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Neural network method for numerical solution of initial value problems of fractional differential equations

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Abstract: In this paper, the cosine basis neural network algorithm is introduced for the initial value problem of fractional differential equations. By training the neural network algorithm, we get the numerical solution of the initial value problem of fractional differential equations successfully.

Keywords: Fractional Differential Equations, Cosine Basis Neural Network Algorithm, Initial Value Problem

1. Introduction

Fractional differential equations (FDE) is a suitable model for material with the property of memory and genetics. It has the advantage of simple modeling and accurate description of complex systems with the property of memory and genetics. The physical meaning of its parameters is clear. It can solve problems that cannot be solved by the integer order differential equation. Thus FDE becomes one of the important tools of the modeling of complex mechanics and physical processes. Recently, FDE is used to solve problems in the optical and thermal system, rheology, mechanical system, signal processing, system identification and many other problems. Because FDE has been applied in more and more areas, it is urgently needed to have methods to find the numerical solutions of FDE. Now, many effective method had been developed for solving FDE such as the method of random walk [1], the method of matrix approach [2], the fixed point theorem [3], the method of HAM [4], finite difference method and so on.

In this paper the neural network method is used for numerical solution of initial value problems of FDE. It has been proved to be an effective method for various examples. The Artificial neural network (ANN) method is a description of the characteristics of the first-order of the human brain system. This method has the following three features: the representation of information distribution, the global parallel algorithm and local operation, and the nonlinear processing of information. Due to these three features, the machine of classification which is composed of artificial neural network, is powerful for data fitting and the ability of generalization, and therefore is widely used in the field of pattern recognition and machine learning.

In this paper, we first consider the numerical solution of linear and nonlinear fractional equation of the form

$$D_{0+}^{\alpha}y(x)=f(x,y(x)),0<x<1,0<\alpha\le 1 \tag{1}$$

with initial condition as follows $y(0)=C$,where D_{0+}^{α} the Riemann-Liouville fractional derivative of order α .

2. Definitions and Lemma

Definition 2.1 [5] The Riemann-Liouville fractional integral of order $\alpha\in R$, $\alpha>0$ of a function $f(x)\in C_{\mu}$, $\mu\ge -1$ is defined as

$$(I_{0+}^{\alpha}f(t))(x):\frac{1}{\Gamma(\alpha)}\int_0^x\frac{f(t)dt}{(x-t)^{1-\alpha}},(x>0).$$

Definition 2.2 [5] The Riemann-Liouville fractional derivative of order $\alpha\in R$, $\alpha>0$, on the usual Lebesgue space L1[a, b] is given by
$f(x)\in C_{\mu}$, $\mu\ge -1$, where (n = [α] + 1, x > 0).

Definition 2.3 [5] The classical Mittag-Leffler function is defined by

$$E_{\alpha}:=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\alpha k+1)},(x\in C,\alpha>0)$$

The generalized Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(x):=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\alpha k+\beta)},(x,\beta\in C,\alpha>0)$$

Definition 2.4 The functions $\sin_{\alpha,\beta}(x)$, $\cos_{\alpha,\beta}(x)$, $(x,\beta\in C,\alpha>0)$ are defined by $\sin_{\alpha,\beta}(x)=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^{2k-1}}{\Gamma(\alpha(2k-1)+\beta)}$ $\cos_{\alpha,\beta}(x)=\sum_{k=0}^{\infty}(-1)^{k}\frac{x^{2k}}{\Gamma(\alpha(2k)+\beta)}$.

Obviously, the Euler's equations have the forms

$$E_{\alpha,\beta}(ix)=\cos_{\alpha,\beta}(x)+i\sin_{\alpha,\beta}(x),E_{\alpha,\beta}(-ix)=\cos_{\alpha,\beta}(x)-i\sin_{\alpha,\beta}(x)$$

Lemma 2.1 If $\sin_{\alpha,\beta}(x)$ and $\cos_{\alpha,\beta}(x)$ are defined as in Definition 2.6, then

$$\begin{aligned}&(D_{0+}^{\alpha}(t-a)^{\beta-1}\sin_{\mu,\beta}[\lambda(t-a)^{\mu}])(x)\\&=(x-a)^{\beta-\alpha-1}\sin_{\mu,\beta-\alpha}[\lambda(t-a)^{\mu}]\end{aligned}\tag{3}$$

$$\begin{aligned}&(D_{0+}^{\alpha}(t-a)^{\beta-1}\cos_{\mu,\beta}[\lambda(t-a)^{\mu}])(x)\\&=(x-a)^{\beta-\alpha-1}\cos_{\mu,\beta-\alpha}[\lambda(t-a)^{\mu}]\end{aligned}\tag{4}$$

Proof. We denote beta function by $\widetilde{\beta}$, then according to the definition of Riemann-Liouville fractional integral, we have

$$\begin{aligned}&(D_{0+}^{\alpha}(t-a)^{\beta-1}\sin_{\mu,\beta}[\lambda(t-a)^{\mu}])(x)\\
&=\frac{1}{\Gamma(n-\alpha)}(\frac{d}{dx})^{n}\int_{a+}^{x}\frac{(t-a)^{\beta-1}\sin_{\mu,\beta}[\lambda(t-a)^{\mu}]}{(x-t)^{\alpha-n+1}}dt\\
&=\frac{1}{\Gamma(n-\alpha)}(\frac{d}{dx})^{n}\int_{0+}^{x-a}\frac{\xi^{\beta-1}\sin_{\mu,\beta}[\lambda\xi^{\mu}]}{(x-\xi-a)^{\alpha-n+1}}d\xi\\
&=\frac{1}{\Gamma(n-\alpha)}(\frac{d}{dx})^{n}\int_{0+}^{x-a}\frac{\xi^{\beta-1}\sin_{\mu,\beta}[\lambda\xi^{\mu}]}{(x-\xi-a)^{\alpha-n+1}}d\xi,(\xi=t-a)\\
&=\frac{1}{\Gamma(n-\alpha)}(\frac{d}{dx})^{n}\int_{0+}^{1}\frac{t^{\beta-1}(x-a)^{\beta-1}}{(x-a)^{\alpha-n+1}}\frac{\sin_{\mu,\beta}[\lambda t^{\mu}(x-a)^{\mu}]}{(1-t)^{\alpha-n+1}}(x-a)dt,(\xi=t(x-a))\\
&=(\frac{d}{dx})^{n}\frac{(x-a)^{\beta-\alpha+n-1}}{\Gamma(n-\alpha)}\int_{0+}^{1}\frac{t^{\beta-1}}{(1-t)^{\alpha-n+1}}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}[\lambda t^{\mu}(x-a)^{\mu}]^{2k-1}}{\Gamma(\mu(2k-1)+\beta)}dt\\
&=(\frac{d}{dx})^{n}\frac{(x-a)^{\beta-\alpha+n-1}}{\Gamma(n-\alpha)}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}[\lambda(x-a)^{\mu}]^{2k-1}}{\Gamma(\mu(2k-1)+\beta)}\frac{\Gamma(n-a)\Gamma(\mu(2k-1)+\beta)}{\Gamma(\mu(2k-1)+\beta-\alpha+n)}\\
&=(x-a)^{\beta-\alpha-1}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}[\lambda(x-a)^{\mu}]^{2k-1}}{\Gamma(\mu(2k-1)+\beta-\alpha)}\\
&=(x-a)^{\beta-\alpha-1}\sin_{\mu,\beta-\alpha}[\lambda(t-a)^{\mu}]\end{aligned}$$

Similarly, we obtain the equation (6). In particular, when $\beta=1,\mu=1$,we have

$$\begin{aligned}&(D_{0+}^{\alpha}\sin_{1,1}[\lambda(t-a)])(x)=(D_{0+}^{\alpha}\sin[\lambda(t-a)](x)\\&=x^{-\alpha}\sin_{1,1-\alpha}[\lambda(x-a)]\end{aligned}$$

and

$$\begin{aligned}&(D_{0+}^{\alpha}\cos_{1,1}[\lambda(t-a)])(x)=(D_{0+}^{\alpha}\cos[\lambda(t-a)](x)\\&=x^{-\alpha}\cos_{1,1-\alpha}[\lambda(x-a)]\end{aligned}$$

3. Illustration of the Method and Application

To describe the method, we consider the equation (1) with initial condition y (0) = C. The j-th trial solution for the problem is written as $y_j(x)=\sum_{i=1}^{M}w_{i,j}\cos(ix)+(C-\sum_{i=1}^{M}w_{i,j})\cos((M+1)x)$, where N represents the number of samples and $w_{i,j}$ are unknown weights of the network determined in training procedures to reduce error

$$E_j(x)=(e_j(1),e_j(2),...e_j(N))^T,J=\frac{1}{2}\|E_j\|_2^2=\frac{1}{2}\sum_{k=1}^{N}e_j(k)$$

$$e_j(k)=f(x_k,y_j(x_k))-D_{0+}^{\alpha}y_j(x_k)=f(x_k,y_j(x_k))-x_k^{-\alpha}\left(\sum_{i=1}^{M}w_{i,j}\cos_{1,1-\alpha}(ix_k)+\left(C-\sum_{i=1}^{M}w_{i,j}\right)\cos_{1,1-\alpha}((M+1)x_k)\right)$$

then we can adjust the weights $w_{i,j}$ by

$$w_{i,j+1} = w_{i,j} + \Delta w_{i,j}$$

$$\Delta w_{i,j} = -\mu \frac{\partial J}{\partial w_{i,j}}$$

$$= -\mu \sum_{k=1}^{N} \frac{\partial J}{\partial e_j(k)} \frac{\partial e_j(k)}{\partial w_{i,j}}$$

$$= -\mu \sum_{k=1}^{N} e_j(k)(f_y(x_k, y_j(x_k))(\cos(x_k) - \cos((M+1)x_k)) - (x_k)^{(-\alpha)}(\cos_{1,1-\alpha} x_k - \cos_{1,1-\alpha}((M+1)x_k)))$$

3.2. Example

Example 1: We first consider the following linear fractional differential equation

$$D_{0^+}^{\alpha} x^3 = x^7 + \frac{6}{\Gamma(4-\alpha)} x^{3-\alpha} - xy^2(x)$$

with condition y(0) = 0. The exact solution is $y(x) = x^3$. We set the parameters α = 0.9, μ = 0.0005, M = 7, N = 10.

Example 2: We second consider the following linear fractional differential equation

$$D_{0+}^{\alpha} x^4 = x^4 + \frac{24}{\Gamma(5-\alpha)} x^{4-\alpha} - y(x)$$

with condition y(0) = 0. The exact solution is $y(x) = x^4$. We set the parameters α = 0.9, μ = 0.0005, M = 7, N = 11.

Example 3: We thirdly consider the following linear fractional differential equation

$$D_{0+}^{\alpha} x^4 = x^5 + \frac{6}{\Gamma(4-\alpha)} x^{3-\alpha} - x^2 y(x)$$

with condition y(0) = 0. The exact solution is $y(x) = x^3$. We set the parameters α = 0.9, μ = 0.0005, M = 7, N = 11.

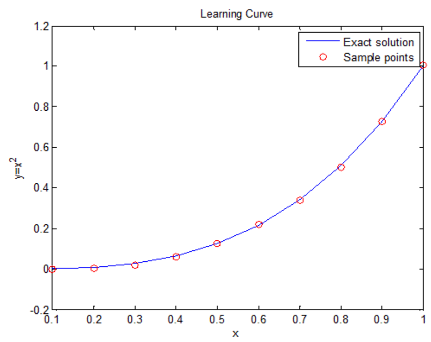

Figure 1. *The learning curve for example 1*

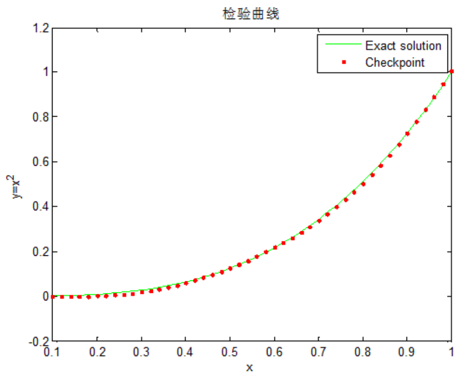

Figure 2. *The inspection curve for example 1*

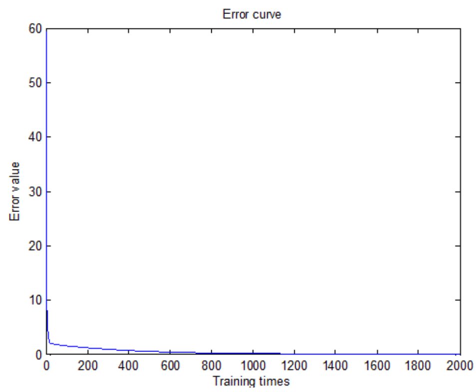

Figure 3. *The error curve for example 1*

4. Conclusion

In this paper, the parameters is continuously adjusted such that the errors are changed by the neural network algorithm, so that the numerical solution of FDE can approximate the exact solutions. From the examples, when the error is infinitely close to zero, the effect of neural network algorithm for solving Fractional differential equations can be observed.

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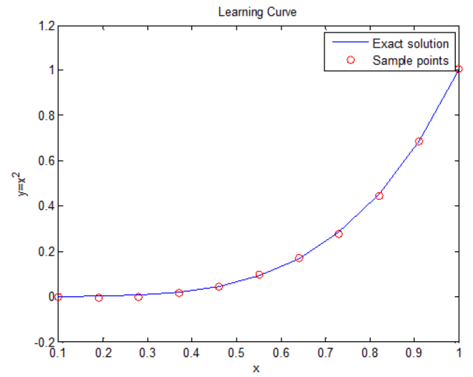

Figure 4. *The learning curve for example 2*

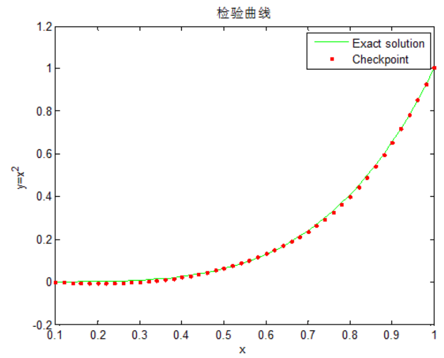

Figure 5. *The inspection curve for example 2*

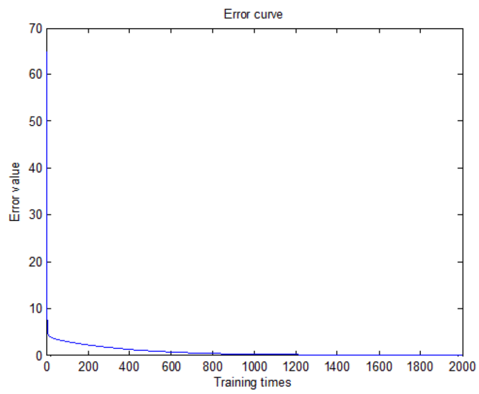

Figure 6. *The error curve for example 2*

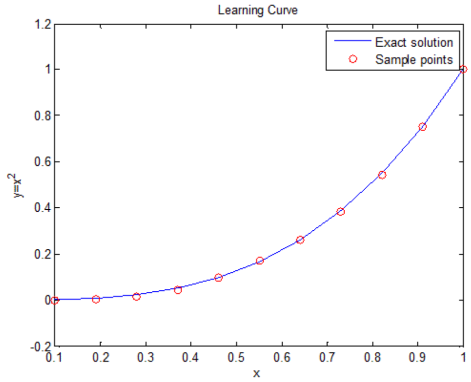

Figure 7. *The learning curve for example 3*

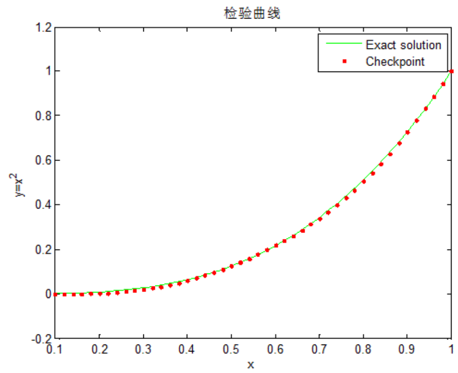

Figure 8. *The inspection curve for example 3*

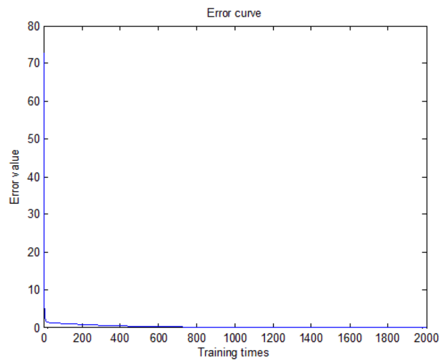

Figure 9. *The error curve for example 3*

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