Advances in Applied Sciences
2016; 1(2): 24-29
http://www.sciencepublishinggroup.com/j/aas
doi: 10.11648/j.aas.20160102.11

Approximation of the Cut Function by Some Generic Logistic Functions and Applications

Nikolay Kyurkchiev, Svetoslav Markov

Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia, Bulgaria

Email address:
nkyurk@math.bas.bg (N. Kyurkchiev), smarkov@bio.bas.bg (S. Markov)

To cite this article:
Nikolay Kyurkchiev, Svetoslav Markov. Approximation of the Cut Function by Some Generic Logistic Functions and Applications. *Advances in Applied Sciences.* Vol. 1, No. 2, 2016, pp. 24-29. doi: 10.11648/j.aas.20160102.11

Received: August 17, 2016; **Accepted**: August 27, 2016; **Published**: September 12, 2016

Abstract: In this paper we study the uniform approximation of the cut function by smooth sigmoid functions such as Nelder and Turner–Blumenstein–Sebaugh growth functions. To illustrate the use of one of the models we have fitted the model to the "classical Verhulst data". Several numerical examples are presented throughout the paper using the contemporary computer algebra system MATHEMATICA.

Keywords: Sigmoid Functions, Cut Function, Step Function, Nelder Growth Function, Turner–Blumenstein–Sebaugh Generic Function, Uniform Approximation

1. Introduction

We study the uniform approximation of the cut function by Nelder and Turner–Blumenstein–Sebaugh growth functions. We find an expression for the error of the best uniform approximation.

The estimates obtained give more insight on the lag phase, growth phase and plateau phase in the growth process [1–4].

2. Preliminaries

2.1. Sigmoid Functions

In this work we consider sigmoidal functions of a single variable defined on the real line, that is functions of the form $\mathbb{R} \to \mathbb{R}$. Sigmoid functions can be defined as bounded monotone non-decreasing functions on $\mathbb{R}$. One usually makes use of normalized sigmoidal functions defined as monotone non-decreasing functions $s(t), t \in \mathbb{R}$, such that $\lim s(t)_{t\to-\infty} = 0$ and $\lim s(t)_{t\to\infty} = 1$ (in some applications the left asymptote is assumed to be -1: $\lim s(t)_{t\to-\infty} = -1$).

2.2. The Cut and the Nelder and Turner–Blumenstein–Sebaugh Growth Functions

The cut (ramp) function is the simplest piece-wise linear sigmoidal function. Let $\Delta = [\gamma-\delta, \gamma+\delta]$ be an interval on the real line $\mathbb{R}$ with centre $\gamma \in \mathbb{R}$ and radius $\delta \in \mathbb{R}$. A cut function is defined as follows:

Definition. The cut function $c_{\gamma,\delta}$ is defined for $t \in \mathbb{R}$ by

$$c_{\gamma,\delta}(t) = \begin{cases} 0, & \text{if} \quad t < \gamma-\delta, \\ \dfrac{t-\gamma+\delta}{2\delta}, & \text{if} \quad |t-\gamma| < \delta, \\ 1, & \text{if} \quad t > \gamma+\delta. \end{cases} \tag{1}$$

Note that the slope of function $c_{\gamma,\delta}(t)$ on the interval Δ is $1/(2\delta)$ (the slope is constant in the whole interval Δ).

Two special cases are of interest for present discussion in the sequel.

Special case 1. For $\gamma = 0$ we obtain the special cut function on the interval $\Delta = [-\delta, \delta]$:

$$c_{0,\delta}(t) = \begin{cases} 0, & \text{if} \quad t < -\delta, \\ \dfrac{t+\delta}{2\delta}, & \text{if} \quad -\delta \le t \le \delta, \\ 1, & \text{if} \quad \delta < t. \end{cases}$$

Special case 2. For $\gamma = \delta$ we obtain the special cut function on the interval $\Delta = [0, 2\delta]$:

$$c_{\delta,\delta}(t) = \begin{cases} 0, & \text{if} \quad t < 0, \\ \dfrac{t}{2\delta}, & \text{if} \quad 0 \le t \le 2\delta, \\ 1, & \text{if} \quad 2\delta < t. \end{cases}$$

In 1961 Nelder [6] consider the differential equation

$$x' = -\beta k^{-m} x (k^m - x^m)$$

with the solution

$$x(t) = \frac{k}{\left(1 + \left(k^m x_0^{-m} - 1\right) e^{-\beta m t}\right)^{\frac{1}{m}}}. \quad (2)$$

When $m = 1$ the ordinary logistic equation is obtained. An attractive choice for $k = k(t)$ is given by Turner–Blumenstein–Sebaugh in [5]:

$$k(t) = \frac{K}{\left(1 + \alpha e^{-Bmt}\right)^{\frac{1}{m}}}$$

The generalization (*the generic logistic equation*) of the Verhulst logistic equation has the form [5]:

$$T(t) = \frac{K}{\left(1 + \left(K^m x_0^{-m} - 1 - \dfrac{\alpha\beta}{\beta - B}\right) e^{-\beta m t} + \dfrac{\alpha\beta}{\beta - B} e^{-Bmt}\right)^{\frac{1}{m}}}, \quad (3)$$

where m, α, β, K, B are growth parameters. If $m = 1$ and either $\alpha = 0$ or $B \longrightarrow \infty$ then (3) reduces to the ordinary logistic equation.

Definition. Define the following shifted modification of (3) with jump at point γ as:

$$T_\gamma(t) = \frac{1}{2x_0} \times \frac{K}{\left(1 + \left(K^m x_0^{-m} - 1 - \dfrac{\alpha\beta}{\beta - B}\right) e^{-\beta m (t-\gamma)} + \dfrac{\alpha\beta}{\beta - B} e^{-Bm(t-\gamma)}\right)^{\frac{1}{m}}} \quad (4)$$

for which $T_\gamma(\gamma) = \dfrac{1}{2}$.

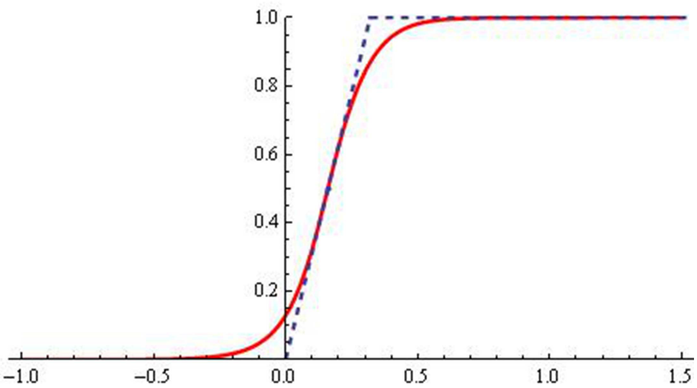

Figure 1. *The cut and the function* $T_\gamma(t)$ *with* $K = 1$, $m = 3$, $\beta = 10$, $\alpha = 2$, $B = 4$, $x_0 = K/2$, $\gamma = \dfrac{1}{2l} = 0.16$, *uniform distance* $\rho = 0.128684$.

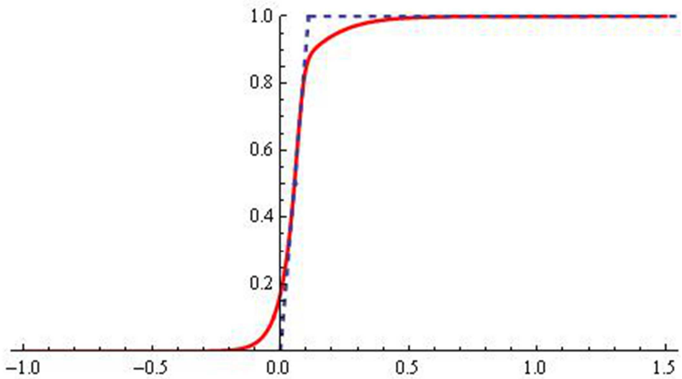

Figure 2. *The cut and the function* $T_\gamma(t)$ *with* $K = 1$, $m = 4.5$, $\beta = 20$, $\alpha = 1.05$, $B = 2$, $x_0 = K/2$, $\gamma = \dfrac{1}{2l} = 0.0549812$, *uniform distance* $\rho = 0.170226$.

3. Approximation of the Cut Function by Shifted Turner–Blumenstein–Sebaugh Function (4)

We next focus on the approximation of the cut function (1) by shifted Turner–Blumenstein–Sebaugh (STBS) growth function $T_\gamma(t)$ defined by (4).

Note that the slope of $T_\gamma(t)$ at $t=\gamma$ is

$$T_\gamma'(\gamma)=\frac{1}{2K^m x_0^{-m}}\left(\beta\left(K^m x_0^{-m}-1-\frac{\alpha\beta}{\beta-B}\right)+B\frac{\alpha\beta}{\beta-B}\right)=l$$

and the slope of $c_{\delta,\delta}(t)$ at $t=\gamma$ is $c'_{\delta,\delta}(\gamma)=\frac{1}{2\delta}$.

Let choose $l=\frac{1}{2\delta}$. The function defined by (4) has an inflection at point $(\gamma,\frac{1}{2})$ (see Fig. 1 and Fig. 2).

Consider functions (1) and (4) with same centres $\gamma=\delta$, that is functions $c_{\delta,\delta}$ and T_δ.

In addition chose c and T to have same slopes at their coinciding centres.

Then, noticing that the largest uniform distance ρ between the cut and (STBS) functions is achieved at the endpoints of the underlying interval $[0,2\delta]$ we have:

$$\rho=T_\delta(0)-c_{\delta,\delta}(0)=\frac{K}{2x_0}\frac{1}{\left(1+\left(K^m x_0^{-m}-1-\frac{\alpha\beta}{\beta-B}\right)e^{\beta m\gamma}+\frac{\alpha\beta}{\beta-B}e^{B m\gamma}\right)^{\frac{1}{m}}}=\frac{K}{2x_0}\frac{1}{\left(1+\left(K^m x_0^{-m}-1-\frac{\alpha\beta}{\beta-B}\right)e^{\frac{\beta m}{2l}}+\frac{\alpha\beta}{\beta-B}e^{\frac{Bm}{2l}}\right)^{\frac{1}{m}}}.$$

The above can be summarized in the following

Theorem 1. The function $T_\gamma(t)$ defined by (4): i) is the (STBS) function of best uniform one-sided approximation to function $c_{\gamma,\delta}$ in the interval $[\gamma,\infty)$ (as well as in the interval $(-\infty,\gamma]$); ii) approximates the cut function $c_{\gamma,\delta}(t)$ in uniform metric with an error

$$\rho=\frac{K}{2x_0}\times\frac{1}{\left(1+\left(K^m x_0^{-m}-1-\frac{\alpha\beta}{\beta-B}\right)e^{\frac{\beta m}{2l}}+\frac{\alpha\beta}{\beta-B}e^{\frac{Bm}{2l}}\right)^{\frac{1}{m}}}. \quad (5)$$

4. Approximation of the Cut Function by Nelder Function [6]

Definition. Define the special shifted Nelder growth function $N_\gamma(t)$ with jump at point γ as:

$$N_\gamma(t)=\frac{1}{2x_0}\frac{K}{\left(1+\left(K^m x_0^{-m}-1\right)e^{-\beta m(t-\gamma)}\right)^{\frac{1}{m}}}. \quad (6)$$

Then $N_\gamma(\gamma)=\frac{1}{2}$.

We next focus on the approximation of the cut function (1) by shifted Nelder growth function defined by (6).

Note that the slope of $N_\gamma(t)$ at $t=\gamma$ is $N'_\gamma(\gamma)=\frac{1}{2K^m x_0^{-m}}\beta\left(K^m x_0^{-m}-1\right)=l_1$ and the slope of $c_{\delta,\delta}(t)$ at $t=\gamma$ is $c'_{\delta,\delta}(\gamma)=\frac{1}{2\delta}$.

Let choose $l_1=\frac{1}{2\delta}$. The function defined by (5) has an inflection at point $(\gamma,\frac{1}{2})$ (see Fig. 3 and Fig. 4).

Consider functions (1) and (6) with same centres $\gamma=\delta$, that is functions $c_{\delta,\delta}$ and N_δ.

In addition chose c and N to have same slopes at their coinciding centres.

Then, noticing that the largest uniform distance ρ_1 between the cut and Nelder functions is achieved at the endpoints of the underlying interval $[0,2\delta]$:

$$\rho_1=N_\delta(0)-c_{\delta,\delta}(0)=\frac{1}{2x_0}\frac{K}{\left(1+\left(K^m x_0^{-m}-1\right)e^{\frac{\beta m}{2l_1}}\right)^{\frac{1}{m}}}=\frac{1}{2x_0}\frac{K}{\left(1+\left(K^m x_0^{-m}-1\right)e^{\frac{mK^m x_0^{-m}}{K^m x_0^{-m}-1}}\right)^{\frac{1}{m}}}.$$

The above can be summarized in the following

Theorem 2. The function $N_\gamma(t)$ defined by (6): i) is the Nelder function of best uniform one-sided approximation to function $c_{\gamma,\delta}$ in the interval $[\gamma,\infty)$ (as well as in the interval $(-\infty,\gamma]$); ii) approximates the cut function $c_{\gamma,\delta}(t)$ in uniform metric with an error

$$\rho_1 = \frac{1}{2x_0} \frac{K}{\left(1+\left(K^m x_0^{-m}-1\right) e^{\frac{mK^m x_0^{-m}}{K^m x_0^{-m}-1}}\right)^{\frac{1}{m}}}. \qquad (7)$$

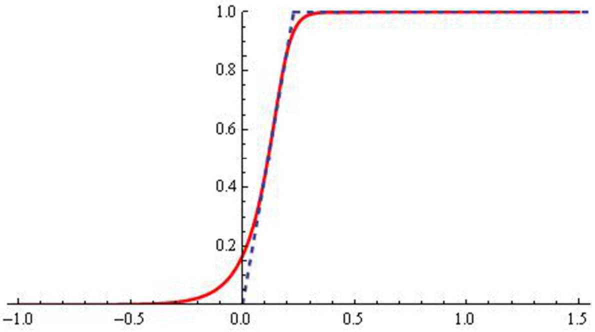

Figure 3. *The cut and the function (6) with* $K=1$, $m=3$, $\beta=10$, $x_0=K/2$, $\gamma=0.114286$, *uniform distance* $\rho_1=0.166454$.

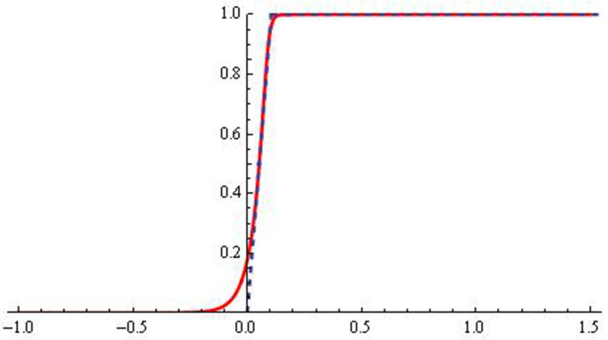

Figure 4. *The cut and the function (6) with* $K=1$, $m=4.5$, $\beta=20$, $x_0=K/2$, $\gamma=0.0523119$, *uniform distance* $\rho_1=0.177385$.

Remark. Let us point out that estimate (7) does not make use of the parameter β.

5. Computational Issues and Fitting the Model (3) (with Jump at γ) Against Verhulst Data

To illustrate the use of the model (3) we have fitted the model to the Verhulst data by use of software module in programming environment *CAS Mathematica*.

In his famous work [7] Verhulst applies the logistic model to fit census data for the population in France. The given data in column 3 (Fig. 5) will be briefly called *Verhulst data*.

The appropriate fitting of Verhulst data by the function (3) with $K=81$, $\beta=59$, $\alpha=0.054$, $m=0.166$, $\gamma=1.817$, $B=57$, $x_0=29{,}981$ is visualized on Fig. 6.

The following scales are appropriate: on the abscissa 1000 units correspond to one division; on the ordinate 1000000 units correspond to one division.

We may expect somewhat more accuracy in predicting the population of metropolitan France for 1841, 1851 and 2010 (see Table 1).

Table 1. *Average of population.*

Year	Population by TBS model	Population — National Inst. of Statistics
1841	34559900	34912000
1851	36489500	36472000
2010	62850000	62765000

Let us note that the results are unexpectedly reliable, especially in relation to extrapolation for the year 2010. In this case the relative error is on the order of 0.135%.

The experts from the National Institute of Statistics - France, when studying in detail the population of metropolitan France pay special attention to existing two centuries of population growth: First World War 1914–1918 and Second World War 1939–1945, (see Fig. 7; https://en.wikipedia.org/wiki/Demographics of France).

It can be concluded from Fig. 7 that knowledge of the lag-time is very substantial for choosing the correct empirical growth model.

We hope that the results on the approximation of the cut function by smooth sigmoidal functions can be useful for choosing the correct population model.

For some approximation, computational and modelling aspects, see [8–25].

Tableau des progrès de la population de la France depuis 1817 *jusqu'à* 1831, *d'après l'Annuaire pour* 1834.

ANNÉES.	D'APRÈS L'ÉTAT CIVIL.	D'APRÈS LA FORMULE.	ERREUR proportionl^le.	Logarithmes de la population calculée.
1817	29,981,336 195,902	29,981,336 208,281	0,0000	7,4768490
1818	30,177,238 161,948	30,189,500 204,500	+0,0004	7,4798565
1819	30,339,186 199,863	30,394,000 200,500	+0,0018	7,4827875
1820	30,539,049 188,227	30,594,500 197,300	+0,0018	7,4856461
1821	30,727,276 212,144	30,791,800 192,700	+0,0021	7,4884310
1822	30,939,420 198,634	30,984,500 189,500	+0,0014	7,4911453
1823	31,138,054 221,286	31,174,000 185,223	+0,0012	7,4937907
1824	31,359,340 220,546	31,359,340 182,777	0,0000	7,4963719
1825	31,579,886 175,974	31,542,000 178,000	—0,0012	7,4988859
1826	31,755,860 157,533	31,720,000 175,000	—0,0011	7,5013366
1827	31,913,393 189,071	31,895,000 172,000	—0,0005	7,5037257
1828	32,102,464 139,402	32,067,000 168,000	—0,0011	7,5060547
1829	32,241,866 161,074	32,235,000 164,500	—0,0002	7,5083251
1830	32,402,940 157,994	32,399,500 161,434	0,0000	7,5105385
1831 1r janv.	32,560,934 (Chiffre du recensl.)	32,560,934	0,0000	7,5126965

Figure 5. *Verhulst data [7].*

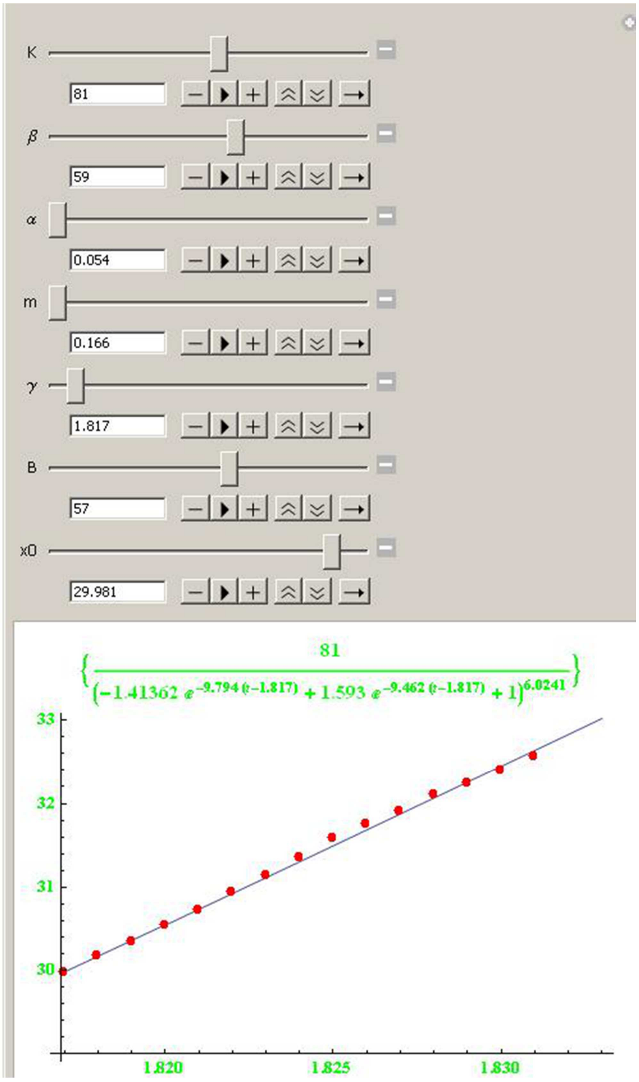

Figure 6. *Software tools in CAS Mathematica.*

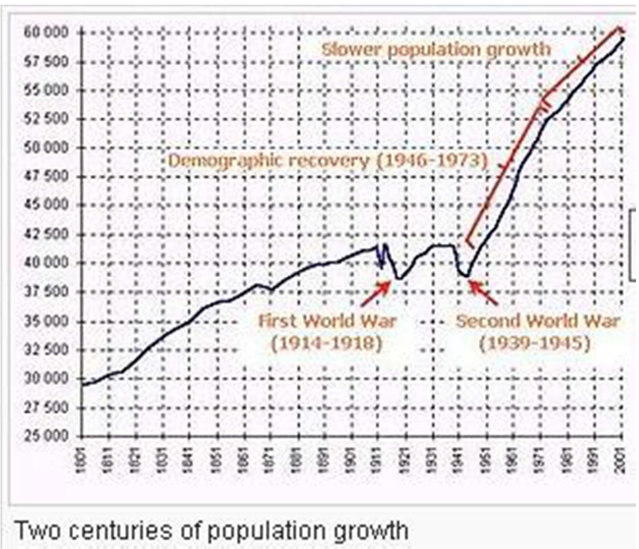

Figure 7. *Two centuries of population growth (First World War 1914–1918; Second World War 1939–1945); https://en.wikipedia.org/wiki/Demographics of France.*

Acknowledgments

The authors would like to thank the anonymous reviewers for their helpful comments that contributed to improving the final version of the presented paper.

References

[1] S. Shoffner and S. Schnell, Estimation of the lag time in a subsequent monomer addition model for fibril elongation, bioRxiv The preprint server for biology, 2015, pp. 1–8, doi: 10.1101/034900.

[2] P. Arosio, T. P. J. Knowles, and S. Linse, On the lag phase in amyloid fibril formation, Physical Chemistry Chemical Physics, vol. 17, 2015, pp. 7606–7618, doi: 10.1039/C4CP05563B.

[3] N. Kyurkchiev, A note on the new geometric representation for the parameters in the fibril elongation process, Compt. rend. Acad. bulg. Sci., vol. 69 (8), 2016, pp. 963–972.

[4] S. Markov, Building reaction kinetic models for amyloid fibril growth, BIOMATH, vol. 5, 2016, http://dx.dpi.org/10.11145/j.biomath.2016.07.311.

[5] M. Turner, B. Blumenstein, and J. Sebaugh, A Generalization of the Logistic Law of Growth, Biometrics, vol. 25 (3), 1969, pp. 577–580.

[6] J. A. Nelder, The fitting of a generalization of the logistic curve, Biometrics, vol. 17, 1961, pp. 89–110.

[7] P. F. Verhulst, Notice sur la loi que la population poursuit dans son accroissement, Correspondance mathematique et physique, vol. 10, 1838, pp. 113–121.

[8] N. Kyurkchiev and S. Markov, On the Hausdorff distance between the Heaviside step function and Verhulst logistic function, J. Math. Chem., vol. 54 (1), 2016, pp. 109–119, doi: 10.1007/S10910-015-0552-0.

[9] N. Kyurkchiev and S. Markov, Sigmoidal functions: some computational and modelling aspects, Biomath Communications, vol. 1 (2), 2014, pp. 30–48, doi: 10.11145/j.bmc.2015.03.081.

[10] A. Iliev, N. Kyurkchiev and S. Markov, On the approximation of the cut and step functions by logistic and Gompertz functions, Biomath, vol. 4 (2), 2015, pp. 2–13.

[11] N. Kyurkchiev and S. Markov, On the approximation of the generalized cut function of degree $p+1$ by smooth sigmoid functions, Serdica J. Computing, vol. 9 (1), 2015, pp. 101–112.

[12] A. Iliev, N. Kyurkchiev, and S. Markov, On the Approximation of the step function by some sigmoid functions, Mathematics and Computers in Simulation, 2015, doi: 10.1016/j.matcom.2015.11.005.

[13] N. Kyurkchiev and A. Iliev, On some growth curve modeling: approximation theory and applications, Int. J. of Trends in Research and Development, vol. 3 (3), 2016, pp. 466–471, http://www.ijtrd.com/papers/IJTRD3869.pdf

[14] N. Kyurkchiev and S. Markov, Sigmoid functions: Some Approximation and Modelling Aspects, LAP LAMBERT Academic Publishing, Saarbrucken, 2015, ISBN 978-3-659-76045-7.

[15] N. Kyurkchiev and A. Iliev, A note on some growth curves arising from Box-Cox transformation, Int. J. of Engineering Works, vol. 3 (6), 2016, pp. 47–51, ISSN: 2409-2770.

[16] N. Kyurkchiev, S. Markov, and A. Iliev, A note on the Schnute growth model, Int. J. of Engineering Research and Development, vol. 12 (6), 2016, pp. 47–54, ISSN: 2278-067X, http://www.ijerd.com/paper/vol12-issue6/Verison-1/G12614754.pdf

[17] A. Iliev, N. Kyurkchiev, and S. Markov, On the Hausdorff distance between the shifted Heaviside step function and the transmuted Stannard growth function, BIOMATH, 2016, (accepted).

[18] N. Kyurkchiev, On the Approximation of the step function by some cumulative distribution functions, Compt. rend. Acad. bulg. Sci., vol. 68 (12), 2015, pp. 1475–1482.

[19] V. Kyurkchiev and N. Kyurkchiev, On the Approximation of the Step function by Raised-Cosine and Laplace Cumulative Distribution Functions, European International Journal of Science and Technology, vol. 4 (9), 2016, pp. 75–84.

[20] D. Costarelli and R. Spigler, Approximation results for neural network operators activated by sigmoidal functions, Neural Networks, vol. 44, 2013, pp. 101–106.

[21] D. Costarelli and G. Vinti, Pointwise and uniform approximation by multivariate neural network operators of the max-product type, Neural Networks, 2016, doi: 10.1016/j.neunet.2016.06.002.

[22] Costarelli, D., R. Spigler, Solving numerically nonlinear systems of balance laws by multivariate sigmoidal functions approximation, Computational and Applied Mathematics, 2016, doi: 10.1007/s40314-016-0334-8.

[23] D. Costarelli and G. Vinti, Convergence for a family of neural network operators in Orlicz spaces, Mathematische Nachrichten, 2016, doi: 10.1002/mana.20160006.

[24] N. Guliyev and V. Ismailov, A single hidden layer feedforward network with only one neuron in the hidden layer san approximate any univariate function, Neural Computation, vol. 28, 2016, pp. 1289–1304.

[25] J. Dombi and Z. Gera, The Approximation of Piecewise Linear Membership Functions and Lukasiewicz Operators, Fuzzy Sets and Systems, vol. 154 (2), 2005, pp. 275–286.