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The study of heat transfer phenomena using PM for approximate solution with Dirichlet and mixed boundary conditions

U. Filobello-Nino[1], H. Vazquez-Leal[1, *], A. Sarmiento-Reyes[2], A. Perez-Sesma[1], L. Hernandez-Martinez[2], A. Herrera-May[3], V. M. Jimenez-Fernandez[1], A. Marin-Hernandez[4], D. Pereyra-Diaz[1], A. Diaz-Sanchez[2]

[1]Electronic Instrumentation and Atmospheric Sciences School, Universidad Veracruzana, Circuito Gonzalo Aguirre Beltrán S/N, Xalapa, Veracruz, Mexico 91000
[2]National Institute for Astrophysics, Optics and Electronics, Luis Enrique Erro #1, Sta. María Tonantzintla. Puebla, México, 72840
[3]Micro and Nanotechnology Research Center, Universidad Veracruzana, Calzada, Ruiz Cortines, Boca del Rio 94292, Veracruz, Mexico
[4]Department of Artificial Intelligence, Universidad Veracruzana, Sebastián Camacho No. 5, C.P. 91000 Xalapa, Veracruz, Mexico

Email address:
hvazquez@uv.mx (H. Vazquez-Leal)

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Abstract: In this paper, we present Perturbation Method (PM) to solve nonlinear problems. As case study PM is employed to obtain approximate solutions for differential equations related with heat transfer phenomena. Comparing figures between approximate and exact solutions, show the effectiveness of the method.

Keywords: Dirichlet Boundary conditions, Mixed Boundary Conditions, Nonlinear Differential Equation, Perturbation Method, Approximate Solutions

1. Introduction

The heat transfer laws are of paramount importance in the design and operation of equipment in many industrial applications as well as in pure sciences. Therefore it is important to search for accurate approximate solutions to the equations describing these phenomena. However, it is well known that nonlinear differential equations that describe them, are difficult to solve.

The perturbation method (PM) is a well established method; it is among the pioneer techniques to approach various kinds of nonlinear problems. This procedure was originated by S. D. Poisson and extended by J. H. Poincare. Although the method appeared in the early 19th century, the application of a perturbation procedure to solve nonlinear differential equations was performed later on that century. The most significant efforts were focused on celestial mechanics, fluid mechanics, and aerodynamics [2, 37].

In a broad sense, it is possible to express a nonlinear differential equation in terms of one linear part and other nonlinear. The nonlinear part is considered as a small perturbation through a small parameter (the perturbation parameter). The assumption that the nonlinear part is small compared to the linear is considered as a disadvantage of the method. There are other modern alternatives to find approximate solutions to the differential equations that describe some nonlinear problems such as those based on: variational approaches [5-7, 29], tanh method [8], exp-function [9, 10], Adomian's decomposition method [11-16], parameter expansion [17], homotopy perturbation method [3,4,16,18-29,31-36,38,42], homotopy analysis method [30], Homotopy asymptotic method [1], and perturbation method [39,41] among many others. Also, a few exact solutions to nonlinear differential equations have been reported occasionally [40].

Although the PM method provides in general, better results for small perturbation parameters $\varepsilon << 1$, we will see that our approximations, besides to be handy, have a good accuracy, even for relatively large values of the perturbation parameter [39,41].

This paper is organized as follows. Section 2 presents a survey about the works related with this paper. In Section 3, we introduce the basic idea of the PM method. For Section 4, we provide an application of PM method, by solving a couple of examples with applications in sciences and engineering. Section 5 discusses the main results obtained. Finally, a brief conclusion is given in Section 6.

2. Survey of the Related Works

Given the importance of phenomena involving heat flow, other sophisticated methods have been used to find approximate solutions to these problems, such as Homotopy Perturbation Method (HPM) [16,43,44], in [16] is compared a modified version of HPM and Adomian decomposition method in order to evaluate the temperature distribution of a straight rectangular fin with temperature dependent surface heat flux for all possible types of heat transfer, in [43] HPM is employed to solve for the temperature distribution in lumped system of combined convection radiation, in [44] HPM and PM are used to find analytical approximate solutions, for an unsteady nonlinear convective-radiative equation and a nonlinear convective-radiative conduction equation containing two small parameters. Laplace Transform Homotopy Perturbation Method (LT-HPM) [42], [42] employs LT-HPM, to approximately solve the differential equation which describes the temperature distribution in a uniformly thick rectangular fin radiation to free space with nonlinearity of high order (see below, this study proposes an approximate solution for this equation), Homotopy Asymptotic Method [1] (where is analyzed the cooling of a system with variable specific heat, also is analysed the temperature distribution in a uniformly thick rectangular fin radiation to free space with nonlinearity of high order), Adomian's decomposition method [16], among others.

3. Basic Idea of Perturbation Method

Let the differential equation of one dimensional nonlinear system be in the form

$$L(x)+\varepsilon N(x)=0, \tag{1}$$

where we assume that x is a function of one variable $x=x(t)$, $L(x)$ is a linear operator which, in general, contains derivatives in terms of t, $N(x)$ is a nonlinear operator, and ε is a small parameter.

Considering the nonlinear term in (1) to be a small perturbation and assuming that the solution for (1) can be written as a power series in the small parameter ε,

$$x(t)=x_0(t)+\varepsilon x_1(t)+\varepsilon^2 x_2(t)+\ldots \tag{2}$$

Substituting (2) into (1) and equating terms having identical powers of ε, we obtain a number of differential equations that can be integrated, recursively, to find the values for the functions: $x_0(t)$, $x_1(t)$, $x_2(t)$

4. Approximate Solution of Equations

4.1. Case study 1

The following example, *emphasizes* the use of HPM for solving nonlinear ODEs with Dirichlet boundary conditions.

We will consider the following nonlinear differential equation, which describes the cooling of a system with variable specific heat [1].

$$\frac{d^2y(x)}{dx^2}+\varepsilon\left[y(x)\frac{d^2y(x)}{dx^2}+\left(\frac{dy(x)}{dx}\right)^2\right]=0\ ,0\le x\le 1,\ y(0)=1,\ y(1)=0, \tag{3}$$

where ε is a positive parameter.

It is possible to find a handy solution for (3) by applying the PM method.

Identifying terms:

$$L(y)=y''(x), \tag{4}$$

$$N(y)=y(x)y''(x)+y'^2(x), \tag{5}$$

where prime denotes differentiation respect to x.

Identifying ε with the PM parameter, we assume a solution for (3) in the form

$$y(x)=y_0(x)+\varepsilon y_1(x)+\varepsilon^2 y_2(x)+\varepsilon^3 y_3(x)+\varepsilon^4 y_4(x)+\ldots \text{,(see (2)).} \tag{6}$$

On comparing the coefficients of like powers of ε it can be solved for $y_0(x)$, $y_1(x)$, $y_2(x)$, $y_3(x)$,..,and so on. Later it will be seen that, a very good handy result is obtained, by keeping up to fifth order approximation.

$$\varepsilon^0)\ \ y_0''=0,\quad y_0(0)=1,\quad y_0(1)=0, \tag{7}$$

$$\varepsilon^1)\ y_1''+y_0'^2+y_0y_0''=0\ , y_1(0)=0,\quad y_1(1)=0, \tag{8}$$

$$\varepsilon^2)\ y_2''+y_1y_0''+y_0y_1''+2y_1'y_0'=0,\quad y_2(0)=0\ ,\ y_2(1)=0, \tag{9}$$

$$\varepsilon^3)\ y_3''+y_2y_0''+y_0y_2''+y_1y_1''+y_1'^2+2y_2'y_0'=0\ \ y_3(0)=0\ \ y_3(1)=0, \tag{10}$$

$$\varepsilon^4)\ y_4''+y_3y_0''+y_1y_2''+y_0y_3''+y_2y_1''+2y_3'y_0'+2y_1'y_2',\ \ y_4(0)=0\ ,\ \ y_4(1)=0, \tag{11}$$

$$\varepsilon^5)\ \begin{array}{c} y_5''+y_0y_4''+\left(y_2'\right)^2+2y_0'y_4'+y_1y_3'' \\ +y_2y_2''+2y_1'y_3'+y_3y_1''+y_4y_0''=0 \end{array},y_5(0)=0,\quad y_5(1)=0, \tag{12}$$

Thus, the results obtained are

$$y_0(x)=-x+1, \tag{13}$$

$$y_1(x)=-\frac{1}{2}x^2+\frac{1}{2}x, \tag{14}$$

$$y_2(x)=-\frac{1}{2}x^3+x^2-\frac{1}{2}x, \tag{15}$$

$$y_3(x)=-\frac{5}{8}x^4+\frac{7}{4}x^3-\frac{13}{8}x^2+\frac{1}{2}x\ , \tag{16}$$

$$y_4(x)=-\frac{7}{8}x^5+\frac{25}{8}x^4-\frac{33}{8}x^3+\frac{19}{8}x^2-\frac{1}{2}x\ , \tag{17}$$

$$y_5(x)=-\frac{21}{16}x^6+\frac{91}{16}x^5-\frac{155}{16}x^4+\frac{129}{16}x^3-\frac{13}{4}x^2+\frac{1}{2}x. \tag{18}$$

By substituting (13)-(18) into (6) we obtain a fifth order approximation for the solution of (3).

We consider as a case study, the value of parameter $\varepsilon=1$; to obtain a handy approximate solution

$$y(x)=-\frac{1}{2}x-2x^2+\frac{83}{16}x^3-\frac{115}{16}x^4+\frac{77}{16}x^5-\frac{21}{16}x^6+1. \tag{19}$$

As a matter of fact, we will see that (19) is also, highly accurate.

4.2. Case Study 2

The following example, *emphasizes* the use of HPM for solving nonlinear ODEs with mixed boundary conditions.

We will consider the following nonlinear differential equation, which describes the temperature distribution in a uniformly thick rectangular fin radiation to free space with nonlinearity of high order [1].

$$\frac{d^2y(x)}{dx^2}-\varepsilon y(x)^4=0,\ \ 0\le x\le 1,\ \ y'(0)=0,\ \ y(1)=1, \tag{20}$$

where ε is a positive parameter.

To apply PM method, we identify terms as follows

$$L(y)=y''(x), \tag{21}$$

$$N(y)=-y(x)^4, \tag{22}$$

where prime denotes differentiation respect to x.

Identifying ε with the PM parameter, we assume a solution for (20) in the form

$$y(x)=y_0(x)+\varepsilon y_1(x)+\varepsilon^2y_2(x)+\varepsilon^3y_3(x)+\varepsilon^4y_4(x)+\ldots,\ \text{see (2)).} \tag{23}$$

Equating the terms with identical powers of ε it can be solved for $y_0(x)$, $y_1(x)$, $y_2(x)$,…, and so on. We will see that a handy result is obtained, by keeping up to fifth order approximation.

$$\varepsilon^0)\ \ y_0''=0,\quad y_0'(0)=0,\ \ y_0(1)=1, \tag{24}$$

$$\varepsilon^1)\ y_1''-y_0^4=0,\quad y_1'(0)=0,\quad y_1(1)=0, \tag{25}$$

$$\varepsilon^2)\ y_2''-4y_0^3y_1=0,\quad y_2'(0)=0\ ,\quad y_2(1)=0, \tag{26}$$

$$\varepsilon^3)\ y_3''-4y_0^2y_1^2-2y_0^2\left(y_1^2+2y_0y_2\right)=0,\ \ y_3'(0)=0\ ,\quad y_3(1)=0, \tag{27}$$

$$\varepsilon^4)\ y_4''-2y_0^2\left(2y_3y_0+2y_1y_2\right)-4\left(y_1^2+2y_2y_0\right)y_0y_1=0,\ \ y_4'(0)=0\ ,\quad y_4(1)=0, \tag{28}$$

$$\varepsilon^5)\ \begin{array}{l} y_5''-2y_0^2\left(y_2^2+2y_0y_4+2y_1y_3\right) \\ -4y_0y_1\left(2y_0y_3+2y_1y_2\right)-\left(y_1^2+2y_0y_2\right)^2=0 \end{array}\ ,y_5'(0)=0\ ,\quad y_5(1)=0, \tag{29}$$

After solving the above equations, one obtains

$$y_0(x)=1, \tag{30}$$

$$y_1(x)=\frac{1}{2}x^2-\frac{1}{2}, \tag{31}$$

$$y_2(x)=\frac{1}{6}x^4-x^2+\frac{5}{6}, \tag{32}$$

$$y_3(x)=\frac{13}{180}x^6-\frac{7}{12}x^4+\frac{29}{12}x^2-\frac{343}{180}, \tag{33}$$

$$y_4(x)=\frac{23}{720}x^8-\frac{13}{36}x^6+\frac{133}{72}x^4-\frac{1181}{180}x^2+\frac{3631}{720}, \tag{34}$$

$$y_5(x)=\frac{929}{64800}x^{10}-\frac{299}{1440}x^8+\frac{611}{432}x^6-\frac{12677}{2160}x^4+\frac{27601}{1440}x^2-\frac{940859}{64800}. \tag{35}$$

By substituting (30)-(35) into (23) we obtain a fifth order approximation for the solution of (20).

Considering as a case study, the value of parameter $\varepsilon=0.30$; we obtain a handy approximate solution

$$\begin{aligned}y(x)=&\frac{70329323}{80000000}+\frac{1898907}{16000000}x^2-\frac{393}{8000000}x^4\\&+\frac{3939}{1600000}x^6-\frac{3933}{16000000}x^8+\frac{2787}{80000000}x^{10}.\end{aligned} \tag{36}$$

We will see that (36) has also, a good precision.

5. Discussion

The fact that the PM depends on a parameter which is assumed small, suggests that the method is for a limited use. In this work, PM method has been applied, successfully to the problem of finding approximate solutions for nonlinear differential equations with Dirichlet and mixed boundary conditions, that describe some heat transfer phenomena. Although the solutions reported for other sophisticated methods (see section 2) have good accuracy, they are more complicated for applications than PM.

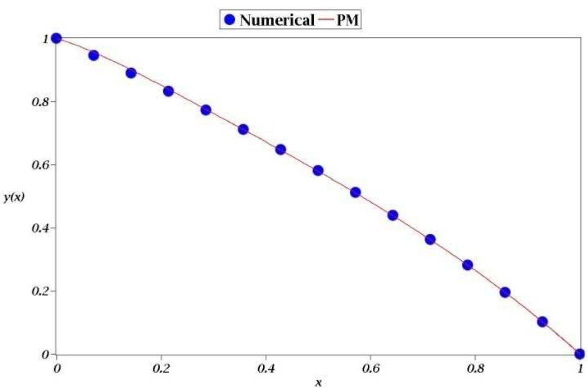

Fig 1. *Approximation of (3) and RK4 comparison for* $\varepsilon=1$

Fig.1 shows the comparison between approximate solution (19) for differential equation with Dirichlet boundary conditions (3), with the four order Runge Kutta (RK4) numerical solution. It can be noticed that figures are in good agreement showing the accuracy of (19). This proves the efficiency of PM method in this case, despite of the fact that it was only considered the fifth-order approximation.

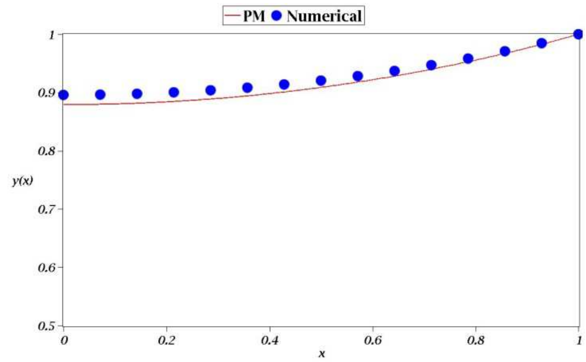

Fig 2. *Approximation of (20) and RK4 comparison for* $\varepsilon=0.30$

On the other hand, Fig.2 compares approximate solution (36) for (20) with the four order Runge Kutta (RK4) numerical solution. From figures it is deduced that (36) has an acceptable precision, taking into account that problems with mixed boundary conditions, have the added difficulty of not provide one of the endpoints of the interval, in this case $y(0)$. It is noted that (36) is only the fifth-order approximation to the equation to be solved. Therefore, accuracy can be increased, considering higher order approximations.

It is well known that PM method provides in general, better results for small perturbation parameters $\varepsilon<<1$ (see (1)) and when are included the most number of terms from (2). To be precise, ε is a parameter of smallness, that measure how greater is the contribution of linear term $L(x)$ than the one of $N(x)$ in (1). Fig. 1 and Fig. 2 show a noticeable fact, that (19) and (36) turn out to be a good approximation of (3) and (20) respectively, although perturbation parameters $\varepsilon=1$ and $\varepsilon=0.30$ cannot be considered small.

Finally, our approximate solutions (19) and (36) do not depend of any adjustment parameter, for which, are in principle, general expressions for proposed problems.

6. Conclusions

In this study, PM was presented to construct analytical approximate solutions for nonlinear differential equations in the form of rapidly convergent series. In order to prove the versatility of this method, we proposed as examples a couple of equations governed by Dirichlet and mixed boundary conditions, with good results. The success of PM for this case, it has to be considered as a possibility to apply it in other non linear problems, instead of using other sophisticated and difficult methods. From Fig. 1 and Fig. 2, it is deduced that the proposed solutions have good precision.

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