Analytical expressions for commensal-host ecological model: Homotopy perturbation method

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Abstract: In this paper a mathematical commensal-host ecological model with replenishment rate for both species is discussed. This model is characterized by a pair of first order non-linear coupled differential equations. The non-linear coupled system-equations are solved analytically by using Homotopy perturbation method. Further, our results are compared with the previous work and a satisfactory agreement is noted.

Keywords: Mathematical Model, Commensal, Host, Replenishment Rate, Non-Linear Differential Equations, Homotopy Perturbation Method

1. Introduction

The analytical solutions for the first order non-linear coupled differential equations are not possible directly because of intractability of non-linear terms. Techniques like perturbation to linearizing the non-linear coupled differential equations to give qualitative nature of solutions in treatises of Cushing [3], Phanikumar Kapoor [4], Meyer [6], Phanikumar and Pattabhi Ramacharyulu [7].

The previous investigation presented [1] a four stage recursive procedure to give an approximate solution for a first order non-linear coupled differential equations and same is applied to solve the commensal-host ecological model with replenishment rate for both the species. A similar algorithm was earlier developed by Langlosis and Rivlin [5] to obtain approximate solution for the flows of slightly visco- elastic fluids. Srinivas [8] and Bhaskara Rama Sarma [2] adopted this algorithm for a general two species eco system.

The purpose of this paper is to derive the approximate analytical solution of the populations of the two species for all values of the dimensionless parameters using the Homotopy perturbation method.

2. The Basic Model Equation

The model equations are obtained as

$$\frac{dN_1(t)}{dt} = a_1 N_1(t) - a_{11} N_1^2(t) + a_{12} N_1(t) N_2(t) + h_1 \qquad (1)$$

$$\frac{dN_2(t)}{dt} = a_2 N_2(t) - a_{22} N_2^2(t) + h_1 \qquad (2)$$

With the initial conditions

$$N_1(0) = N_{10}; \quad N_2(0) = N_{20} \qquad (3)$$

where $N_1(t)$ and $N_2(t)$ are the population of the species S_1 and S_2 respectively at time t, a_1 and a_2 are the natural growth rates of species S_1 and S_2 respectively, a_{11} and a_{22} are the rates of decrease of species S_1 and S_2 respectively due to limited resources, a_{12} is the inhibition coefficient, h_1 and h_2 are the replenishment rates of S_1 and S_2 respectively. All the coefficients $a_1, a_2, a_{11}, a_{12}, a_{22}, h_1$ and h_2 are positive values.

3. Solution of the Non-Linear Initial Value Problem Using the Homotopy Perturbation Method (HPM)

Recently, many authors have applied the Homotopy perturbation method (HPM) to solve the non-linear problem in physics and engineering sciences [9-12]. This method is also used to solve some of the non-linear problem in physical

sciences [13-15]. This method is a combination of Homotopy in topology and classic perturbation techniques. Ji-Huan He used to solve the Lighthill equation [13], the Duffing equation [14] and the Blasius equation [15-16]. The HPM is unique in its applicability, accuracy and efficiency. The HPM uses the imbedding parameter p as a small parameter, and only a few iterations are needed to search for an asymptotic solution. Using this method [17-19], we can obtain solution to the eqns. (1) - (3) as follows:

$$N_1(t) = \left(k_0\, e^{a_1 t} - k_1\right) + A\, e^{a_1 t} + \left[\frac{a_{11}\, k_1^2 - a_{12}\, k_1\, u_1}{a_1}\right] + \left[2\, a_{11} k_0\, k_1 - a_{12}\, k_0\, u_1\right] t\, e^{a_1 t}$$
$$- \left[\frac{a_{12}\, k_1\, u_0}{a_2 - a_1}\right] - \left[\frac{a_{11}\, k_0^2 e^{a_2 t}}{a_1}\right] + \left[\frac{a_{12} k_0\, u_0\, e^{(a_1 + a_2)\, t}}{a_2}\right] \tag{4}$$

$$N_2(t) = \left(u_0\, e^{a_2 t} - u_1\right) + \left(a_{22}\big/a_2\right)\left[\left(u_0^2 - u_1^2\right) e^{a_2 t} - \left(u_1^2 - u_0^2\, e^{2 a_2\, t}\right)\right] + 2\, a_{22}\, u_0\, u_1\, e^{a_2\, t} \tag{5}$$

where

$$k_0 = N_{10} + \frac{h_1}{a_1},\ k_1 = \frac{h_1}{a_1},\ u_0 = N_{20} + \frac{h_2}{a_2}, u_1 = \frac{h_2}{a_2} \tag{6}$$

$$A = \left[\frac{a_{12} k_1 u_1 + a_{11}\, k_0^2 - a_{11}\, k_1^2}{a_1}\right] + \left[\frac{a_{12}\, k_1\, u_0}{a_2 - a_1}\right] - \left[\frac{a_{12}\, k_0\, u_0}{a_2}\right] \tag{7}$$

4. Results and Discussions

The variation of N_1 and N_2 verses time t in the interval [0, 1] is computed analytically employing Homotopy perturbation method for a wide range of values of the characterizing the parameters a_1, a_2, a_{11}, a_{12} and a_{22} as shown in Table-1. From Fig.1-8, it is observed that when the growth rate of host increases the N_2 curves are falling with increasing steepness. Both N_1 and N_2 are decreases with time t because of utilization of energy. For $a_{12} < 1$, weak commensalism of host over commensal N_2 falls with slower rate than N_1. However with increasing growth rate of host steepness increase for the N_2 curves and falling rate is decreased comparison with N_1. With increasing a_{22} steepness decreases.

For $a_{12} = 1$, the steepness of N_2 increases with a_2 however falling of N_2 is faster along with increasing a_{22} than N_1. For $a_{12} > 1$, strong commensalism of host over commensal it is observed that N_1 increasing to some extent and then start falling. This tendency is observed at slow rate in case of $a_{12} = 1$. Because of growth rate of the specie S_1, N_1 falls much faster than N_2. However this fall becomes slower in case of $a_{12} > 1$.

5. Conclusion

Investigate some relation-chains between the species such as commensal-host ecological model with replenishment rate between two species (S_1, S_2) with the population relations. The present paper deals with an investigation on analytical approach of a typical two species syn eco-system. The system of coupled non-linear differential equations of populations of two species S_1 and S_2 has been solved analytically using Homotopy perturbation method. Our analytical results are compared with previous work and the results are good agreement with the previous work. This method can be extended to solve a mathematical model of a population of three species and four species with Syn-Ecological system.

Nomenclature

Symbol	Meaning
N_1	Population of the specie S_1
N_2	Population of the specie S_2
a_1	Natural growth rate of the specie S_1
a_2	Natural growth rate of the specie S_2
a_{11}	Rate of decrease of the specie S_1
a_{22}	Rate of decrease of the specie S_2
a_{12}	Inhibition coefficient
h_1	Replenishment rate of the specie S_1
h_2	Replenishment rate of the specie S_1

Appendix A

Basic Concept of the Homotopy Perturbation Method (HPM) [9-16]

To explain this method, let us consider the following function:

$$D_o(u) - f(r) = 0, \quad r \in \Omega \tag{A.1}$$

with the boundary conditions of

$$B_o\left(u,\ \frac{\partial u}{\partial n}\right) = 0, \qquad r \in \Gamma \tag{A.2}$$

where D_o is a general differential operator, B_o is a boundary operator, $f(r)$ is a known analytical function and Γ is the boundary of the domain Ω. In general, the operator D_o can be divided into a linear part L and a non-linear part N. The eqn. (A.1) can therefore be written as

$$L(u) + N(u) - f(r) = 0 \qquad (A.3)$$

By the Homotopy technique, we construct a Homotopy $v(r, p) : \Omega \times [0,1] \to \Re$ that satisfies

$$H(v, p) = (1 - p)[L(v) - L(u_0)] + p[D_o(v) - f(r)] = 0. \qquad (A.4)$$

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0. \qquad (A.5)$$

where $p \in [0, 1]$ is an embedding parameter, and u_0 is an initial approximation of eqn. (A.1) that satisfies the boundary conditions. From eqns. (A.4) and (A.5), we have

$$H(v,0) = L(v) - L(u_0) = 0 \qquad (A.6)$$

$$H(v,1) = D_o(v) - f(r) = 0 \qquad (A.7)$$

When $p=0$, the eqns. (A.4) and (A.5) become linear equations. When $p =1$, they become non-linear equations. The process of changing p from zero to unity is that of $L(v) - L(u_0) = 0$ to $D_o(v) - f(r) = 0$. We first use the embedding parameter p as a "small parameter" and assume that the solutions of eqns. (A.4) and (A.5) can be written as a power series in p :

$$v = v_0 + pv_1 + p^2 v_2 + ... \qquad (A.8)$$

Setting $p = 1$ results in the approximate solution of the eqn. (A.1):

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + ... \qquad (A.9)$$

This is the basic idea of the HPM.

Appendix B

Solution of the Non-Linear Differential eqns. (1)-(3) Using the Homotopy Perturbation Method[9-19]

In this Appendix, we indicate how the eqns. (4) and (5) are derived in this paper. To find the solution of eqns. (1) – (3), we construct the Homotopy as follows:

$$(1 - p)\left[\frac{dN_1}{dt} - a_1 N_1 - h_1 \right] + p\left[\frac{dN_1}{dt} - a_1 N_1 - h_1 + a_{11} N_1^2 - a_{12} N_1 N_2 \right] = 0 \qquad (B.1)$$

$$(1 - p)\left[\frac{dN_2}{dt} - a_2 N_2 - h_2 \right] + p\left[\frac{dN_2}{dt} - a_2 N_2 - h_2 + a_{22} N_2^2 \right] = 0 \qquad (B.2)$$

The analytical solutions of (B.1) and (B.2) is

$$N_1 = N_{1_0} + pN_{1_1} + p^2 N_{1_2} + ... \qquad (B.3)$$

$$N_2 = N_{2_0} + pN_{2_1} + p^2 N_{2_2} + ... \qquad (B.4)$$

Substituting the eqns. (B.3) and (B.4) in (B.1) and (B.2) respectively we get

$$(1 - p)\left[\frac{d(N_{1_0} + pN_{11} + p^2 N_{1_2} + ...)}{dt} - a_1 (N_{1_0} + pN_{11} + p^2 N_{1_2} + ...) - h_1 \right] +$$

$$p\left[\begin{array}{l} \dfrac{d(N_{1_0} + pN_{11} + p^2 N_{1_2} + ...)}{dt} - a_1 (N_{1_0} + pN_{11} + p^2 N_{1_2} + ...) - h_1 \\ + a_{11} (N_{1_0} + pN_{11} + p^2 N_{1_2} + ...)^2 \\ - a_{12} (N_{1_0} + pN_{11} + p^2 N_{1_2} + ...) (N_{2_0} + pN_{21} + p^2 N_{2_2} + ...) \end{array} \right] = 0 \qquad (B.5)$$

$$(1 - p)\left[\frac{d(N_{2_0} + pN_{21} + p^2 N_{2_2} + ...)}{dt} - a_2 (N_{2_0} + pN_{21} + p^2 N_{1_2} + ...) - h_2 \right] +$$

$$p\left[\begin{array}{l} \dfrac{d(N_{2_0} + pN_{21} + p^2 N_{2_2} + ...)}{dt} - a_2 (N_{2_0} + pN_{21} + p^2 N_{2_2} + ...) - h_2 \\ + a_{22} (N_{2_0} + pN_{21} + p^2 N_{2_2} + ...)^2 \end{array} \right] = 0 \qquad (B.6)$$

Comparing the coefficients of like powers of p in (B.5) and (B.6) we get

$$p^0 : \frac{dN_{1_0}}{dt} - a_1 N_{1_0} - h_1 = 0 \qquad (B.7)$$

$$p^0 : \frac{dN_{2_0}}{dt} - a_2 N_{2_0} - h_2 = 0 \qquad (B.8)$$

$$p^1 : \frac{dN_{1_1}}{dt} - a_1 N_{1_1} + a_{11} N_1^2 - a_{12} N_1 N_2 = 0 \qquad (B.9)$$

$$p^1 : \frac{dN_{2_1}}{dt} - a_2 N_{2_1} + a_{22} N_2^2 = 0 \qquad (B.10)$$

The initial approximations is as follows

$$N_1(0) = N_{10}, \; N_2(0) = N_{20} \qquad (B.11)$$

$$N_{1i}(0) = N_{2i}(0) = 0, \; i = 1,2,3...... \qquad (B.12)$$

Solving the eqns. (B.7) - (B.10) and using the boundary conditions (B.11) and (B.12) we obtain the following results:

$$N_{1_0}(t) = \left(k_0 e^{a_1 t} - k_1 \right) \qquad (B.13)$$

$$N_{1_1}(t) = A e^{a_1 t} + \left[\frac{a_{11} k_1^2 - a_{12} k_1 u_1}{a_1} \right] + [2 a_{11} k_0 k_1 - a_{12} k_0 u_1] t e^{a_1 t} - \left[\frac{a_{12} k_1 u_0}{a_2 - a_1} \right]$$
$$- \left[\frac{a_{11} k_0^2 e^{a_2 t}}{a_1} \right] + \left[\frac{a_{12} k_0 u_0 e^{(a_1 + a_2) t}}{a_2} \right] \tag{B.14}$$

$$N_{2_0}(t) = \left(u_0 e^{a_2 t} - u_1 \right) \tag{B.15}$$

$$N_{2_1}(t) = \left(\frac{a_{22}}{a_2} \right) \left[\left(u_0^2 - u_1^2 \right) e^{a_2 t} - \left(u_1^2 - u_0^2 e^{2 a_2 t} \right) \right] + 2 a_{22} u_0 u_1 e^{a_2 t} \tag{B.16}$$

Where k_0, k_1, u_0, u_1 and A are defined in the text (6) and (7).

According to the HPM, we can conclude that
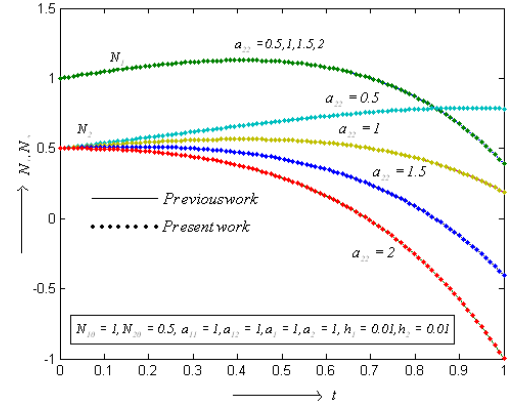
$$N_1 = \lim_{p \to 1} N_1(t) = N_{1_0} + N_{1_1} \tag{B.17}$$

$$N_2 = \lim_{p \to 1} N_2(t) = N_{2_0} + N_{2_1} \tag{B.18}$$

After putting the eqns. (B.13) and (B.14) into an eqn. (B.17) and (B.15) and (B.16) into an eqn. (B.18), we obtain the solutions in the text (4) - (7).
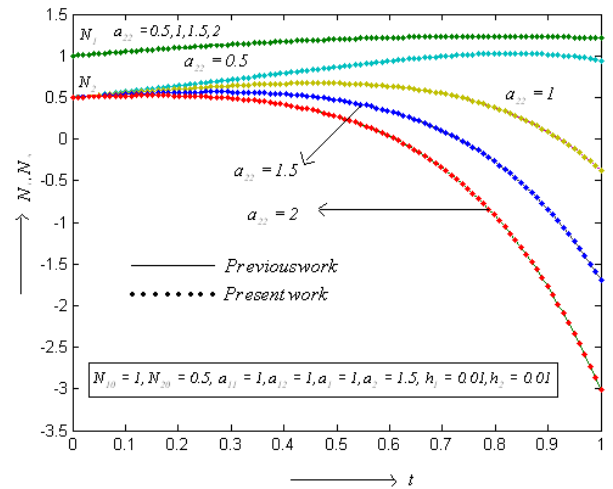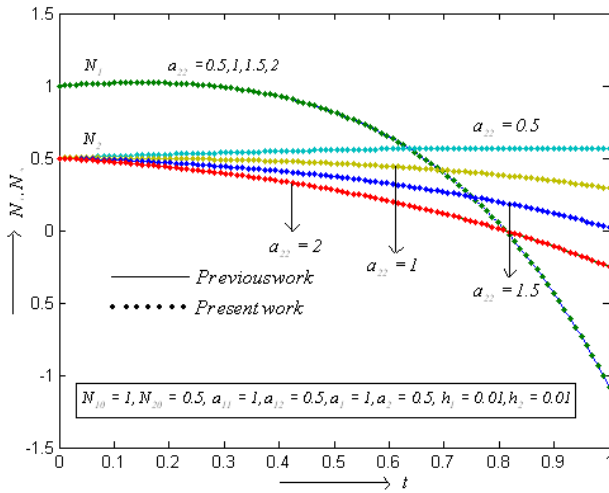
Appendix C
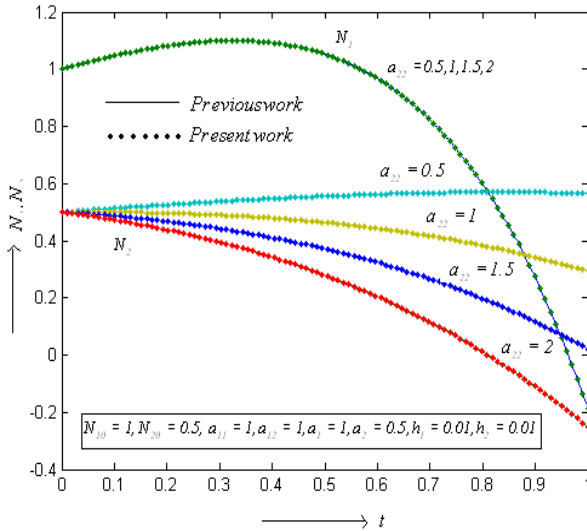
Fig 1. The populations $N_1(t)$ and $N_2(t)$ versus the time t . The populations were computed using the eqns. (4) and (5) for various values of the dimensionless parameters.
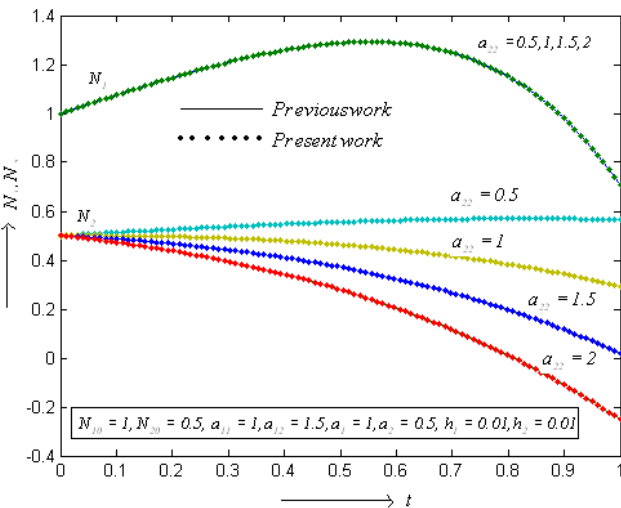
Fig 2. The populations $N_1(t)$ and $N_2(t)$ versus the time t . The populations were computed using the eqns. (4) and (5) for various values of the dimensionless parameters.
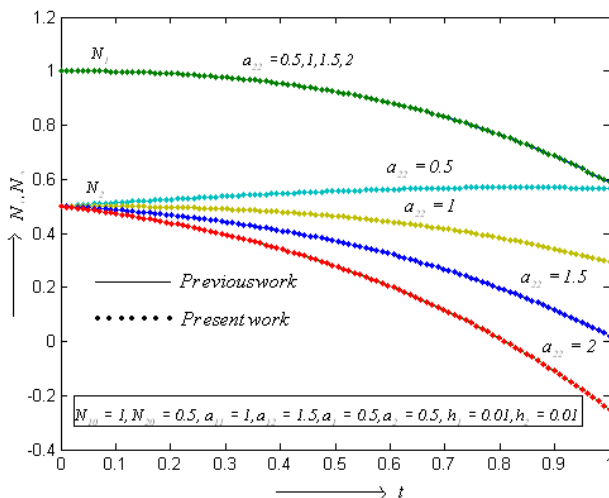
Fig 3. The populations $N_1(t)$ and $N_2(t)$ versus the time t . The populations were computed using the eqns. (4) and (5) for various values of the dimensionless parameters.

Fig 4. *The populations $N_1(t)$ and $N_2(t)$ versus the time t. The populations were computed using the eqns. (4) and (5) for various values of the dimensionless parameters.*

Fig 7. *The populations $N_1(t)$ and $N_2(t)$ versus the time t. The populations were computed using the eqns. (4) and (5) for various values of the dimensionless parameters.*
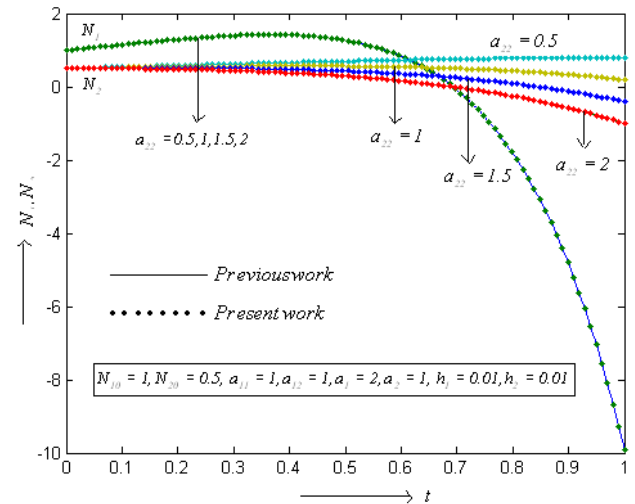

Fig 5. *The populations $N_1(t)$ and $N_2(t)$ versus the time t. The populations were computed using the eqns. (4) and (5) for various values of the dimensionless parameters.*
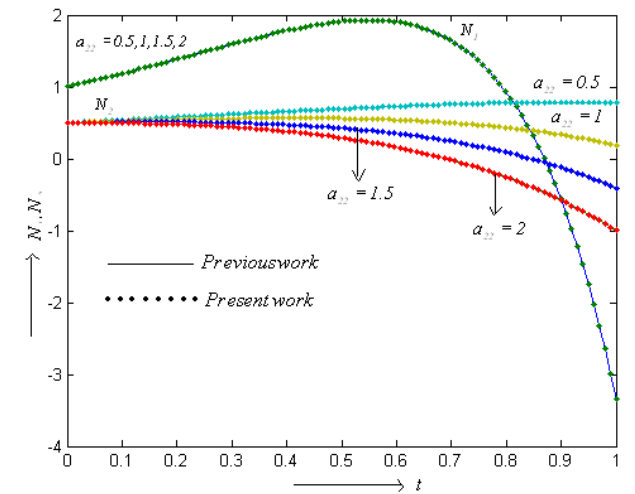

Fig 8. *The populations $N_1(t)$ and $N_2(t)$ versus the time t. The populations were computed using the eqns. (4) and (5) for various values of the dimensionless parameters.*

Fig 6. *The populations $N_1(t)$ and $N_2(t)$ versus the time t. The populations were computed using the eqns. (4) and (5) for various values of the dimensionless parameters.*

Table 1. Numerical values of the dimensionless parameters for previous and present work.

1	$N_{10}=1$	$N_{20}=0.5$	$a_{11}=1$	$a_{12}=1$	$a_1=1$	$a_2=0.5$	$h_1=0.01$	$h_2=0.01$	$a_{22}:0.5,1,1.5,2$
2	$N_{10}=1$	$N_{20}=0.5$	$a_{11}=1$	$a_{12}=1$	$a_1=1$	$a_2=0.5$	$h_1=0.01$	$h_2=0.01$	$a_{22}:0.5,1,1.5,2$
3	$N_{10}=1$	$N_{20}=0.5$	$a_{11}=1$	$a_{12}=1$	$a_1=1$	$a_2=0.5$	$h_1=0.01$	$h_2=0.01$	$a_{22}:0.5,1,1.5,2$
4	$N_{10}=1$	$N_{20}=0.5$	$a_{11}=1$	$a_{12}=1$	$a_1=1$	$a_2=0.5$	$h_1=0.01$	$h_2=0.01$	$a_{22}:0.5,1,1.5,2$
5	$N_{10}=1$	$N_{20}=0.5$	$a_{11}=1$	$a_{12}=1.5$	$a_1=1$	$a_2=0.5$	$h_1=0.01$	$h_2=0.01$	$a_{22}:0.5,1,1.5,2$
6	$N_{10}=1$	$N_{20}=0.5$	$a_{11}=1$	$a_{12}=1$	$a_1=1$	$a_2=0.5$	$h_1=0.01$	$h_2=0.01$	$a_{22}:0.5,1,1.5,2$
7	$N_{10}=1$	$N_{20}=0.5$	$a_{11}=1$	$a_{12}=1$	$a_1=1$	$a_2=0.5$	$h_1=0.01$	$h_2=0.01$	$a_{22}:0.5,1,1.5,2$
8	$N_{10}=1$	$N_{20}=0.5$	$a_{11}=1$	$a_{12}=1$	$a_1=1$	$a_2=0.5$	$h_1=0.01$	$h_2=0.01$	$a_{22}:0.5,1,1.5,2$

Acknowledgement

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