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Implicit Runge-Kutta method for Van der pol problem

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Abstract: In this manuscript the implicit Runge-Kutta (IRK) method, with three slopes of order five has been explained, and is applied to Van der pol stiff differential equation. Truncation error, of order five, has been estimated. Stability of the procedure for the Van der pol equation, is analyzed by the Lyapunov method. To illustrate the structure of the method, an Algorithm is presented to solve this stiff problem. Results confirm the validity and the ability of this approach.

Keywords: Implicit Method, Taylor Series, Legendre Orthogonal Polynomial, Van Der Pol Equation, Lyapunov Function

1. Introduction

An Implicit Runge-Kutta method for solving differential equation $y'=f(t,y)$ with ν slopes is defined by the following equation:

$$y_{n+1} = y_n + \sum_{i=1}^{\nu} w_i K_i$$

Where $K_i = hf\left(t_n + c_i h, y_n + \sum_{j=1}^{\nu} a_{ij} K_j\right)$

and $c_i = \sum_{i=1}^{\nu} a_{ij}$, $i = 1, 2, \ldots, \nu$

And a_{ij} , $1 \le i, j \le \nu$, $w_1, w_2, \ldots, w_\nu$ are parameters that will be determined. The function K_j is defined by a set of ν implicit equation.(see [1] and [6])

In electronic, the Van der Pol oscillator is a non-conservative oscillator with non-linear damping. This problem was originally introduced by Van der pol (1926) in the study of electronic circuit by the following second order stiff differential equation

$$u'' + \varepsilon(u^2 - 1)u' + u = 0.$$

Where u is a function of the time t , and ε is a positive scalar parameter indicating the nonlinearity and the strength of the damping. (see [4] p-121 & [1] p-16)

2. The Implicit Runge-Kutta Method

In this implicit method let's $\nu = 3$, then we have the following equation

$$\begin{cases} K_i = h\,f(t_n + c_i h, y_n + a_{i1}K_1 + a_{i2}K_2 + a_{i3}K_3), \\ y_{n+1} = y_n + w_1 K_1 + w_2 K_2 + w_3 K_3 , \\ c_i = a_{i1} + a_{i2} + a_{i3}, \quad i = 1, 2, 3 . \end{cases} \tag{1}$$

Where $a_{i1}, a_{i2}, a_{i3}, w_i$, $i = 1, 2, 3$ are twelve arbitrary parameters, which should be determined. The Taylor series gives

$$y(t_{n+1}) = y(t_n) + hy'(t_n) + \frac{1}{2}h^2 y''(t_n) + \frac{1}{6}h^3 y'''(t_n) + \frac{1}{24}h^4 y^{iv}(t_n) + \ldots \tag{2}$$

Where

$$y'(t_n) = f(t_n, y_n)$$
$$y''(t_n) = (f_t + ff_y)_n = (\frac{\partial}{\partial t} + f_n \frac{\partial}{\partial y})f = Df$$
$$y'''(t_n) = [(f_{tt} + 2ff_{ty} + f^2 f_{yy}) + f_y (f_t + ff_y)] = D^2 f + f_y Df$$
$$y^{iv}(t_n) = [(f_{ttt} + 3ff_{tty} + 3f^2 f_{tyy} + f^3 f_{yyy}) + f_y (f_{tt} + 2ff_{ty} + f^2 f_{yy}) + (f_t + ff_y)(3f_{ty} + 3ff_{yy} + f_y^2)]_n = D^3 f + f_y D^2 f + 3Df\, Df_y + f_y^2 Df$$

Based on expansion of two-variable function, the equation K_i in (1) changes to:

$$\begin{aligned} K_i = h[& f_n + (c_i h f_t + (a_{i1}K_1 + a_{i2}K_2 + a_{i3}K_3) f_y) \\ &+ \frac{1}{2}(c_i^2 h^2 f_{tt} + 2c_i h (a_{i1}K_1 + a_{i2}K_2 + a_{i3}K_3) f_{ty} + (a_{i1}K_1 + a_{i2}K_2 + a_{i3}K_3)^2 f_{yy}) \\ &+ \frac{1}{6}(c_i^3 h^3 f_{ttt} + 3c_i^2 h^2 (a_{i1}K_1 + a_{i2}K_2 + a_{i3}K_3) f_{tty} \\ &+ 3c_i h (a_{i1}K_1 + a_{i2}K_2 + a_{i3}K_3)^2 f_{tyy} + (a_{i1}K_1 + a_{i2}K_2 + a_{i3}K_3)^3 f_{yyy}) \\ &+ \frac{1}{24}(c_i^4 h^4 f_{tttt} + 4c_i^3 h^3 (a_{i1}K_1 + a_{i2}K_2 + a_{i3}K_3) f_{ttty} \\ &+ 6c_i^2 h^2 (a_{i1}K_1 + a_{i2}K_2 + a_{i3}K_3)^2 f_{ttyy} + 4c_i h (a_{i1}K_1 + a_{i2}K_2 + a_{i3}K_3)^3 f_{tyyy} \\ &+ (a_{i1}K_1 + a_{i2}K_2 + a_{i3}K_3)^4 f_{yyyy}) + \ldots] \quad , i = 1, 2, 3 \end{aligned} \tag{3}$$

These equations are implicit and we cannot easily obtain the explicit expression for K_1, K_2, and K_3. In order to determine $K_1, K_2,$ and K_3 explicitly, we assume the following form

$$K_i = hA_i + h^2 B_i + h^3 E_i + h^4 D_i + \ldots \quad , i = 1, 2, 3 \tag{4}$$

Where A_i, B_i, E_i, D_i, and F_i are unknown to be determined.

Substituting for $K_1, K_2,$ and K_3 from (4) into (3), and on equating the terms with identical powers of h in Taylors series, we obtain the following results:

$$A_i = f_n$$
$$B_i = c_i f_t + (a_{i1}A_1 + a_{i2}A_2 + a_{i3}A_3) f_y = c_i (f_t + f_n f_y) = c_i Df$$
$$E_i = (a_{i1}c_1 + a_{i2}c_2 + a_{i3}c_3) f_y Df + \frac{1}{2} c_i^2 D^2 f$$
$$\begin{aligned} D_i = [& a_{i1}(a_{11}c_1 + a_{12}c_2 + a_{13}c_3) + a_{i2}(a_{21}c_1 + a_{22}c_2 + a_{23}c_3) \\ &+ a_{i3}(a_{31}c_1 + a_{32}c_2 + a_{33}c_3)] f_y^2 Df_n + c_i (a_{i1}c_1 + a_{i2}c_2 + a_{i3}c_3) Df_n Df_y \\ &+ \frac{1}{2}(a_{i1}c_1^2 + a_{i2}c_2^2 + a_{i3}c_3^2) f_y D^2 f_n + \frac{1}{6} c_i^3 D^3 f_n \end{aligned}$$

$$\begin{aligned} F_i = [& a_{i1}[(a_{11}(a_{11}c_1 + a_{12}c_2 + a_{13}c_3) + a_{12}(a_{21}c_1 + a_{22}c_2 + a_{23}c_3) \\ &+ a_{13}(a_{31}c_1 + a_{32}c_2 + a_{33}c_3)) f_y^2 Df_n + c_1 (a_{11}c_1 + a_{12}c_2 + a_{13}c_3) Df_n Df_y \\ &+ \frac{1}{2}(a_{11}c_1^2 + a_{12}c_2^2 + a_{13}c_3^2) f_y D^2 f_n + \frac{1}{6} c_1^3 D^3 f_n] \\ &+ a_{i2}[(a_{21}(a_{11}c_1 + a_{12}c_2 + a_{13}c_3) + a_{22}(a_{21}c_1 + a_{22}c_2 + a_{23}c_3) \\ &+ a_{23}(a_{31}c_1 + a_{32}c_2 + a_{33}c_3)) f_y^2 Df_n + c_2 (a_{21}c_1 + a_{22}c_2 + a_{23}c_3) Df_n Df_y \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}(a_{21}c_1^{\,2}+a_{22}c_2^{\,2}+a_{23}c_3^2)f_yD^2f_n+\frac{1}{6}c_2^3D^3f_n]+a_{i3}[(a_{31}(a_{11}c_1+a_{12}c_2+a_{13}c_3)\\
&+a_{32}(a_{21}c_1+a_{22}c_2+a_{23}c_3)+a_{33}(a_{31}c_1+a_{32}c_2+a_{33}c_3))f_y^{\,2}Df_n\\
&+c_3(a_{31}c_1+a_{32}c_2+a_{33}c_3)Df_nDf_y+\frac{1}{2}(a_{31}c_1^{\,2}+a_{32}c_2^{\,2}+a_{33}c_3^2)f_yD^2f_n+\frac{1}{6}c_3^3D^3f_n]]f_y\\
&+c_i[a_{i1}((a_{11}c_1+a_{12}c_2+a_{13}c_3)f_yDf_n+\frac{1}{2}c_1^2Df_n)\\
&+a_{i2}((a_{21}c_1+a_{22}c_2+a_{23}c_3)f_yDf_n+\frac{1}{2}c_2^2Df_n)\\
&+a_{i3}((a_{31}c_1+a_{32}c_2+a_{33}c_3)f_yDf_n+\frac{1}{2}c_3^2Df_n)]f_{ty}+\frac{1}{2}(a_{i1}c_1+a_{i2}c_2+a_{i3}c_3)^2f_{yy}D^2f_n\\
&+\frac{1}{2}c_i^2(a_{i1}c_1+a_{i2}c_2+a_{i3}c_3)f_{tty}Df_n+c_i^2(a_{i1}c_1+a_{i2}c_2+a_{i3}c_3)f_nf_{tyy}Df_n\\
&+\frac{1}{2}c_i^2f_n^2(a_{i1}c_1+a_{i2}c_2+a_{i3}c_3)f_{yyy}Df_n+\frac{1}{24}c_i^4D^4f_n \qquad ,i=1,2,3
\end{aligned}
\tag{5}
$$

Numerical method in equation (1) with the help of (4) may be written as

$$
\begin{aligned}
y_{n+1} &= y_n + h(w_1A_1+w_2A_2+w_3A_3)+h^2(w_1B_1+w_2B_2+w_3B_3)\\
&\quad + h^3(w_1E_1+w_2E_2+w_3E_3)+h^4(w_1D_1+w_2D_2+w_3D_3)+\dots
\end{aligned}
\tag{6}
$$

Where A_i, B_i, E_i, D_i, and F_i are given by.(5)

By equating the coefficients of the terms with the identical powers of h in (6) and (2), the following equations are obtained. which are the same as the system of equations Butcher introduced in ([1] & [6])

$(7a)$: $\sum_{i=1}^{3} w_i = 1,$

$(7b)$: $\sum_{i=1}^{3} w_i c_i = \frac{1}{2},$

$(7c)$: $\sum_{i=1}^{3} w_i c_i^2 = \frac{1}{3},$

$(7d)$: $\sum_{i=1}^{3} w_i c_i^3 = \frac{1}{4},$

$(7e)$: $\sum_{i,j=1}^{3} w_i a_{ij} c_j = \frac{1}{6},$

$(7f)$: $\sum_{i,j=1}^{3} w_i c_i a_{ij} c_j = \frac{1}{8},$

$(7g)$: $\sum_{i,j=1}^{3} w_i c_i a_{ij} c_j = \frac{1}{8},$

$(7h)$: $\sum_{i,j=1}^{3} w_i a_{ij} c_j^2 = \frac{1}{12},$

$(7i)$: $\sum_{i,j,k=1}^{3} w_i a_{ij} a_{jk} c_k = \frac{1}{24}.$

w_i , $i=1,2,3$ can be determined from $(7a-7d)$, and from $(7e-7i)$ and three more the following equations $c_i = a_{i1}+a_{i2}+a_{i3}$, $i=1,2,3$, a_{ij}, $i,j=1,2,3$ will be calculated.

Butcher (1964) introduced RK method based on the Radau and Lobatto quadrature formulas. In this procedure the coefficients c_i are taken the Radau's roots of the Legendre polynomial of degree three:

$$\frac{d^2}{dx^2}\left(x^2(x-1)^3\right)=0 \quad \Rightarrow c_1=\frac{4-\sqrt{6}}{10},\ c_2=\frac{4+\sqrt{6}}{10},\ c_3=1.$$

The solution of the system (7), after substitution of the values of c_i results in the following coefficient of Radau formula of order five:

$$
C=\begin{bmatrix}\frac{4-\sqrt{6}}{10}\\ \frac{4+\sqrt{6}}{10}\\ 1\end{bmatrix},\quad
W=\begin{bmatrix}\frac{16-\sqrt{6}}{36}\\ \frac{16+\sqrt{6}}{36}\\ \frac{1}{9}\end{bmatrix}\quad
A=\begin{bmatrix}\frac{88-7\sqrt{6}}{360} & \frac{296-169\sqrt{6}}{1800} & \frac{-2+3\sqrt{6}}{225}\\ \frac{296+169\sqrt{6}}{1800} & \frac{88+7\sqrt{6}}{360} & \frac{-2-3\sqrt{6}}{225}\\ \frac{16-\sqrt{6}}{36} & \frac{16+\sqrt{6}}{36} & \frac{1}{9}\end{bmatrix}
$$

Substitutions of these values in (1), leads to:

$$\begin{aligned}
K_1 &= hf\left(t_n + \frac{4-\sqrt{6}}{10}h,\, y_n + \frac{88-7\sqrt{6}}{360}K_1 + \frac{296-169\sqrt{6}}{1800}K_2 + \frac{-2+3\sqrt{6}}{225}K_3\right),\\
K_2 &= hf\left(t_n + \frac{4+\sqrt{6}}{10}h,\, y_n + \frac{296+169\sqrt{6}}{1800}K_1 + \frac{88+7\sqrt{6}}{360}K_2 + \frac{-2-3\sqrt{6}}{225}K_3\right),\\
K_3 &= hf\left(t_n + h,\, y_n + \frac{16-\sqrt{6}}{36}K_1 + \frac{16+\sqrt{6}}{36}K_2 + \frac{1}{9}K_3\right).\\
y_{n+1} &= y_n + \frac{16-\sqrt{6}}{36}K_1 + \frac{16+\sqrt{6}}{36}K_2 + \frac{1}{9}K_3.
\end{aligned} \tag{8}$$

3. Numerical Example

To illustrate the method, let's apply IRK on the following stiff problem, which is known as Van der pol equation:

$$y'' + \varepsilon(y^2-1)y' + y = 0,\ y(0) = 2,\ y'(0) = 0.$$

In the first step by considering the new dependent variables, Van der pol equation is written, equivalently, as the following system of two first order differential equation:

$$\begin{cases} u' = f(u,v) = v - \varepsilon\left(\dfrac{u^3}{3} - u\right), & u(0) = 2\\ v' = g(u,v) = -u, & v(0) = \dfrac{2}{3}\varepsilon \end{cases} \tag{9}$$

For applying iterative formula (8), to the system (9) the parameters K_{ui}, K_{vi} should be computed from the following formulas:

$$\begin{cases} K_{ui} = hf\,(t_n + c_i h,\, u_n + \sum_j a_{i,j}K_{uj},\, v_n + \sum_j a_{i,j}K_{vj})\\ K_{vi} = hg\,(t_n + c_i h,\, u_n + \sum_j a_{i,j}K_{uj},\, v_n + \sum_j a_{i,j}K_{vj}) \end{cases}$$

Where $i, j = 1,2,3$. Then

$$\begin{cases} u_{n+1} = u_n + \dfrac{16-\sqrt{6}}{36}K_{u1} + \dfrac{16+\sqrt{6}}{36}K_{u2} + \dfrac{1}{9}K_{u3}\\ v_{n+1} = v_n + \dfrac{16-\sqrt{6}}{36}K_{v1} + \dfrac{16+\sqrt{6}}{36}K_{v2} + \dfrac{1}{9}K_{v3} \end{cases} \tag{10}$$

Pseudo code Algorithm: Let's explain the above method with following Pseudo code of Newton iterative procedure for solving example (9) :(take $i, j = 1,2,3$)

IRK_NEWTON ALG(ε, step size h, u_0, v_0)

for $i \leftarrow 2$ **to** *time.lenght*
  do *input* $U^{(0)}, V^{(0)}$.
    for $k \leftarrow 1$ **to** *infinite*
      do for $j \leftarrow 1$ **to** 3
        do $\varphi_u(j) \leftarrow U^{(k)}[j] - u_n - h\cdot A[j, 1..3]\cdot\left[f\left(U^{(k)}[1], V^{(k)}[1]\right); \ldots\right.$
          $\left.f\left(U^{(k)}[2], V^{(k)}[2]\right); f\left(U^{(k)}[3], V^{(k)}[3]\right)\right]$
        $\varphi_v(j) \leftarrow V^{(k)}[j] - v_n + h\cdot A[j, 1..3]\cdot U^{(k)}$
      for $s \leftarrow 1$ **to** 3
        do for $r \leftarrow 1$ **to** 3
          do $M_u(s,r) \leftarrow \delta_{s,r} - h\cdot A(s,r)\cdot\dfrac{df}{du}\left(U^{(k)}[r], V^{(k)}[r]\right)$
          $M_v(s,r) \leftarrow \delta_{s,r}$
      $\Delta_u \leftarrow M_u^{-1}\cdot\varphi_u$
      $\Delta_v \leftarrow M_v^{-1}\cdot\varphi_v$
      $U^{(k+1)} \leftarrow U^{(k)} - \Delta_u$
      $V^{(k+1)} \leftarrow V^{(k)} - \Delta_v$
      if $\left\| U^{(k+1)} - U^{(k)} \right\| <$ *precise1* **and** $\left\| V^{(k+1)} - V^{(k)} \right\| <$ *precise2*
        break
  $K_u \leftarrow A^{-1}\left(U^{(final)} - u_n\cdot[1,1,1]^t\right)$
  $K_v \leftarrow A^{-1}\left(V^{(final)} - v_n\cdot[1,1,1]^t\right)$
  $u_{n+1} \leftarrow u_n + W^t\cdot K_u$
  $v_{n+1} \leftarrow v_n + W^t\cdot K_v$

The results of applying this algorithm to the Van der pol equation, are plotted for $\varepsilon = 10, h = 0.1$, in the following Fig1.

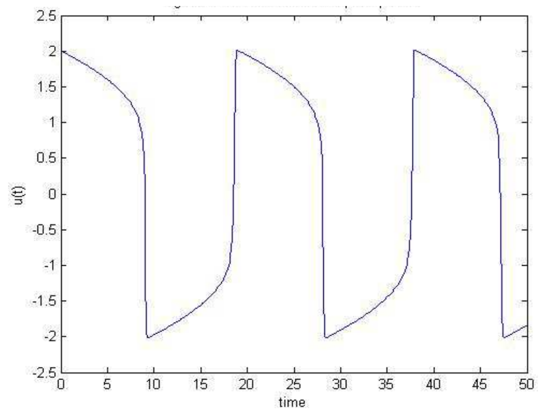

Fig. 1. *The solution of Vsn der pol equation*

must be added to the computed quantity in order that the result be exactly equal to the quantity that we are looking for. This means:

$$y\left(\text{true computed quantity}\right) + \mathbf{T} = y\left(\text{exact solution}\right)$$

Then the exact value of $y(t)$ will satisfy

$$y(t_{n+1}) = y(t_n) + h\varphi(t_n, y(t_n), h) + T_n$$

Where $\varphi(t_n, y(t_n), h)$ is a function of the argument t, y, and h, and is called the increment function, and T_n is the local truncation error. Let's taken $y(t) = t^6$ as computed quantity of (8) we get:

4. Truncation Error Analysis

Definition:[6] truncation error is the quantity T which

$$f(t,y) = 6t^5$$

$$y(t_{n+1}) = y(t_n) + \frac{16-\sqrt{6}}{36}K_1 + \frac{16+\sqrt{6}}{36}K_2 + \frac{1}{9}K_3 + T_n$$

$$t_{n+1}^6 = (t_n + h)^6 = t_n^6 + \frac{16-\sqrt{6}}{36}6h(t_n + \frac{4-\sqrt{6}}{10}h)^5 + \frac{16+\sqrt{6}}{36}6h(t_n + \frac{4+\sqrt{6}}{10}h)^5$$

$$+\frac{1}{9}6h(t_n + h)^5 + T_n \Rightarrow T_n = -\frac{1}{100}h^6$$

The traditional value of the truncation error is usually considered as: $T_n = C_6 h^6 y^{(6)}(\zeta)$. comparing with the above value of truncation error results in $C_6 = -\frac{1}{72000}$.

So the truncation error is of $O(h^6)$, i.e. the method (8) is of order five. (see [6] and [1])

5. Stability Analysis of the Van Der Pol System

Definition 1. Stability and asymptotic stability: The solution of a system of equation say $X' = F(X)$, is stable, if for all $t \geq t_0$ we get: *[9]*

$$\forall\varepsilon > 0 \quad \exists\delta_\varepsilon > 0 \quad (\ \|\bar{x}(t_0) - x(t_0)\| < \delta \Rightarrow \|\bar{x}(t) - x(t)\| < \varepsilon)$$

and the solution is asymptotic stable, if it is stable and also

$$\exists\delta_0 > 0 \quad (\|\bar{x}(t_0) - x(t_0)\| < \delta_0 \Rightarrow \|\bar{x}(t) - x(t)\| \to 0), \text{ as } t \to \infty$$

Definition 2. Invariant set**:** A set such as $M \subset R^n$ is invariant set of system of equation $X' = F(X)$, if from any $X_0 \in M$ conclude that $X(t, X_0) \in M$, for all $t \in R$ *[9].*

Theorem l. If the scalar positive definite function $V(X)$, called Lyapunov function, defined on the set $S_p = \{X \in R^n : \|X\| < P\}$, and $V'(X) \leq 0$ then the zero solution of $X' = F(X)$ is stable *[9].*

Theorem 2. Let's the positive definite function $V(X)$ exists as such $V'(X) \leq 0$ on the open set Ω, in $X' = F(X)$, F is a function $F: \Omega \to R^n$, M includes all invariant subsets of $E \subset c_\lambda$, where $E = \{X \in R^n : V' = 0\}$ and $c_\lambda = \{X \in R^n : V(X) \leq \lambda\}$. Then any solution $X(t, X_0) \in C_\lambda$ of $X' = F(X)$, converges into the M *[11].*

Corollary. Let's the assumptions of the theorem 2 hold. If zero is only invariant point of E, then the zero solution of $X' = F(X)$ is asymptotically stable *[11].*

Stability of the system of equation (9) is proved in the following:

Let's $V(u,v) = \frac{1}{2}(u^2 + v^2)$, which is positive definite on R^2 and $V'(u,v) = -\varepsilon u^2(\frac{u^2}{3} - 1)$, then on the strip $\Omega = \{(u,v) \in R^2 : -\sqrt{3} < u < \sqrt{3}, -\infty < v < \infty\}$, we have $V'(X) \leq 0$. It is obvious that for $E = \{(u,v) \in \Omega : u = 0\}$, system (9) is converted into the following system:

$$\begin{cases} u' = v \\ v' = 0 \end{cases}$$

Regarding the last corollary the set including zero is the only invariant subset of E. And $(0,0)$ is the asymptotic stable point of (9). To determine the asymptotic stability area, Let's consider the set of curves $V(u,v) = \lambda$, where $\lambda \geq 0$, $(u,v) \in \Omega$. This set is obviously closed and the curves are symmetric with respect to the $u-axis$. The function $\frac{u^2}{2}$ is decreasing on the interval $(-\sqrt{3}, 0]$ and increases on the $0 < u < \sqrt{3}$. For the constant λ, the curve $V(u,v) = \lambda$ cuts the borders at one of the points $(-\sqrt{3},0)\, or\, (\sqrt{3},0)$. So the best value for the parameter λ is equal to $\bar{\lambda} = min(\frac{(-\sqrt{3})^2}{2}, \frac{(\sqrt{3})^2}{2}) = \frac{3}{2}$, and asymptotic stability area consists of the pointes in the closed circle $C_{\bar{\lambda}} = \{(u,v) \in \Omega : u^2 + v^2 \leq 3\}$. Then any limit cycle of the Van der pol equation is out of the circle $u^2 + v^2 = 3$ (see [1] p-16).

6. Conclusions

Since there are three different topics studied, conclusion is also divided in different parts;

A-From truncation error section, we conclude that order of this implicit method is five, and this means that this numerical method is precise for polynomials of degree less than six.

B- The disadvantage of the Runge-Kutta methods is that they involve considerably more computations, but have the advantage of self starting.

C- Method (10) with $h = 0.1$ can only used until $\varepsilon = 33$, and for $\varepsilon > 33$, the order of the method should increased or related step size decreased.

D- From stability analysis section, we conclude that the method applied to the Van der pol equation is stable, And this means the formula of the numerical method is insensitive to small change in the local errors.

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