that

$$\|Ax + x\|^{2\alpha} \leq \|Ax\|^\alpha \|Ax - x\|^\alpha + \|x\|^{2\alpha},$$

$$\forall x \in D^* \text{ and } Ax \notin \overline{D},$$

where

$$D^* = \{x \in D : \text{dist}(x, \partial G) < \delta\}$$

for some $\delta > 0$, then A has at least one fixed point in $\overline{D}$.

Proof Take $\beta = \alpha > 1$. It follows from Theorem 2.2 that the conclusion of Corollary 2.5 holds true.

Corollary 2.6 Let X be a real Banach space and let D be a bounded open and strictly star-shaped subset of X with $\theta \in D$. Let $A: \overline{D} \rightarrow X$ be a condensing mapping. Suppose also that A satisfies the following condition

$$\|Ax + x\|^3 \leq (\|Ax\| \|Ax - x\|)^{\frac{3}{2}} + \|x\|^3,$$

$$\forall x \in D^* \text{ and } Ax \notin \overline{D},$$

where

$$D^* = \{x \in D : \text{dist}(x, \partial G) < \delta\}$$

for some $\delta > 0$, then A has at least one fixed point in $\overline{D}$.

Proof Take $\alpha = \beta = 3/2$. It follows from Theorem 2.2 that the conclusion of Corollary 2.6 holds true.

Corollary 2.7 Let X be a real Banach space and let D be a bounded open and strictly star-shaped subset of X with $\theta \in D$. Let $A: \overline{D} \rightarrow X$ be a condensing mapping.

Suppose also that there exists $\alpha \geq 1$ such that

$$\|Ax + x\|^\alpha \leq \|Ax - x\|^\alpha + \|x\|^\alpha,$$

$$\forall x \in D^* \text{ and } Ax \notin \overline{D},$$

where

$$D^* = \{x \in D : \text{dist}(x, \partial G) < \delta\}$$

for some $\delta > 0$, then A has at least one fixed point in $\overline{D}$.

Proof Take $\beta = 0$. It follows from Theorem 2.2 that the conclusion of Corollary 2.7 holds true.

Corollary 2.8 Let X be a real Banach space and let D be a bounded open and strictly star-shaped subset of X with $\theta \in D$. Let $A: \overline{D} \rightarrow X$ be a condensing mapping. Suppose also that A satisfies the following condition

$$\|Ax + x\|^2 \leq \|Ax - x\|^2 + \|x\|^2,$$

$$\forall x \in D^* \text{ and } Ax \notin \overline{D},$$

where

$$D^* = \{x \in D : \text{dist}(x, \partial G) < \delta\}$$

for some $\delta > 0$, then A has at least one fixed point in $\overline{D}$.

Proof Take $\alpha = 1, \beta = 1$. It follows from Theorem 2.2 that the conclusion of Corollary 2.8 holds true.

Remark 1 We can see that the formula which satisfies the condition of Corollary 2.8 above, must also satisfies the famous Altman's theorem, namely,

$$\|Ax - x\|^2 \geq \|Ax\|^2 + \|x\|^2, \forall x \in D^* \text{ and } Ax \notin \overline{D}$$

Therefore, Corollary 2.8 and Theorem 2.2 are the useful supplements of the famous Altman's theorem.

Corollary 2.9 Let X be a real Banach space and let D be a bounded open and strictly star-shaped subset of X with $\theta \in D$. Let $A: \overline{D} \rightarrow X$ be a condensing mapping. Suppose also that A satisfies the following condition

$$\|Ax + x\|^2 \leq \|Ax - x\|^{\frac{3}{2}} \|Ax\|^{\frac{1}{2}} + \|x\|^2,$$

$$\forall x \in D^* \text{ and } Ax \notin \overline{D},$$

where

$$D^* = \{x \in D : \text{dist}(x, \partial G) < \delta\}$$

for some $\delta > 0$, then A has at least one fixed point in $\overline{D}$.

Proof Take $\alpha = \frac{3}{2}, \beta = \frac{1}{2}$. It follows from Theorem 2.2 that the conclusion of Corollary 2.9 holds true.

Theorem 2.3 Let X be a real Banach space and let D be a bounded open and strictly star-shaped subset of X with $\theta \in D$. Let $A: \overline{D} \rightarrow X$ be a condensing mapping. Suppose also that there exist $\alpha \geq 1, \beta \geq 0$, such that

$$\|Ax\|^\alpha \|Ax + x\|^\beta \leq \|Ax\|^\beta \|Ax - x\|^\alpha,$$

$$\forall x \in D^* \text{ and } Ax \notin \overline{D}; \quad (2.3)$$

where

$$D^* = \{x \in D : \text{dist}(x, \partial G) < \delta\}$$

for some $\delta > 0$, then A has at least one fixed point in $\overline{D}$.

Proof If the operator A has a fixed point on ∂D , then A has at least one fixed point on D . Now suppose that A has no fixed points on ∂D . Next we shall prove that the condition (I-C) is satisfied.

Suppose it is not true. Then there exist $x_0 \in \partial D$ and $\mu_0 \geq 1$ such that $Ax_0 = \mu_0 x_0$. It is easy to see that $\mu_0 > 1$. Now, consider the function defined by

$$f(t) = t^\alpha (t+1)^\beta - t^\beta (t-1)^\alpha, \forall t \geq 1.$$

Since

$$f'(t) = \alpha [t^{\alpha-1} (t+1)^\beta - (t-1)^{\alpha-1} t^\beta] + \beta [t^\alpha (t+1)^{\beta-1} - t^{\beta-1} (t-1)^\alpha] > 0$$

by formal differentiation, $f(t)$ is a strictly decreasing function in $[1, \infty)$. And so $f(t) > f(1)$ for $t > 1$. Thus, $t^\alpha (t+1)^\beta > t^\beta (t+1)^\alpha$, for any $t > 1$. Consequently,

noting that $\|x_0\| \neq 0, \mu_0 > 1$, we have

$$\begin{aligned} & \|Ax_0\|^\alpha \|Ax_0 + x_0\|^\beta \\ &= \|\mu_0 x_0\|^\alpha \|\mu_0 x_0 + x_0\|^\beta \\ &= \mu_0^\alpha (\mu_0 + 1)^\beta \|x_0\|^{\alpha+\beta} \\ &> \mu_0^\beta (\mu_0 - 1)^\alpha \|x_0\|^{\alpha+\beta} \end{aligned}$$

$$\begin{aligned} &= \|\mu_0 x_0\|^\beta \|\mu_0 x_0 - x_0\|^\alpha \\ &= \|Ax_0\|^\beta \|Ax_0 - x_0\|^\alpha, \end{aligned}$$

which is a contradiction to (2.3), and so the condition (I-C) is satisfied. Therefore, it follows from Lemma 2.1 that the conclusion of Theorem 2.3 holds.

From Theorem 2.3 we can easily get the subsequent such a corollary.

Corollary 2.10 Let X be a real Banach space and let D be a bounded open and strictly star-shaped subset of X with $\theta \in D$. Let $A: \overline{D} \rightarrow X$ be a condensing mapping. Suppose also that A satisfies the following condition

$$\|Ax\| \leq \|Ax - x\|, \forall x \in D^* \text{ and } Ax \notin \overline{D};$$

where

$$D^* = \{x \in D : \text{dist}(x, \partial G) < \delta\}$$

for some $\delta > 0$, then A has at least one fixed point in $\overline{D}$.

Proof Take $\alpha = 1, \beta = 0$. It follows from Theorem 2.3 that the conclusion of Corollary 2.10 holds true.

Remark 2 Corollary 2.10 is the famous Petryshyn's theorem, so the Theorem 2.3 extends the famous Petryshyn's theorem to the case of condensing mappings satisfying the interior condition.

Theorem 2.4 Let X be a real Banach space and let D be a bounded open and strictly star-shaped subset of X with $\theta \in D$. Let $A: \overline{D} \rightarrow X$ be a condensing mapping. Suppose also that there exist $\alpha \geq 1, \beta \geq 0$, such that

$$\begin{aligned} \|Ax - x\|^\alpha \|x\|^{\alpha+\beta} &\geq \|Ax\|^\alpha \|Ax + x\|^{\alpha+\beta} - \|x\|^{2\alpha+\beta}, \\ \forall x \in D^* \text{ and } Ax &\notin \overline{D}, \end{aligned} \quad (2.4)$$

where

$$D^* = \{x \in D : \text{dist}(x, \partial G) < \delta\}$$

for some $\delta > 0$, then A has at least one fixed point in $\overline{D}$.

Proof If the operator A has a fixed point on ∂D , then A has at least one fixed point on D . Now suppose that A has no fixed points on ∂D . Next we shall prove that the condition (I-C) is satisfied.

Suppose it is not true. Then there exist $x_0 \in \partial D$ and $\mu_0 \geq 1$ such that $Ax_0 = \mu_0 x_0$. It is easy to see that $\mu_0 > 1$. Now, consider the function defined by

$$f(t) = (t-1)^\alpha - t^\alpha (t+1)^{\alpha+\beta} + 1, \forall t \geq 1.$$

Since

$$f'(t) = \alpha \left[(t-1)^{\alpha-1} - t^{\alpha-1} (t+1)^{\alpha+\beta} - t^\alpha (t+1)^{\alpha+\beta-1} \right] - \beta \left[t^\alpha (t+1)^{\alpha+\beta-1} \right]$$

< 0 by formal differentiation, $f(t)$ is a strictly decreasing function in $[1, \infty)$. And so $f(t) < f(1)$ for $t > 1$. Thus, $(t-1)^\alpha < t^\alpha (t+1)^{\alpha+\beta} - 1$, for any $t > 1$. Consequently, noting that $\|x_0\| \neq 0, \mu_0 > 1$, we have

$$\begin{aligned} &\|x_0\|^{\alpha+\beta} \|\mu_0 x_0 - x_0\|^\alpha \\ &= \|Ax_0 - x_0\|^\alpha \|x_0\|^{\alpha+\beta} \\ &= (\mu_0 - 1)^\alpha \|x_0\|^{2\alpha+\beta} \\ &< \left[\mu_0^\alpha (\mu_0 + 1)^{\alpha+\beta} - 1 \right] \|x_0\|^{2\alpha+\beta} \\ &= \|\mu_0 x_0\|^\alpha \|\mu_0 x_0 + x_0\|^{\alpha+\beta} - \|x_0\|^{2\alpha+\beta} \\ &= \|Ax_0\|^\alpha \|Ax_0 + x_0\|^{\alpha+\beta} - \|x_0\|^{2\alpha+\beta}, \end{aligned}$$

which is a contradiction to (2.4), and so the condition (I-C) is satisfied. Therefore, it follows from Lemma 2.1 that the conclusion of Theorem 2.4 holds.

From Theorem 2.4, we can easily obtain the subsequent three corollaries.

Corollary 2.11 Let X be a real Banach space and let D be a bounded open and strictly star-shaped subset of X with $\theta \in D$. Let $A: \overline{D} \rightarrow X$ be a condensing mapping. Suppose also that there exists $\alpha \geq 1$ such that

$$\begin{aligned} \|x\|^\alpha \|Ax - x\|^\alpha &\geq \|Ax\|^\alpha \|Ax + x\|^\alpha - \|x\|^{2\alpha}, \\ \forall x \in D^* \text{ and } Ax &\notin \overline{D}, \end{aligned}$$

where

$$D^* = \{x \in D : \text{dist}(x, \partial G) < \delta\}$$

for some $\delta > 0$, then A has at least one fixed point in $\overline{D}$.

Proof Take $\beta = 0$. It follows from Theorem 2.4 that the conclusion of Corollary 2.11 holds true.

Corollary 2.12 Let X be a real Banach space and let D be a bounded open and strictly star-shaped subset of X with $\theta \in D$. Let $A: \overline{D} \rightarrow X$ be a condensing mapping. Suppose also that A satisfies

$$\begin{aligned} \|x\| \|Ax - x\| &\geq \|Ax\| \|Ax + x\| - \|x\|^2, \\ \forall x \in D^* \text{ and } Ax &\notin \overline{D}, \end{aligned}$$

where

$$D^* = \{x \in D : \text{dist}(x, \partial G) < \delta\}$$

for some $\delta > 0$, then A has at least one fixed point in $\overline{D}$.

Proof Take $\alpha = 1, \beta = 0$. It follows from Theorem 2.4

that the conclusion of Corollary 2.12 holds true.

Corollary 2.13 Let X be a real Banach space and let D be a bounded open and strictly star-shaped subset of X with $\theta \in D$. Let $A: \overline{D} \rightarrow X$ be a condensing mapping. Suppose also that A satisfies

$$\|Ax - x\| \|x\|^2 \geq \|Ax\| \|Ax + x\|^2 - \|x\|^3,$$

$$\forall x \in D^* \text{ and } Ax \notin \overline{D},$$

where

$$D^* = \{x \in D : \text{dist}(x, \partial G) < \delta\}$$

for some $\delta > 0$, then A has at least one fixed point in $\overline{D}$.

Proof Take $\alpha=1, \beta=1$. It follows from Theorem 2.4 that the conclusion of Corollary 2.13 holds true.

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