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A Unified Singular Solution of Laplace's Equation with Neumann and Dirichlet Boundary Conditions

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Abstract: Laplace's equation is one of the important equations in studying applied mathematics and engineering problems including the study of temperature distribution of steady-state heat conduction or the concentration distribution of steady-state diffusion problems. In this study, the analytical method has been applied to solve the Laplace's equation in a two-dimensional domain. For the specified Neumann or Dirichlet boundary conditions, the analytical solution of temperature distribution in the quarter-plane can be found by several methods including the Fourier transform method, similarity method, and the method of Green's function with images. For different boundary conditions, the solution of temperature distribution of the Laplace's equation will be in a totally different form. Nevertheless, the merit of this research is that the solutions of steady-state temperature distribution in the quarter plane with Neumann and Dirichlet boundary conditions are unified under the singular similarity solution with source type singularity. With the typical benchmarked examples for finding the temperature distribution by the numerical integral method, it is shown that Gibbs phenomenon behaves at a jump discontinuity, where serious oscillation result was found especially near the singular points of the boundary. In addition, the temperature distribution in the domain can be easily calculated without oscillation phenomenon near the singular points from the similarity solutions.

Keywords: Laplace's Equation, Fourier Transform, Green's Function, Similarity Solution

1. Introduction

Many important physical problems described by mathematical models such as the heat equation, the wave equation, and others are taught in college courses [1-3]. The application of mathematical techniques to solve such a model described by the partial differential equations (PDE) is various, such as the numerical methods including finite difference method, finite element method and boundary element method for obtaining the approximate solutions [4-7], and the analytical methods for finding the exact solutions. General technique for finding the exact solutions of the linear PDE is the method of separation of variables, which is an approach by identifying the solution that depends on each of the independent variables separately [1-3]. There are also several techniques by reducing the PDE to an ordinary differential equation (ODE) which include various integral transforms and eigenfunction expansions [8]. However, there is an approach by identifying the PDE in terms of the solution depends on certain grouping of the independent variables. This approach to finding the solution is so-called the similarity method [9, 10]. As we know, the basic spirit of similarity transformations is the transformations by which an n-independent variable PDE can be converted to a system with (n-1) independent variables. The best situation is n=2 which is an approach for finding the solution of an ODE instead of finding the solution of a PDE.

Laplace's equation is one of the important equations in studying applied mathematics and engineering problems including the study of temperature distribution of steady-state heat conduction problems. For some boundary-value problems of semi-infinite strip or in the quarter-plane, it cannot be solved by the method of separation of variables [1-3, 8]. However, in these special domains, the operational method [11, 12], and the method of Green's function with images [13, 14] are appropriate to solve these kinds of boundary-value problems. In general, the solutions from these methods are expressed by an integral form with which the solution has to be carried out through numerical calculations.

Similarity method plays an important role in studying PDE for problems in mathematical physics and applied mechanics [9, 10]. The general similarity solution of Laplace's equation by the method of infinitesimal transformation groups was studied by Feng [15]. Moreover, similarity analysis of boundary value problem of Poisson's equation and Helmholtz equation were also studied by Feng [16], and Feng and Lee [17].

In this paper, the two-dimensional steady-state heat conduction problem, such as Laplace's equation for temperature distribution in a domain with semi-infinite strip and quarter-plane has been investigated by the analytical method of Fourier transform. For the specified Neumann and Dirichlet boundary conditions, the analytical solutions of temperature in quarter-plane were found by Fourier transform [1] and the method of Green's function with images [13, 14]. All these solutions of integral form are related to the singular similarity solutions.

As we know that for a solution of PDE, if with different boundary conditions then different solutions will turn out to be. Based on this study, nevertheless, all these solutions of two different boundary-value problems with Neumann and Dirichlet conditions are the same from the similarity point of view. In other words, these two kinds of solutions are unified under the singular similarity solution with the same source type singularity near some singular points in the domain as shown in the following sections.

2. Solution of Laplace's Equation by Fourier Transform Method

2.1. Semi-infinite Strip with Neumann Condition

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, 0 < x < L, 0 < y < \infty \tag{1}$$

$$\text{B.C. } T(0,y) = \begin{Bmatrix} T_0, 0 < y \le 1 \\ 0, 1 < y < \infty \end{Bmatrix}, \ \frac{\partial T}{\partial y}(x,0) = 0 \tag{2}$$

$$T(L,y) = 0, \ T(x,\infty) < \infty \tag{3}$$

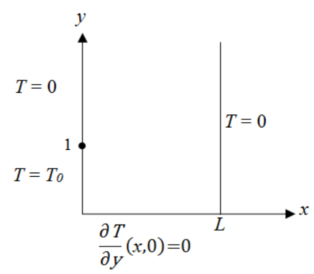

Figure 1. *Domain of semi-infinite trip for Neumann condition.*

where $T(x, y)$ is the temperature in the domain and T_0 is a constant as shown in Figure 1. This boundary value problem cannot be solved by the method of separation of variables for this complicated boundary condition. However, it is easy to solve by the method of integral transform such as Fourier transform.

From Fourier cosine transform, we obtain the solution for equation (1)-(3) with Neumann condition in Figure 1 as follows [1]:

$$T(x,y) = \frac{2T_0}{\pi}\int_0^\infty \frac{\sin\alpha}{\alpha}\frac{\sinh\alpha(L-x)}{\sinh\alpha L}\cos\alpha y \, d\alpha \tag{4}$$

2.2. Quarter-Plane with Neumann Condition

For the case of quarter-plane, the domain of the same problem can be depicted as in Figure 2.

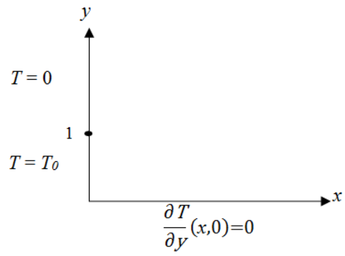

Figure 2. *Domain of quarter-plane for Neumann condition.*

As the semi-infinite strip length $L \to \infty$, we have the approximate result

$$\lim_{L\to\infty}\frac{\sinh\alpha(L-x)}{\sinh\alpha L} \to \lim_{L\to\infty}\frac{(e^{\alpha L}e^{-\alpha x} - e^{-\alpha L}e^{\alpha x})}{(e^{\alpha L} - e^{-\alpha L})} \to e^{-\alpha x} \tag{5}$$

Substituting (5) into (4), then the solution in the quarter-plane can be expressed as follows.

$$T(x,y) = \frac{2T_0}{\pi}\int_0^\infty \frac{\sin\alpha}{\alpha} e^{-\alpha x}\cos\alpha y \, d\alpha \tag{6}$$

Equation (6) can be calculated by the traditional numerical computation method such as the mean-value theorem for integration, which by setting very small interval, $d\alpha$ between 0 to α in equation (6), then the converged result is the sum of the area of all the intervals [4]. The accuracy of temperature distribution is based on the value, α, as long as it is approach infinitive. However, in this study, with $d\alpha$=0.001, the value, α which is larger than 500 is accurate enough for finding the converged result in the domain. For T_0=10, the distribution of temperature was calculated and depicted in Figure 3, with no matter how large the integration upper bound will be, the oscillation phenomena (so-called Gibbs phenomena) exist near the singular point (0,1). The 2-D contour plot for the temperature distribution is shown in Figure 4.

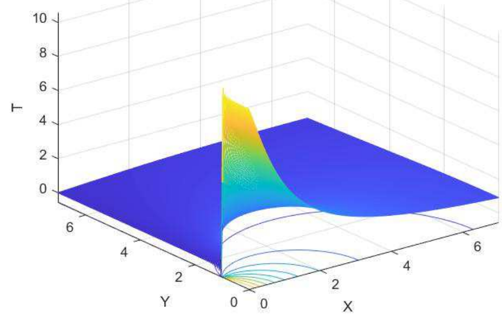

Figure 3. *The distribution of temperature from equation (6) by numerical calculation.*

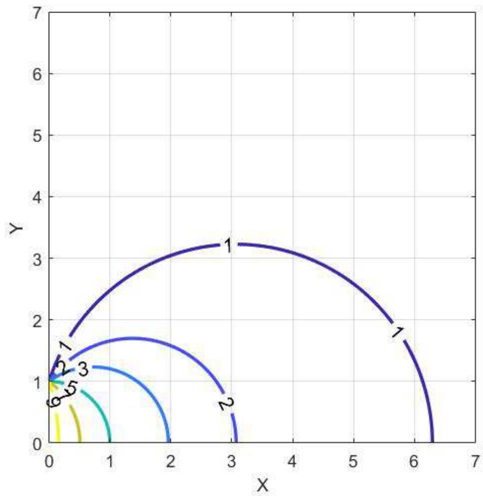

Figure 4. *The 2-D contour of temperature distribution in the domain.*

2.3. Semi-infinite Strip with Dirichlet Condition

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0,\ 0 < x < L, 0 < y < \infty \tag{7}$$

$$\text{B.C. } T(0,y) = \begin{Bmatrix} T_0,\ 0 < y \le 1 \\ 0,\ 1 < y < \infty \end{Bmatrix},\ T(x,0) = 0 \tag{8}$$

$$T(L,y) = 0,\ T(x,\infty) < \infty \tag{9}$$

where T(x, y) is the temperature in the domain and T_0 is a constant.

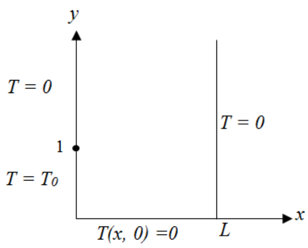

Figure 5. *Domain of semi-infinite strip for Dirichlet condition.*

From Fourier sine transform, we obtain the solution for Dirichlet condition in Figure 5 as follows [1].

$$T(x,y) = \frac{2T_0}{\pi} \int_0^\infty \frac{(1-\cos\alpha)}{\alpha} \frac{\sinh\alpha(L-x)}{\sinh\alpha L} \sin\alpha y\, d\alpha \tag{10}$$

2.4. Quarter-Plane with Dirichlet Condition

For considering the domain of interest is in the first quarter-plane, with Dirichlet condition at y=0 was depicted as shown in Figure 6.

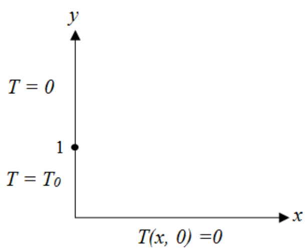

Figure 6. *Domain of quarter-plane for Dirichlet condition.*

As the semi-infinite strip length $L \to \infty$, we have the approximate result shown in equation (5). Substituting equation (5) into (10), the solution in the quarter-plane becomes

$$T(x,y) = \frac{2T_0}{\pi} \int_0^\infty \frac{(1-\cos\alpha)}{\alpha} e^{-\alpha x} \sin\alpha y\, d\alpha \tag{11}$$

Equation (11) can be calculated by the same numerical integration method as mentioned for computation of equation (6), the converged result of temperature distribution for $T_0 = 10$, in the domain was shown in Figure 7 and the 2-D contour plot for the temperature distribution is shown in Figure 8.

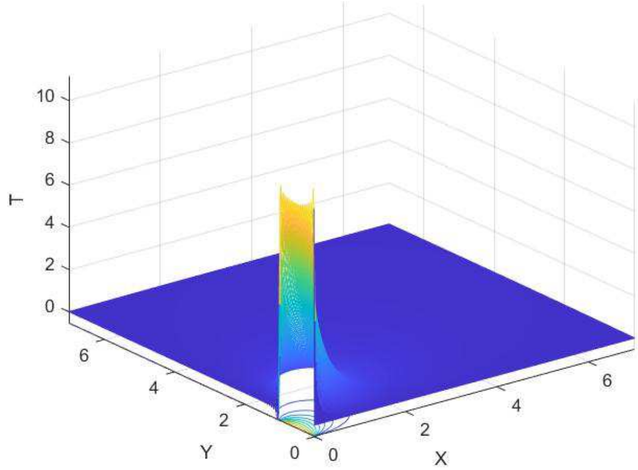

Figure 7. *The distribution of temperature from equation (11) by numerical calculation.*

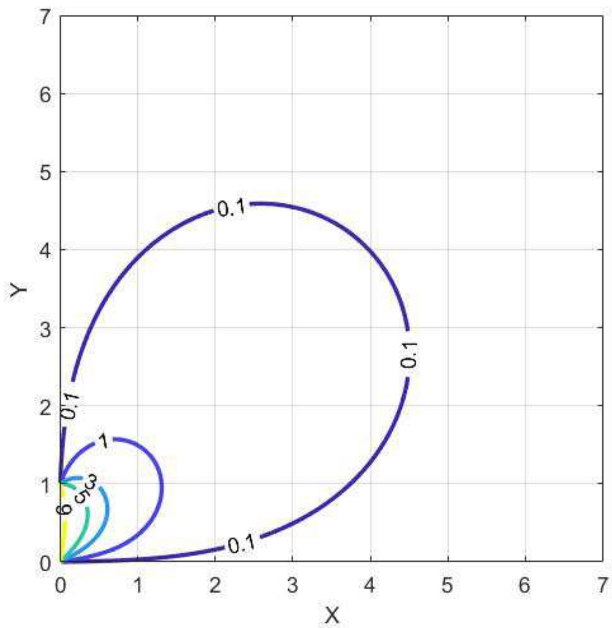

Figure 8. *The 2-D contour of temperature distribution in the domain.*

By comparison with the distribution of temperature depicted between Figure 3 and Figure 7, it is clear that the Gibbs phenomenon exists near the singular points for both (0,0) and (0,1).

3. Solution of Laplace's Equation by Similarity Method

3.1. Quarter-Plane with Neumann Condition

The solution obtained from Fourier cosine transformation, equation (6), is expressed as follows:

$$T(x,y) = \frac{2T_0}{\pi}\left(\int_0^\infty \frac{\sin\alpha}{\alpha} e^{-\alpha x}\cos\alpha y\, d\alpha\right) = \frac{2T_0}{\pi}\mathrm{I}_1(x,y) \tag{12}$$

Where

$$\mathrm{I}_1(x,y) = \int_0^\infty \frac{\sin\alpha}{\alpha} e^{-\alpha x}\cos\alpha y\, d\alpha \tag{13}$$

From the Integral table [18], equation (13) gives

$$\mathrm{I}_1(x,y) = \int_0^\infty \frac{\sin\alpha}{\alpha} e^{-\alpha x}\cos\alpha y\, d\alpha = \frac{1}{2}\tan^{-1}\left(\frac{2x}{x^2+y^2-1}\right) \tag{14}$$

Substituting (14) into (12), we obtain the temperature solution in the quarter-plane,

$$T(x,y) = \frac{2T_0}{\pi}\left(\int_0^\infty \frac{\sin\alpha}{\alpha} e^{-\alpha x}\cos\alpha y\, d\alpha\right) = \frac{T_0}{\pi}\tan^{-1}\left(\frac{2x}{x^2+y^2-1}\right) \tag{15}$$

From trigonometric relation, we have

$$\tan^{-1}\frac{2x}{x^2+y^2-1} + \tan^{-1}\frac{x^2+y^2-1}{2x} = \frac{\pi}{2} \tag{16}$$

Then, the solution (15) becomes

$$T(x,y) = T_0\left[\frac{1}{2} - \frac{1}{\pi}\tan^{-1}\left(\frac{x^2+y^2-1}{2x}\right)\right] \tag{17}$$

By setting similarity variable with $\eta(x,y) = \frac{(x^2+y^2-1)}{2x}$, then equation (17) becomes the similarity solution.

$$T(x,y) = T_0\left(\frac{1}{2} - \frac{1}{\pi}\tan^{-1}\eta\right) = f(\eta), -\infty < \eta < \infty. \tag{18}$$

and the similarity function $f(\eta)$ satisfies Laplace's equation, the following second order ODE can be derived through similarity transform.

$$(1+\eta^2)\frac{d^2f}{d\eta^2} + 2\eta\frac{df}{d\eta} = 0,\ -\infty < \eta < \infty. \tag{19}$$

with the boundary conditions

$$f(\infty) = 0,\ f(-\infty) = T_0 \tag{20}$$

The general solution of equation (19) is

$$f(\eta) = c_1\tan^{-1}\eta + c_2 \tag{21}$$

From the boundary conditions (20), yields

$$c_1 = \frac{-T_0}{\pi},\ c_2 = \frac{T_0}{2} \tag{22}$$

Finally, we obtain the similarity solution of *T*(x, y) in the quarter-plane as follows:

$$T(x,y) = \frac{2T_0}{\pi}\left(\int_0^\infty \frac{\sin\alpha}{\alpha} e^{-\alpha x}\cos\alpha y\, d\alpha\right) = T_0\left(\frac{1}{2} - \frac{1}{\pi}\tan^{-1}\eta\right),\ -\infty < \eta < \infty. \tag{23}$$

For $T_0 = 10$, the temperature distribution of Laplace's equation in a quarter-plane with Neumann condition can be easily calculated from the similarity solution (23), as shown in Figure 9. And the 2-D contour of temperature distribution is shown in Figure 10.

By comparing the results between Figure 3 and Figure 9, it's clear that near the singular point (0,1), where the numerical solution shown in Figure 3 presents the Gibbs phenomenon due to the peculiar manner in which the Fourier series of a piecewise continuous function behaves at a jump discontinuity. Nevertheless, this particular phenomenon does not exist in the similarity solution as shown in Figure 9.

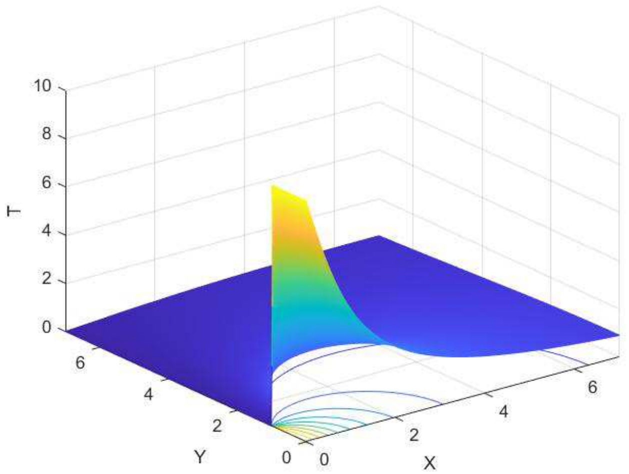

Figure 9. *The distribution of temperature in the domain from equation (23) by similarity method.*

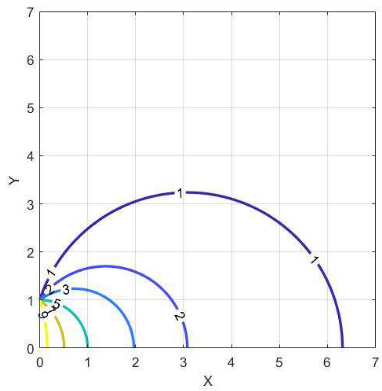

Figure 10. *The 2-D contour of temperature distribution in the domain.*

3.2. Quarter-Plane with Dirichlet Condition

From equation (11), the solution obtained from Fourier Sine transform is expressed as follows:

$$T(x,y)=\frac{2T_0}{\pi}\left(\int_0^\infty \frac{(1-\cos\alpha)}{\alpha}e^{-\alpha x}\sin\alpha y\,d\alpha\right)=\frac{2T_0}{\pi}[\mathrm{I}_2(x,y)-\mathrm{I}_3(x,y)] \tag{24}$$

where

$$\mathrm{I}_2(x,y)=\int_0^\infty \frac{e^{-\alpha x}}{\alpha}\sin\alpha y\,d\alpha \tag{25}$$

$$\mathrm{I}_3(x,y)=\int_0^\infty \frac{\cos\alpha}{\alpha}e^{-\alpha x}\sin\alpha y\,d\alpha \tag{26}$$

From Integral Table [18], we obtain

$$\mathrm{I}_2(x,y)=\int_0^\infty \frac{e^{-\alpha x}}{\alpha}\sin\alpha y\,d\alpha=\tan^{-1}\frac{y}{x} \tag{27}$$

$$\mathrm{I}_3(x,y)=\int_0^\infty \frac{\cos\alpha}{\alpha}e^{-\alpha x}\sin\alpha y\,d\alpha=\frac{1}{2}\tan^{-1}\left(\frac{2xy}{x^2-y^2+1}\right) \tag{28}$$

Thus, we have

$$\mathrm{I}_2(x,y)-\mathrm{I}_3(x,y)=\left[\tan^{-1}\left(\frac{y}{x}\right)-\frac{1}{2}\tan^{-1}\left(\frac{2xy}{x^2-y^2+1}\right)\right] \tag{29}$$

From trigonometry [18], we have

$$\tan^{-1}\mathrm{A}-\tan^{-1}\mathrm{B}=\tan^{-1}\left(\frac{\mathrm{A-B}}{1+\mathrm{AB}}\right) \tag{30}$$

Based on equation (30), yields

$$\mathrm{I}_2(x,y)-\mathrm{I}_3(x,y)=\frac{1}{2}\tan^{-1}\left(\frac{y}{x}\right)+\frac{1}{2}\left[\tan^{-1}\left(\frac{y}{x}\right)-\tan^{-1}\left(\frac{2xy}{x^2-y^2+1}\right)\right]=\frac{1}{2}\tan^{-1}\frac{2xy}{(x^2+y^2)^2+(x^2-y^2)} \tag{31}$$

Substituting (31) into (24), we obtain the temperature solution in the quarter-plane,

$$T(x,y)=\frac{2T_0}{\pi}\int_0^\infty \frac{(1-\cos\alpha)}{\alpha}e^{-\alpha x}\sin\alpha y\,d\alpha=\frac{T_0}{\pi}\tan^{-1}\frac{2xy}{(x^2+y^2)^2+(x^2-y^2)} \tag{32}$$

Similar to equation (16), solution (32) becomes

$$T(x,y)=T_0\left[\frac{1}{2}-\frac{1}{\pi}\tan^{-1}\frac{(x^2+y^2)^2+(x^2-y^2)}{2xy}\right] \tag{33}$$

By setting similarity variable $\xi(x,y)=\frac{(x^2+y^2)^2+(x^2-y^2)}{2xy}$, then (33) becomes the similarity solution of the form

$$T(x,y)=T_0\left(\frac{1}{2}-\frac{1}{\pi}\tan^{-1}\xi\right)=g(\xi),\ -\infty<\xi<\infty \tag{34}$$

and the similarity function $g(\xi)$ satisfies the 2nd order O.D.E

$$(1+\xi^2)\frac{d^2g}{d\xi^2}+2\xi\frac{dg}{d\xi}=0,\ -\infty<\xi<\infty \tag{35}$$

with the boundary conditions

$$g(\infty)=0,\ g(-\infty)=T_0 \tag{36}$$

The general solution of (36) is

$$g(\xi)=c_3\tan^{-1}\xi+c_4 \tag{37}$$

From the boundary condition (36), we obtain

$$c_3=\frac{-T_0}{\pi},\ c_4=\frac{T_0}{2} \tag{38}$$

Finally, we obtain the similarity solution of T(x, y) in the quarter-plane as follows:

$$T(x,y)=\frac{2T_0}{\pi}\left(\int_0^\infty\frac{(1-\cos\alpha)}{\alpha}e^{-\alpha x}\sin\alpha y\,d\alpha\right)=T_0\left(\frac{1}{2}-\frac{1}{\pi}\tan^{-1}\xi\right),\ -\infty<\xi<\infty \tag{39}$$

For $T_0=10$, the solution of temperature distribution of the quarter-plane with Dirichlet condition can be easily calculated from the similarity solution, equation (39), as shown in Figure 11. And the 2-D contour of temperature distribution is shown in Figure 12.

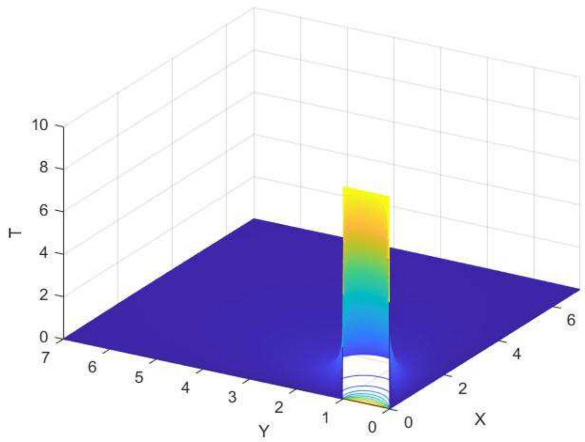

Figure 11. *The distribution of temperature in the domain from equation (39) by similarity method.*

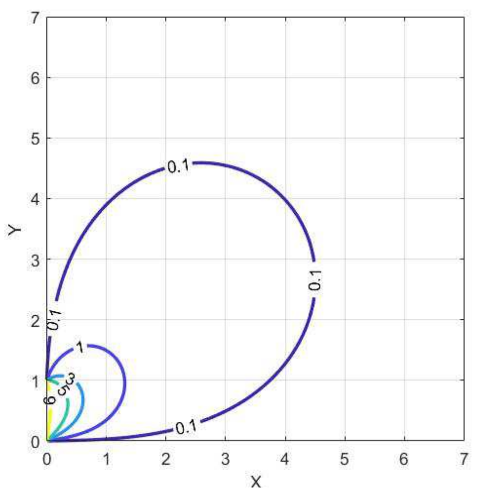

Figure 12. *The 2-D contour of temperature distribution in the domain.*

By comparison of the results between Figure 7 and Figure 11, it's clear that near the two singular points (0,0) and (0,1), the numerical solution presents the Gibbs phenomenon due to a particular manner in which the Fourier series of a piecewise continuous function behaves at a jump discontinuity.

4. Solution of Laplace's Equation by the Method of Images

By using the method of Green's function with images mentioned in [13, 14], Laplace's equation in the quarter-plane with Neumann and Dirichlet conditions can be solved and discussed in the following section.

4.1. Quarter-Plane with Neumann Condition

$$\frac{\partial^2T}{\partial x^2}+\frac{\partial^2T}{\partial y^2}=0,\ 0<x<\infty,\ 0<y<\infty \tag{40}$$

$$\text{B.C.}\qquad T(0,y)=f(y) \tag{41}$$

$$\frac{\partial T}{\partial y}(x,0)=0 \tag{42}$$

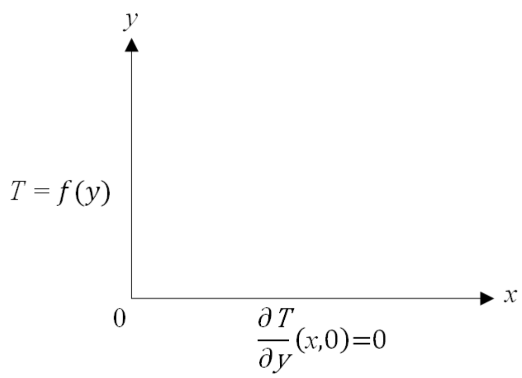

Figure 13. *Domain of quarter-plane for Neumann condition.*

The general solution in the quarter-plane as shown in Figure 13 with Neumann condition from [13, 14] is

$$T(x,y) = \frac{x}{\pi}\int_0^{\infty} f(y_0)\left[\frac{1}{x^2+(y_0-y)^2} + \frac{1}{x^2+(y_0+y)^2}\right] dy_0 = \frac{1}{\pi}[\mathrm{I}_4(x,y) + \mathrm{I}_5(x,y)] \tag{43}$$

with

$$\mathrm{I}_4(x,y) = \int_0^{\infty} \frac{x\, f(y_0)}{x^2+(y_0-y)^2} dy_0 \tag{44}$$

$$\mathrm{I}_5(x,y) = \int_0^{\infty} \frac{x\, f(y_0)}{x^2+(y_0+y)^2} dy_0 \tag{45}$$

If the boundary condition is $T(0,y) = f(y) = \begin{Bmatrix} T_0, & 0 < y \le 1 \\ T_1, & 1 < y < \infty \end{Bmatrix}$, we obtain

$$\mathrm{I}_4(x,y) = T_0 \int_0^1 \frac{x}{x^2+(y_0-y)^2} dy_0 + T_1 \int_1^{\infty} \frac{x}{x^2+(y_0-y)^2} dy_0 = T_0 \tan^{-1}\left(\frac{y_0-y}{x}\right)\Big|_0^1 + T_1 \tan^{-1}\left(\frac{y_0-y}{x}\right)\Big|_1^{\infty} = T_0\left[\tan^{-1}\left(\frac{1-y}{x}\right) + \tan^{-1}\frac{y}{x}\right] + T_1\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{1-y}{x}\right)\right] \tag{46}$$

$$\mathrm{I}_5(x,y) = T_0 \int_0^1 \frac{x}{x^2+(y_0+y)^2} dy_0 + T_1 \int_1^{\infty} \frac{x}{x^2+(y_0+y)^2} dy_0 = T_0\left[\tan^{-1}\left(\frac{1+y}{x}\right) - \tan^{-1}\frac{y}{x}\right] + T_1\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{1+y}{x}\right)\right] \tag{47}$$

Finally, substituting (46) and (47) into (43), we obtain the solution in the quarter-plane with Neumann condition.

$$T(x,y) = \frac{T_0}{\pi}\tan^{-1}\left(\frac{2x}{x^2+y^2-1}\right) + \frac{T_1}{\pi}\left[\pi - \tan^{-1}\left(\frac{2x}{x^2+y^2-1}\right)\right] \tag{48}$$

If $T_1 = 0$, from (48), we obtain the solution

$$T(x,y) = \frac{T_0}{\pi}\tan^{-1}\left(\frac{2x}{x^2+y^2-1}\right) = \frac{T_0}{\pi}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{x^2+y^2-1}{2x}\right)\right] = f(\eta) \tag{49}$$

with similarity variable $\eta(x,y) = \left(\frac{x^2+y^2-1}{2x}\right), -\infty < \eta < \infty$.

The similarity solution (49) is the same as (18) obtained from the Fourier cosine transform. In other words, the similarity solution (49) can be found from the Fourier transform and the method of images from Green's function with Neumann condition.

If $T_1 = T_0$, from (48), we obtain the special solution

$$T(x,y) = T_0 \tag{50}$$

everywhere in the domain with Neumann condition.

4.2. Quarter-Plane with Dirichlet Condition

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0,\ 0 < x < \infty,\ 0 < y < \infty \tag{51}$$

$$\text{B.C.}\quad T(0,y) = f(y) \tag{52}$$

$$T(x,0) = 0 \tag{53}$$

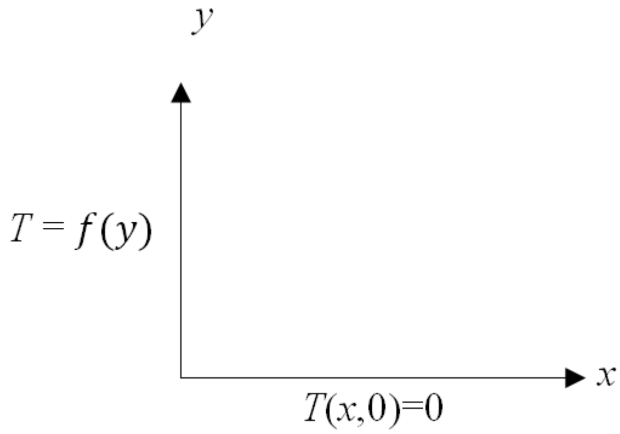

Figure 14. *Domain of quarter-plane for Dirichlet condition.*

The general solution *T*(*x*, *y*) in the quarter-plane as shown in Figure 14 with Dirichlet condition from [13, 14] is

$$T(x,y) = \frac{x}{\pi}\int_0^{\infty} f(y)\left[\frac{1}{x^2+(y_0-y)^2} - \frac{1}{x^2+(y_0+y)^2}\right] dy_0 = \frac{1}{\pi}[\mathrm{I}_4(x,y) - \mathrm{I}_5(x,y)] \tag{54}$$

with $\mathrm{I}_4(x,y) = \int_0^{\infty} \frac{x\, f(y_0)}{x^2+(y_0-y)^2} dy_0$ and $\mathrm{I}_5(x,y) = \int_0^{\infty} \frac{x\, f(y_0)}{x^2+(y_0+y)^2} dy_0$.

If the boundary condition is $T(0,y) = f(y) = \begin{Bmatrix} T_0, & 0 < y \le 1 \\ T_1, & 1 < y < \infty \end{Bmatrix}$, substituting (46) and (47) into (54) we obtain the following solution in the quarter-plane with Dirichlet condition.

$$T(x,y) = \frac{T_0}{\pi}\left[2\tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{2xy}{x^2-y^2+1}\right)\right] + \frac{T_1}{\pi}\tan^{-1}\left(\frac{2xy}{x^2-y^2+1}\right) \tag{55}$$

If $T_1 = 0$, we obtain

$$T(x,y)=\frac{T_0}{\pi}\left[2\tan^{-1}\left(\frac{y}{x}\right)-\tan^{-1}\left(\frac{2xy}{x^2-y^2+1}\right)\right]=T_0\left[\frac{1}{2}-\frac{1}{\pi}\tan^{-1}\frac{\left(x^2+y^2\right)^2+\left(x^2-y^2\right)}{2xy}\right]=g(\xi) \tag{56}$$

with similarity variable $\xi(x,y)=\frac{\left(x^2+y^2\right)^2+\left(x^2-y^2\right)}{2xy}, -\infty<\xi<\infty$.

The similarity solution (56) is the same as (33) obtained from Fourier sine transform. In other words, the similarity solution (56) can be found from Fourier transform and the method of images from Green's function with Dirichlet condition.

If $T_1=T_0$, from (55), we obtain the fundamental similarity solution

$$T(x,y)=\frac{2T_0}{\pi}\tan^{-1}\left(\frac{y}{x}\right)=g(\xi) \tag{57}$$

with similarity variable $\xi=\frac{y}{x},\ 0<\xi<\infty$.

5. A Unified Solution of Laplace's Equation in the Quarter-Plane with Neumann and Dirichlet Conditions

From (17) and (49), the solution obtained from Fourier cosine transform and the method of images from Green's function with Neumann condition is

$$T(x,y)=\frac{2T_0}{\pi}\int_0^\infty\frac{\sin\alpha}{\alpha}e^{-\alpha x}\cos\alpha y\ d\alpha=\frac{T_0}{\pi}\int_0^1\left[\frac{x}{x^2+(y_0-y)^2}+\frac{x}{x^2+(y_0+y)^2}\right]dy_0=\frac{T_0}{\pi}\left[\frac{\pi}{2}-\tan^{-1}\left(\frac{x^2+y^2-1}{2x}\right)\right]=f(\eta) \tag{58}$$

with similarity variable $\eta(x,y)=\left(\frac{x^2+y^2-1}{2x}\right),\ -\infty<\eta<\infty$.

From (33) and (56), the solution obtained from Fourier sine transform and the method of images from Green's function with Dirichlet condition is

$$T(x,y)=\frac{2T_0}{\pi}\int_0^\infty\frac{(1-\cos\alpha)}{\alpha}e^{-\alpha x}\sin\alpha y\,d\alpha=\frac{T_0}{\pi}\int_0^1\left[\frac{x}{x^2+(y_0-y)^2}-\frac{x}{x^2+(y_0+y)^2}\right]dy_0=\frac{T_0}{\pi}\left(\frac{\pi}{2}-\tan^{-1}\xi\right)=g(\xi) \tag{59}$$

with similarity variable $\xi(x,y)=\frac{\left(x^2+y^2\right)^2+\left(x^2-y^2\right)}{2xy},\ -\infty<\xi<\infty$.

Comparing (58) with (59), these two types of integral solutions are completely different and it seems that there is no connection between them. However, we found that these two different solutions have the following similarity form, respectively.

$$T(x,y)=f(\eta),\ -\infty<\eta<\infty \tag{60}$$

$$T(x,y)=g(\xi),\ -\infty<\xi<\infty \tag{61}$$

However, the similarity function $f(\eta)$ in (60) and $g(\xi)$ in (61) satisfy the same ordinary differential equation and the same boundary conditions in (19), (20) and (35), (36) respectively.

Finally, these two different boundary-value problems are the same from the similarity point of view. The reason is that these two different boundary-value problems have the same source type singularity near the singular point at $(x,y)=(0,1)$ as shown below.

From (58), the similarity solution of Laplace's function in the quarter-plane with Neumann condition is

$$T(x,y)=f(\eta)=\frac{T_0}{\pi}\left[\frac{\pi}{2}-\tan^{-1}\left(\frac{x^2+y^2-1}{2x}\right)\right].$$

Near the singular point $(x,y)=(0,1)$, by setting $x=x^*, y=1+y^*$, then in the limit of $x^*\to 0$ and $y^*\to 0$, (58) becomes

$$\lim_{\substack{x\to x^*\\ y\to 1+y^*}}T(x,y)=\lim_{\substack{x\to x^*\\ y\to 1+y^*}}f(\eta)\to\lim_{\substack{x\to x^*\\ y\to 1+y^*}}\tan^{-1}\frac{x^2+y^2-1}{2x}\to\lim_{\substack{x\to x^*\\ y\to 1+y^*}}\tan^{-1}\frac{{x^*}^2+(1+y^*)^2-1}{2x^*} \tag{62}$$

Neglecting the higher order term in (62), then we have source type singularity near the singular point $(x,y)=(0,1)$ for Neumann condition, i.e.

$$\lim_{\substack{x\to x^*\\ y\to 1+y^*}}T(x,y)=\lim_{\substack{x\to x^*\\ y\to 1+y^*}}f(\eta)\to\lim_{\substack{x^*\to 0\\ y^*\to 0}}\tan^{-1}\frac{y^*}{x^*} \tag{63}$$

From (59), the similarity solution of Laplace equation in the quarter-plane with Dirichlet condition is

$$T(x,y) = g(\xi) = \frac{T_0}{\pi}\left[\frac{\pi}{2} - \tan^{-1}\frac{(x^2+y^2)^2+(x^2-y^2)}{2xy}\right].$$

Near the singular point $(x,y) = (0,1)$, by setting $x = x^*, y = 1 + y^*$, then in the limit of $x^* \to 0$ and $y^* \to 0$, (59) becomes

$$\lim_{\substack{x\to x^*\\ y\to 1+y^*}} T(x,y) = \lim_{\substack{x\to x^*\\ y\to 1+y^*}} g(\xi) \to \lim_{\substack{x\to x^*\\ y\to 1+y^*}} \tan^{-1}\frac{(x^2+y^2)^2+(x^2-y^2)}{2xy} \to \lim_{\substack{x\to x^*\\ y\to 1+y^*}} \tan^{-1}\frac{[x^{*2}+(1+y^*)^2]^2+[x^{*2}-(1+y^*)^2]}{2x^*(1+y^*)} \quad (64)$$

Neglecting the higher order terms in (64), we obtain the same source type of singularity as equation (63) near the singular point $(x,y) = (0,1)$ for Dirichlet condition, i.e.

$$\lim_{\substack{x\to x^*\\ y\to 1+y^*}} T(x,y) = \lim_{\substack{x\to x^*\\ y\to 1+y^*}} g(\xi) \to \lim_{\substack{x^*\to 0\\ y^*\to 0}} \tan^{-1}\frac{y^*}{x^*}. \quad (65)$$

Similarly, from (59), there is another source type singularity near the singular point $(x,y) = (0,0)$. By setting $x = x^*, y = y^*$, then in the limit of $x^* \to 0$ and $y^* \to 0$, (59) becomes

$$\lim_{\substack{x^*\to 0\\ y^*\to 0}} T(x,y) \to \lim_{\substack{x^*\to 0\\ y^*\to 0}} \tan^{-1}\frac{(x^2+y^2)^2+(x^2-y^2)}{2xy} \to \lim_{\substack{x^*\to 0\\ y^*\to 0}} \tan^{-1}\frac{(x^{*2}+y^{*2})^2+(x^{*2}-y^{*2})}{2x^*y^*} \quad (66)$$

Neglecting higher-order terms in (66), we obtain another source type singularity near the singular point $(x,y) = (0,0)$ for Dirichlet condition, i.e.

$$\lim_{\substack{x\to 0\\ y\to 0}} T(x,y) = \lim_{\substack{x\to 0\\ y\to 0}} g(\xi) \to \lim_{\substack{x^*\to 0\\ y^*\to 0}} \tan^{-1}\left(\frac{x^{*2}-y^{*2}}{2x^*y^*}\right) \quad (67)$$

In principle, Laplace's equation gives different solutions for different boundary conditions. However, if these solutions of different boundary-value problems have the same type of singularity near some singular points, then these different solutions can be unified as the same type of singular similarity solution as shown above.

6. Conclusion

Laplace's equation for temperature distribution, in a domain with semi-infinite and in the quarter-plane, has been studied by the analytical methods. For the specified Neumann and Dirichlet boundary conditions, the analytical solution in the quarter-plane found by Fourier transform, the method of Green's function with images, and the similarity method are unified. In this study, we found that for different boundary-value problems if the solutions have the same type of singularity near some singular points on the boundary, then these different solutions can be unified as the same type of singular similarity solution.

For the specified Neumann or Dirichlet boundary conditions, the temperature distribution in the quarter-plane domain can be easily calculated from the similarity solutions. However, it is shown that Gibbs phenomenon behaves at a jump discontinuity, where serious oscillation result was found from the integral methods, especially near the singular point on the boundary. This study proves that the similarity solution can be used to check the accuracy of numerical solutions. In addition, this paper provided a new way to evaluate some definite integrals analytically.

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