



Existence and Uniqueness of Weak Solution for Weighted p-bilaplacian (p-biharmonic)

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Abstract: This paper deals with the equation $-\Delta_p^2 u + \lambda \ell(x) |u|^{p-2} u = f(x, y)$ in bounded domain $\Omega \in \mathbb{R}^N$. Relying on Browder theorem, under conditions of the monotonous function f . We obtained the existence and uniqueness of weak solutions for the weighted p-bilaplacian boundary value of the form: $(P) \begin{cases} -\Delta_p^2 u + \lambda \ell(x) |u|^{p-2} u = f(x, u) \text{ in } \Omega \\ u \in W_0^{2,p}(\Omega) \end{cases}$

Keywords: Weak Solutions, p-biharmonic Operator, Browder Theorem

1. Introduction

In this paper, we are concerned with the existence and uniqueness of weak solution for a weighted p-bilaplacian boundary value of the form:

$$(P) \begin{cases} -\Delta_p^2 u + \lambda \ell(x) |u|^{p-2} u = f(x, u) \text{ in } \Omega \\ u \in W_0^{2,p}(\Omega) \end{cases}$$

Let Ω be a bounded domain in \mathbb{R}^N and $f: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a Caratheodry (CAR) Satisfies the following:

(H1) $f(x, s_1) \leq f(x, s_2)$ for a.e. $x \in \Omega$ and $s_1, s_2 \in \mathbb{R}$, $s_1 \geq s_2$.

(H2) $|f(x, s)| \leq f_0(x) + c|s|^{p-1}$ that there exists $f_0 \in L^{p'}(\Omega)$, $c > 0$.

Here $\Delta_p^2 u = \Delta(|\Delta u|^{p-2} \Delta u)$, and $N > 1, 1 < p < \infty, \lambda \in \mathbb{R}$ $\ell \in L'(\Omega)$, $\ell \neq 0$, and $r = r(N, p)$

Satisfying the conditions:

$$(H_3) \begin{cases} r > \frac{N}{2-p} \text{ if } \frac{N}{p} > 2 \\ r > p \text{ if } \frac{N}{p} = 2 \\ r = 1 \text{ if } \frac{N}{p} < 2 \end{cases}$$

We assume that $|\Omega_\ell^+| \neq 0$, where $\Omega_\ell^+ = \{x \in \Omega / \ell(x) > 0\}$.

Our paper is organized as follow: Section 2 contains some basic definitions concerning the nonlinear operators that will be used throughout the paper. Also, we introduce the space setting of the problem and give some basic characteristics, as the equivalent norm and embedding results. In section 3 we state the main result on the existence and uniqueness of weak solutions of the problem (P).

The existence of solutions for the nonlinear eigenvalue problem with p-biharmonic see [3].

2. Preliminaries and Space Setting

First, we introduce some basic definitions concerning the nonlinear operators which we use extensively in this paper 3.

Definition 2.1 [4] Let $A: V \rightarrow V'$ be an operator on a real Banach space V . We say that the operator A is :

(i) bounded iff it maps bounded sets into bounded i.e. for each $r > 0$ there exists $M > 0$ (M depending on r) such that

$$\|u\| \leq r \Rightarrow \|A(u)\| \leq M, \forall u \in V$$

(ii) Coercive : iff

$$\lim_{\|u\| \rightarrow \infty} \frac{\langle A(u), u \rangle}{\|u\|} = \infty;$$

(iii) Monotone iff $\langle A(u_1) - A(u_2), u_1 - u_2 \rangle \geq 0, \forall u_1, u_2 \in V$.

(iv) Strictly monotone iff

$$\langle A(u_1) - A(u_2), u_1 - u_2 \rangle > 0, \text{ for all } u_1, u_2 \in V, u_1 \neq u_2.$$

(v) Strongly monotone iff there exists $k > 0$,

$$\langle A(u_1) - A(u_2), u_1 - u_2 \rangle \geq k \|u_1 - u_2\|, \text{ for all } u_1, u_2 \in V,$$

$u_1 \neq u_2$;

(vi) Continuous iff $u_n \rightarrow u$ implies $A(u_n) \rightarrow A(u)$, for all $u_n, u \in V$.

(vii) Strongly continuous iff $u_n \xrightarrow{w} u$ implies $A(u_n) \rightarrow A(u)$, for all $u_n, u \in V$.

(viii) Demi continuous iff $u_n \rightarrow u$ implies $A(u_n) \xrightarrow{w} A(u)$, for all $u_n, u \in V$.

Theorem 2.1(Browder) [4]

Let A be a reflexive real Banach space. Moreover let $A : V \rightarrow V'$ be an operator which is: bounded, demicontinuous, coercive, and monotone on the space V . Then, the equation $A(u) = f$ has at least one solution $u \in V$ for each $f \in V'$. If moreover, A is strictly monotone operator, then the equation (P) has precisely one solution $u \in V$ for every $f \in V'$.

Proposition 2.1 [6] For any bounded domain Ω and $1 < p < \infty, \Delta_p^2$ satisfies the following:

(i) Δ_p^2 is an hemicontinuous operator from $W_0^{2,p}(\Omega)$ into $W^{-2,p'}(\Omega)$.

(ii) Δ_p^2 is a bounded monotonous, and coercive operator.

(iii) $\Delta_p^2 : W_0^{2,p}(\Omega) \rightarrow W^{-2,p'}(\Omega)$ is a bicontinuous operator.

Proof

See [6].

The norm $\|\Delta\|$ is uniformly equivalent on $W_0^{2,p}$ to the usual norm of $W_0^{2,p}$.

3. Existence and Uniqueness Results

In this section, using Browder theorem, we prove the existence and uniqueness of weak solution for equation (P).

Definition 3.1 We say that $u \in W_0^{2,p}(\Omega)$ is a weak solution to equation (P) if

$$\int_{\Omega} |\Delta u|^{p-2} \Delta u \Delta \phi + \lambda \int_{\Omega} \ell(x) |u|^{p-2} u \phi = \int_{\Omega} f(x, u) \phi, \text{ for all } \phi \in C_0^\infty(\Omega)$$

Our main results concerning problem (P) is the following theorem:

Theorem 3.1 Let $p \geq 2, \lambda > 0$ and $f(x, u) \in C^0(\Omega \times \mathbb{R})$ satisfy (H_1) , (H_2) and (H_3) . Then problem (P) has a

unique weak solution.

Proof:

We define for $\lambda > 0$ the operator $A : W_0^{2,p}(\Omega) \rightarrow W^{-2,p'}(\Omega)$, as $A = J + \lambda G - F$, where the

Operators $J : W_0^{2,p}(\Omega) \rightarrow W^{-2,p'}(\Omega)$, $G : W_0^{2,p}(\Omega) \rightarrow W^{-2,p'}(\Omega)$ and $F : \Omega \times W_0^{2,p}(\Omega) \rightarrow W^{-2,p'}(\Omega)$ are given by

$$\langle J(u), \phi \rangle = \int_{\Omega} |\Delta u|^{p-2} \Delta u \Delta \phi$$

$$\langle G(u), \phi \rangle = \int_{\Omega} \ell(x) |u|^{p-2} u \phi$$

And

$$\langle F(u), \phi \rangle = \int_{\Omega} f(x, u) \phi$$

for all $u, \phi \in W_0^{2,p}(\Omega)$. Thus, to find a weak solution of

(P) is equivalent to finding $u \in W_0^{2,p}(\Omega)$ which satisfies the operator equation $A(u) = 0$. Now, we have the following properties of the operators J, G , and F :

a) J, G and F are well defined. Using Holder's inequality, we have

$$\langle J(u), \phi \rangle = \int_{\Omega} |\Delta u|^{p-2} \Delta u \Delta \phi$$

$$|\langle J(u), \phi \rangle| \leq \int_{\Omega} |\Delta u|^{p-1} |\Delta \phi| dx \leq \left(\int_{\Omega} |\Delta u|^p \right)^{\frac{1}{p}} \left(\int_{\Omega} |\Delta \phi|^p \right)^{\frac{1}{p}} < \infty.$$

$$\langle G(u), \phi \rangle = \int_{\Omega} \ell(x) |u|^{p-2} u \phi$$

First case: $\frac{N}{p} > 2$ and $r > \frac{N}{2p}$. Let $u, \phi \in W_0^{2,p}(\Omega)$ By

Hölder's inequality, we have

$$|\langle G(u), \phi \rangle| \leq \|\ell\|_r \|u\|_s^{p-1} \|\phi\|_{p_2},$$

Where $\frac{1}{p_2} = \frac{1}{p} - \frac{2}{N}$, and s is given by:

$$\frac{p-1}{s} + \frac{1}{p_2} + \frac{1}{r} = 1.$$

Therefore

$$\frac{p-1}{s} = 1 - \frac{1}{r} - \frac{1}{p_2} > 1 - \frac{2p}{N} - \frac{1}{p_2} = \frac{p-1}{p_2}.$$

The nit suffices to take $\max(1, p-1) < s < p_2$ so that G is well defined.

Second case: $\frac{N}{p} = 2$, and $r > p$. In this case $W_0^{2,p}(\Omega)$ embedded into the space $L^q(\Omega)$, for any $q \in [p, +\infty[$, there

is $q \geq p$ such that:

$$\frac{1}{q} + \frac{1}{r} + \frac{p-1}{p} = \frac{1}{q} + \frac{1}{r} + \frac{1}{p'} = 1.$$

We obtain that $\frac{1}{q} = \frac{1}{p} - \frac{1}{r} \leq \frac{1}{p}$. By Hölder's inequality, we arrive at:

$$|\langle G(u), \varphi \rangle| \leq \|\ell\|_r \|u\|_p^{p-1} \|\varphi\|_q$$

For any $u, \varphi \in W_0^{2,p}(\Omega)$, and G is well defined.

Third case: $\frac{N}{p} < 2$, and $r=1$. In this case $W_0^{2,p}(\Omega)$

embedded into the space $C(\bar{\Omega}) \cap L^\infty(\Omega)$ Then for any $u, \varphi \in W_0^{2,p}(\Omega)$ we have:

$$|\langle G(u), \varphi \rangle| \leq \|\ell\|_1 \|u\|_\infty^{p-1} \|\varphi\|_\infty < \infty$$

And G is well defined.

And

$$\langle F(u), \varphi \rangle = \int_\Omega f(x, u) \varphi \, dx$$

$$|\langle F(u), \varphi \rangle| \leq \int_\Omega |f(x, u) \varphi| \leq \int_\Omega (f_0(x) + c|u|^{p-1}) |\varphi|.$$

By Hölder's inequality,

$$|\langle F(u), \varphi \rangle| \leq \left(\int_\Omega |f_0(x)|^{p'} \right)^{\frac{1}{p'}} \left(\int_\Omega |\varphi|^p \right)^{\frac{1}{p}} + c \left(\int_\Omega |u|^p \right)^{\frac{1}{p}} \left(\int_\Omega |\varphi|^p \right)^{\frac{1}{p}} < \infty,$$

And hence J, G, F are well defined.

b) G is completely continuous. Let $(u_n) \subset W_0^{2,p}(\Omega)$ be as sequence such that

$u_n \rightarrow u$ weakly in $W_0^{2,p}(\Omega)$. We have to show that $G(u_n) \rightarrow G(u)$ strongly in $W^{-2,p'}(\Omega)$, i.e.

$$\sup_{\substack{\varphi \in W_0^{2,p}(\Omega) \\ \|\Delta \varphi\|_p \leq 1}} \left| \int_\Omega \ell(|u_n|^{p-2} u_n - |u|^{p-2} u) \varphi \, dx \right| \rightarrow 0, \text{ as } n \rightarrow +\infty$$

For $\frac{N}{p} > 2, r > \frac{N}{2p}$. Let s be as $\max(1, p-1) < s < p_2$ and

$$\frac{1}{p_2} = \frac{1}{p} - \frac{2}{N} \text{ and } \frac{p-1}{s} + \frac{1}{p_2} + \frac{1}{r} = 1.$$

$$\begin{aligned} & \sup_{\substack{\varphi \in W_0^{2,p}(\Omega) \\ \|\Delta \varphi\|_p \leq 1}} \left| \int_\Omega \ell(|u_n|^{p-2} u_n - |u|^{p-2} u) \varphi \, dx \right| \\ & \leq \sup_{\substack{\varphi \in W_0^{2,p}(\Omega) \\ \|\Delta \varphi\|_p \leq 1}} \|\ell\|_r \|u_n|^{p-2} u_n - |u|^{p-2} u\|_{\frac{s}{p-1}} \|\varphi\|_{p_2} \\ & \leq C \|\ell\|_r \|u_n|^{p-2} u_n - |u|^{p-2} u\|_{\frac{s}{p-1}} \end{aligned}$$

where C is the constant of Sobolev. On the other hand, the Nemyskii's operator

$$u \mapsto |u|^{p-2} u \text{ is continuous from } L^s(\Omega) \text{ into } L^{\frac{s}{p-1}}(\Omega),$$

and $u_n \rightarrow u$ weakly in $W_0^{2,p}(\Omega)$. So, we deduce that $u_n \rightarrow u$ strongly in $L^s(\Omega)$ because $s < p_2$. Hence $\|u_n|^{p-2} u_n - |u|^{p-2} u\|_{\frac{s}{p-1}} \rightarrow 0$ as $n \rightarrow +\infty$.

This completes the proof of the claim.

If $\frac{N}{p} = 2, r > p$ then :

$$\begin{aligned} & \left| \int_\Omega \ell(|u_n|^{p-2} u_n - |u|^{p-2} u) \varphi \, dx \right| \\ & \leq \|\ell\|_r \|u_n|^{p-2} u_n - |u|^{p-2} u\|_p^{p-1} \|\varphi\|_q \end{aligned}$$

Where q is given $\frac{1}{q} = \frac{1}{p} - \frac{1}{r}$. By Sobolev's embedding

there exist $c > 0$ such that $\|\varphi\|_q \leq c \|\Delta \varphi\|_p, \forall \varphi \in W_0^{2,p}(\Omega)$. Thus

$$\begin{aligned} & \sup_{\substack{\varphi \in W_0^{2,p}(\Omega) \\ \|\Delta \varphi\|_p \leq 1}} \left| \int_\Omega \ell(|u_n|^{p-2} u_n - |u|^{p-2} u) \varphi \, dx \right| \\ & \leq C \|\ell\|_r \|u_n|^{p-2} u_n - |u|^{p-2} u\|_p^{p-1} \end{aligned}$$

From the continuity of $u \mapsto |u|^{p-2} u$ from $L^p(\Omega)$ into $L^{p'}(\Omega)$ and from the compact embedding of $W_0^{2,p}(\Omega)$ in $L^p(\Omega)$ we have the desired result.

If $\frac{N}{p} < 2, r=1, W_0^{2,p}(\Omega) \subset C(\bar{\Omega})$, then we obtain :

$$\sup_{\substack{\varphi \in W_0^{2,p}(\Omega) \\ \|\Delta \varphi\|_p \leq 1}} \left| \int_\Omega \ell(|u_n|^{p-2} u_n - |u|^{p-2} u) \varphi \, dx \right|$$

$$\leq C \|\ell\| \sup_{\Omega} \| |u_n|^{p-2} u_n - |u|^{p-2} u \|$$

Where C is the constant given by embedding $W_0^{2,p}(\Omega)$ of in $C(\Omega) \cap L^\infty(\Omega)$. It is clear that,

$$\sup_{\Omega} \| |u_n|^{p-2} u_n - |u|^{p-2} u \| \rightarrow 0, \text{ as } n \rightarrow +\infty.$$

Hence G is completely continuous, also in this case. J and F are bounded operators. Indeed, for every u such that $\|u\|_{W_0^{2,p}(\Omega)} \leq M$.

We have

$$\|J(u)\|_{W^{-2,p'}(\Omega)} = \sup_{\|\phi\|_{W_0^{2,p}(\Omega)} \leq 1} |\langle J(u), \phi \rangle| \leq \sup_{\|\phi\|_{W_0^{2,p}(\Omega)} \leq 1} \int_{\Omega} |\Delta u|^{p-1} |\Delta \phi| dx.$$

Using Hölder's inequality, we obtain

$$\|J(u)\|_{W^{-2,p'}(\Omega)} = \sup_{\|\phi\|_{W_0^{2,p}(\Omega)} \leq 1} \left(\int_{\Omega} |\Delta u|^p \right)^{\frac{1}{p'}} \left(\int_{\Omega} |\Delta \phi|^p \right)^{\frac{1}{p}} \leq M^{\frac{p}{p'}}.$$

Also, we get

$$\begin{aligned} \|F(u)\|_{W^{-2,p'}(\Omega)} &= \sup_{\|\phi\|_{W_0^{2,p}(\Omega)} \leq 1} |\langle F(u), \phi \rangle| \\ &\leq \sup_{\|\phi\|_{W_0^{2,p}(\Omega)} \leq 1} \left[\left(\int_{\Omega} |f_0(x)|^{p'} \right)^{\frac{1}{p'}} + \left(\int_{\Omega} |u|^{(p-1)p'} \right)^{\frac{1}{p'}} \right] \left(\int_{\Omega} |\phi|^p \right)^{\frac{1}{p}} \\ &\leq k \left(\|f_0\|_{L^{p'}(\Omega)} + k \|u\|_{W_0^{2,p}(\Omega)}^{\frac{p}{p'}} \right) \leq k \left(\|f_0\|_{L^{p'}(\Omega)} + k M^{\frac{p}{p'}} \right) \end{aligned}$$

Where k is the constant of the embedding of $W_0^{2,p}(\Omega)$ into $L^p(\Omega)$.

c) J and F are continuous operators. If $u_n \rightarrow u$, in $W_0^{2,p}(\Omega)$. Then, we have

$$\|u_n - u\|_{W_0^{2,p}(\Omega)} \rightarrow 0, \|\Delta u_n - \Delta u\|_{L^p(\Omega)} \rightarrow 0$$

Applying Dominated Convergence Theorem, we obtain

$$\| |\Delta u_n|^{p-2} \Delta u_n - |\Delta u|^{p-2} \Delta u \|_{L^p(\Omega)} \rightarrow 0.$$

Hence

$$\begin{aligned} \|J(u_n) - J(u)\|_{W^{-2,p'}(\Omega)} &= \sup_{\|\phi\|_{W_0^{2,p}(\Omega)} \leq 1} |\langle J(u_n) - J(u), \phi \rangle| \\ &\leq \sup_{\|\phi\|_{W_0^{2,p}(\Omega)} \leq 1} \left(\int_{\Omega} (|\Delta u_n|^{p-2} \Delta u_n - |\Delta u|^{p-2} \Delta u) \phi \right)^{\frac{1}{p'}} \left(\int_{\Omega} |\phi|^p \right)^{\frac{1}{p}} \end{aligned}$$

$$\leq k \left(\int_{\Omega} (|\Delta u_n|^{p-2} \Delta u_n - |\Delta u|^{p-2} \Delta u) \phi \right)^{\frac{1}{p'}} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

0 Similarly, we have

$$\begin{aligned} \|F(u_n) - F(u)\|_{W^{-2,p'}(\Omega)} &= \sup_{\|\phi\|_{W_0^{2,p}(\Omega)} \leq 1} |\langle F(u_n) - F(u), \phi \rangle| \\ &\leq k \left(\int_{\Omega} (|f(x, u_n) - f(x, u)|)^{p'} \right)^{\frac{1}{p'}} \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

d) Let $p \geq 2, \forall x_1, x_2 \in \mathfrak{R}^N$, we have the following inequality (see [8])

$$|x_2|^p \geq |x_1|^p + p |x_1|^{p-2} x_1 (x_2 - x_1) + \frac{|x_2 - x_1|^p}{2^{p-1} - 1} \quad (1)$$

Now,

$$\begin{aligned} \langle J(u) - J(\phi), u - \phi \rangle &= \int_{\Omega} [|\Delta u|^{p-2} \Delta u - |\Delta \phi|^{p-2} \Delta \phi] (\Delta u - \Delta \phi) \\ &= \int_{\Omega} |\Delta u|^{p-2} \Delta u (\Delta u - \Delta \phi) - \int_{\Omega} |\Delta \phi|^{p-2} \Delta \phi (\Delta u - \Delta \phi) = I_1 + I_2 \end{aligned}$$

Using (1), we get

$$I_1 + I_2 \geq \frac{2}{p(2^{p-1} - 1)} \int_{\Omega} |\Delta u - \Delta \phi|^p = c(p) \|u - \phi\|_{W_0^{2,p}(\Omega)}^p,$$

for $p \geq 2$.

So

$$\langle J(u) - J(\phi), u - \phi \rangle \geq c(p) \|u - \phi\|_{W_0^{2,p}(\Omega)}^p, \text{ for } p \geq 2. \quad (2)$$

Similarly, we have

$$\begin{aligned} \langle G(u) - G(\phi), u - \phi \rangle &= \int_{\Omega} \ell(x) [|u|^{p-2} u - |\phi|^{p-2} \phi] (u - \phi) \\ &\geq \frac{2}{p(2^{p-1} - 1)} \int_{\Omega} \ell(x) |u - \phi|^p \geq c(p) \|u - \phi\|_{W_0^{2,p}(\Omega)}^p \geq 0. \end{aligned}$$

Hence,

$$\langle G(u) - G(\phi), u - \phi \rangle \geq 0. \quad (3)$$

Also, we get

$$\langle F(u) - F(\phi), u - \phi \rangle = \int_{\Omega} [f(x, u) - f(x, \phi)] (u - \phi).$$

Since f is decreasing with respect to the second variable, we have

$$[f(x, u) - f(x, \phi)] (u - \phi) \leq 0.$$

Consequently

$$\langle F(u) - F(\varphi), u - \varphi \rangle = \int_{\Omega} [f(x, u) - f(x, \varphi)](u - \varphi) \leq 0. \quad (4)$$

Equations (2), (3), and (4) imply that

$$\langle A(u) - A(\varphi), u - \varphi \rangle \geq c(p) \|u - \varphi\|_{W_0^{2,p}(\Omega)}^p, \text{ for } p \geq 2. \quad (5)$$

So A is strongly monotone.

Now, to apply Browder theorem, it remains to prove that A is a coercive operator. From (5), we have

$$\langle A(u), u \rangle \geq \langle A(0), u \rangle + c(p) \|u\|_{W_0^{2,p}(\Omega)}^p.$$

On the other hand

$$\begin{aligned} \langle A(0), u \rangle &= \langle J(0), u \rangle + \lambda \langle G(0), u \rangle - \langle F(0), u \rangle = - \int_{\Omega} f(x, 0) u \\ &\geq - \left(\int_{\Omega} [f_0(x)]^{p'} \right)^{\frac{1}{p'}} \left(\int_{\Omega} |u|^p \right)^{\frac{1}{p}} \geq -k \|f_0\|_{L^{p'}(\Omega)} \|u\|_{W_0^{2,p}(\Omega)}, \end{aligned}$$

Then

$$\langle A(u), u \rangle \leq c(p) \|u\|_{W_0^{2,p}(\Omega)}^p - k \|f_0\|_{L^{p'}(\Omega)} \|u\|_{W_0^{2,p}(\Omega)}.$$

So

$$\lim_{\|u\|_{W_0^{2,p}(\Omega)} \rightarrow \infty} \frac{\langle A(u), u \rangle}{\|u\|_{W_0^{2,p}(\Omega)}^p} = \infty.$$

This proves the coercivity condition and so, the existence of weak solution for (P). The uniqueness of weak solution of (P) is a direct consequence of (5). Suppose that u, φ be a weak solutions of (P) such that $u \neq \varphi$. Now, from (5), we have

$$0 = \langle A(u) - A(\varphi), u - \varphi \rangle \geq c(p) \|u - \varphi\|_{W_0^{2,p}(\Omega)}^p \geq 0.$$

Therefore $u = \varphi$. This completes the proof.

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