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Separation of Angular Momentum

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Abstract: In this paper we speak about angular momentum, we have shown that the separation of the total angular momentum of the electromagnetic field into its orbital and spin parts. It is dictated by quantum mechanics of photons reproduces. Therefore, the results are derived from the proprieties of Fourier and Maxwell fields by Darwin, with the correspondence results that derived heuristically by many authors.

Keywords: Angular Momentum of Light, Quantum Mechanics of Photons, Riemann-Silberstien Vector

1. Introduction

Darwin's theory of evolution is the widely held notion that all life is related and has descended from a common ancestor. Darwin's general theory presumes the development of life from non-life and stresses a purely naturalistic. That is complex creatures evolve from more simplistic ancestors naturally over time. In this work, we will review the work done by Iwo Bialynicki-Birula1 and Zofia Bialynicka-Birula2 [1], to show that the Darwin separation of the total angular momentum for an arbitrary electromagnetic field into two parts follows from the photon picture of the electromagnetic field with some remarks.

2. Generator of the Poincar´e Group

The Poincar´e group is the group of Minkowski spacetime isometries. It is a ten dimensional noncompact Lie group. The abelian group of translations is a normal subgroup, while the Lorentz group is also a subgroup, the stabilizer of the origin. The Poincar´e group itself is the minimal subgroup of the affine group which includes all translations and Lorentz translations. More precisely, it is a semi direct product of the translations and Lorentz group.

In a relativistic theory, we must first of all define the operators representing ten generators of the Poincar´e group, that must obey the following commutation relations.

$$[\hat{H}, \hat{P}_i] = 0, [\hat{H}, \hat{J}_i] = 0, [\hat{H}, \hat{K}_i] = -ihc\hat{P}_i \tag{1}$$

$$[\hat{P}_i, \hat{P}_j] = 0, [\hat{J}_i, \hat{J}_j] = ih\epsilon_{ijk}\hat{J}_k, [\hat{K}_i, \hat{K}_j] == -ihc^2\epsilon_{ijk}\hat{J}_k \tag{2}$$

$$[\hat{J}_i, \hat{P}_j] = ih\epsilon_{ijk}\hat{P}_k, [\hat{J}_i, \hat{K}_j] = ih\epsilon_{ijk}\hat{K}_k, [\hat{K}_i, \hat{P}_j] = ih\delta_{ij}\hat{H} \tag{3}$$

where, $\hat{P}$ is the generators of translation in space (momentum), $\hat{H}$ is the translation in time (energy), $\hat{J}$ is the rotation (angular momentum), $\hat{K}$ are the Lorentz boosts (momentum of energy), and C is speed of light.

The representation of the generators of the Lorentz group for massless particles was given by Lomont and Mose [2]. We will review here a modified version of these generators for photons [3], [4], [6] that exhibits its geometrical meaning. The momentum operator, by definition, acts on the wavefunctions in momentum representation as a multiplication by hk. The complete list of generators also contains the operator of angular momentum and the boost operator [1]

$$\hat{H} = hw_k \tag{4}$$

$$\hat{P} = hk \tag{5}$$

$$\hat{J} = ihD \times k + h\hat{x}n_k \tag{6}$$

$$\hat{K} = ihw_k D \tag{7}$$

Where $n_k = \frac{k}{|k|}$, the photon helicity operator $\hat{x}$ has the eigenvalues ± 1

$$D = \nabla_k - i\hat{x} \propto (k), \nabla_k = \frac{\partial}{\partial k} \tag{8}$$

stands for the covariant derivative on the light cone. These operators act on the two-component photon wavefunctions

$$f(k) = \begin{pmatrix} f_{L(k)} \\ f_R(k) \end{pmatrix} \tag{9}$$

and satisfy the commutation relations (1), (2) and (3) appropriate for the Poincar´e group. The components of the photon wavefunction correspond to two eigenvalues of $\hat{x}$,

$$\hat{x}\begin{pmatrix} f_{L(k)} \\ f_R(k) \end{pmatrix} = \begin{pmatrix} f_{L(k)} \\ -f_R(k) \end{pmatrix} \tag{10}$$

here L and R are used to denote the eigenfunctions of the helicity operator since they correspond to left-handed and right-handed circular polarization.

The properties of the covariant derivative are obtained from the commutation relations for the angular momentum and they read [1],

$$\left[D_{i,}D_j\right] = i\hat{x}\epsilon_{ijl}\frac{n_l}{|k|^2} \tag{11}$$

These conditions determine the vector $\alpha(k)$ up to a gauge transformation

$$\alpha(k) \rightarrow \alpha(k) + \nabla_k\varphi(k) \tag{12}$$

Which is connected to the change of the phase of the wavefunction, in analogy to the theory of charged particles coupled to an electromagnetic field. In order to solve the problem of the total angular momentum separation into two parts for the classical electromagnetic field, we shall employ the correspondence between the fundamental physical quantities (energy, momentum, and angular momentum) in photon quantum mechanics and in Maxwell theory. In the quantum mechanics of photons these quantities are represented by the operators (4)– (7) . The generators (4), (5) ,(6) and (7) are Hermitian with respect to the following Lorentz-invariant scalar product [1],

$$\langle f|g\rangle = \int\frac{d^3k}{hw_k}f^{\dagger}(k)\cdot g(k) = \int\frac{d^3k}{hw_k}\left[f_L^*(k)g_L(k) + f_R^*(k)g_R(k)\right] \tag{13}$$

3. Maxwell's Theory

Maxwell's equations can be cast into covariant form. The Einstein expression of it, is that, the general laws of nature are to be expressed by equations which holds for all systems of coordinates that are covariant with respect to any substitution whatever generally covariant.

Maxwell's theory of electromagnetism is alongside with Einstein's theory of gravitation, on the most classical field theories. The revolutionary work of Maxwell, puplished in 1865 took the individual and seemingly unconnected phenomena of electricity and magnetism and brought them into a Coherent and unified theory [16], [18]. This unified theory of electricity and magnetism depicts the behavior of two fields. Maxwell discussed his idea in terms of model in which vacuum was like an elastic solid, he tried to explain the meaning of his new equations in term of mathematical model. There was much reluctance to accept his theory, first because of the model, and second because there was at first no experimental justification. But today, we understand better that what counts are the equations themselves and not the model used to get them. If we chose units in which $\mu_0 = \epsilon_0 = c = 1$, then the covariant form of Maxwell's equations take the form[7], [8]:

$$\nabla\cdot E = \rho \tag{14}$$

$$\nabla\times B - \frac{\partial E}{\partial t} = J \tag{15}$$

$$\nabla\cdot B = 0 \tag{16}$$

$$\nabla\times E + \frac{\partial B}{\partial t} = 0 \tag{17}$$

Where E is the electric, and B is magnetic field, ρ is the charge density, and J is the current density. Taking the divergence of equation (14) and substituting equation (15) into resulting equation. Now, we can obtain the continuity equation [22], [23]

$$\nabla\times J + \frac{\partial\rho}{\partial t} = 0 \tag{18}$$

Now we have used the fact that for any vector H and scalar ψ the following identities hold:

$$\nabla\cdot(\nabla\times \mathrm{H}) = 0 \tag{19}$$

$$\nabla\times(\nabla\psi) = 0 \tag{20}$$

Also, since equation (16) always holds, this means that B must be curl of a vector function, namely the vector potential A,

$$\mathrm{B} = \nabla\times \mathrm{A} \tag{21}$$

Substituting equation (21) into equation (17) we obtain

$$\nabla\times\left(E + \frac{\partial A}{\partial t}\right) = 0 \tag{22}$$

Which means that the quantity with vanishing curl in equation (22) can be written as the gradient of scalar function, namely the scalar potential ϕ.

$$E = -\nabla\emptyset - \frac{\partial \mathrm{A}}{\partial \mathrm{t}} \tag{23}$$

The minus sign attached to the gradient is for technical convenience. These quantities, in Maxwell theory are given as space integrals of corresponding densities built from quadratic expressions in field vectors. The convenient tool in this construction is a complex vector F, that was called the Riemann–Silberstein (RS) vector in [1], [5], [9]. and given by

$$F = \sqrt{\frac{\epsilon_0}{2}}(E + icB) \tag{24}$$

The Maxwell equations expressed in terms of F are:

$$\partial_t F(r,t) = -ic\nabla\times F(r,t), \nabla\cdot F(r,t) = 0, \tag{25}$$

Now all of the field energy H, the field momentum P, the field angular momentum J, and the field moment of energy K can be constructed from the energy–momentum tensor of the electromagnetic field, and they expressed in terms of the RS vector as follows [1],

$$H = \frac{1}{2}\int d^3r\left[\epsilon_0 E^2 + \frac{B^2}{\mu_0}\right] = \int d^3rF^* \cdot F \tag{26}$$

$$P = \int d^3 r[\epsilon_0 E \times B] = \frac{1}{2i}\int d^3rF^* \times F \tag{27}$$

$$J = \int d^3 rr \times [\epsilon_0 E(r) \times B(r)] = \frac{1}{2i}\int d^3rr \times (F^* \times F) \tag{28}$$

$$K = \frac{1}{2}\int d^3rr\left[\epsilon_0 E^2 + \frac{B^2}{\mu_0}\right] = \int d^3 rr(F^* \cdot F) \tag{29}$$

The above quantities, like their counterparts in photon quantum mechanics (4), (5), (6), and (7) serve as the generators of Poincar´e transformations of the electromagnetic field. They have analogous algebraic properties of the Poincar´e group (1), (2) and (3), with quantum commutators replaced by Poisson brackets, $\frac{[a,b]}{i\hbar} \to [a, b]$.

The solutions of Maxwell equations in vacuum can be decomposed into plane waves with positive and negative frequencies. This decomposition gives the following Fourier representation of $F(r, t)$:

$$\int F(r,t) = \sqrt{N}\int \frac{d^3k}{(2\pi)^{3/2}} e(k)[f_L(k)e^{-iw_kt+ik\cdot r} + f_R^*(k)e^{iw_kt-ik\cdot r}] \tag{30}$$

Where the complex polarization vector

$$e(k) = \frac{1}{\sqrt{2}}[L_1(k) + iL_2(k)] \tag{31}$$

has the following properties:

$$ck \times e(k) = -iw_k\, e(k) \tag{32}$$

$$e(k)\cdot e(k) = 0 \tag{33}$$

$$e^*(k)\cdot e(k) = 1 \tag{34}$$

$$e^*(k)\times e(k) = in_k \tag{35}$$

$$e^*(k)\cdot e(-k) = 0 \tag{36}$$

$$e(k)\times e(k) = 0 \tag{37}$$

$$e_i^*(k)e_j(k) = \frac{1}{2}\left(\delta_{ij} + i\epsilon_{ijl}\frac{k_1}{|k|}\right) \tag{38}$$

4. Fourier Theory

The fact that momentum can be expressed as $p = k\hbar$ allows us to define a "momentum space" wavefunction that is related to the position space wavefunction via the Fourier transform. A function f(x) and its Fourier transform. F(k) are related via the relations:

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)e^{ikx}dk \tag{39}$$

$$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}dk \tag{40}$$

These relations can be expressed in terms of p with a position space wavefunction $\psi(x)$ and momentum space wavefunction $\Phi(p)$ as:

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} \emptyset(p)e^{ipx/\hbar}dp \tag{41}$$

$$\emptyset(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} \psi(x)e^{-ipx/\hbar}dx \tag{42}$$

Parseval's theorem tells us that[6]:

$$\int_{-\infty}^{\infty}|f(x)|^2dx = \int_{-\infty}^{\infty}|F(k)|^2dk \tag{43}$$

These relations tell us that $\Phi(p)$, like $\psi(x)$, represents a probability density. The function $\Phi(p)$ gives us information about the probability of finding momentum between $a \le p \le b$:

$$P(a \le p \le b) = \int_a^b|\emptyset(p)|^2dp \tag{44}$$

Parseval's theorem tells us that if the wavefunction $\psi(x)$ is normalized, then the momentum space wavefunction $\Phi(p)$ is also normalized

$$\int_{-\infty}^{\infty}|\psi(x)|^2dx = 1 \Rightarrow \int_{-\infty}^{\infty}|\emptyset(p)|^2dp = 1 \tag{45}$$

It is a fact of Fourier theory and wave mechanics that the spatial extension of the wave described by $\psi(x)$ and the extension of wavelength described by the Fourier transform $\Phi(p)$ cannot be made arbitrarily small.

5. Spin of the Electron

We shall now see that, unlike the Schrodinger equation where the spin of the electron had to be introduced in an ad hoc manner, the Dirac equation naturally leads to the spin of the electron. To see this let us evaluate the commutator bracket of the orbital angular momentum operator $\hat{L} = \hat{r} \times p$ with the Dirac Hamiltonian. Thus for the commutator of $\hat{L}_1$ we have

$$[\hat{L}_1, \hat{H}] = [(\hat{x}_2\hat{p}_3 - \hat{x}_3\hat{p}_2), \{c(\alpha_1\hat{p}_1 + \alpha_2\hat{p}_2 + \alpha_3\hat{p}_3) + \beta mc^2\}] = i\hbar c(\alpha_2\hat{p}_3 - \alpha_3\hat{p}_2) \tag{46}$$

where we used the position-momentum commutation relations to evaluate various terms in this equation. In a similar manner, we obtain

$$[\hat{L}_2, \hat{H}] = i\hbar c(\alpha_3\hat{p}_1 - \alpha_1\hat{p}_3) \tag{47}$$

$$[\hat{L}_3, \hat{H}] = i\hbar c(\alpha_1\hat{p}_2 - \alpha_2\hat{p}_1) \tag{48}$$

Thus the angular momentum operator $\hat{L}$ does not commute with the Dirac Hamiltonian. But the total angular momentum, being a conserved quantity for a free particle, must commute with the free Hamiltonian. This means the orbital angular momentum cannot be the total angular momentum. The electron must possess an intrinsic angular momentum, which when added to its orbital angular momentum, gives the total angular momentum, which is the conserved quantity.

To identify this intrinsic angular momentum, let us introduce a matrix operator $\sum \equiv (\sum_1, \sum_2, \sum_3,)$ defined by

$$\textstyle\sum_k = \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \tag{49}$$

Where σ_k's are Pauli spin matrices. Then the commutation

$$\left[\sum\nolimits_1, \hat{\mathrm{H}}\right] = c\hat{\mathrm{p}}_2\left[\sum\nolimits_1, \alpha_2\right] + c\hat{\mathrm{p}}_3\left[\sum\nolimits_1, \alpha_3\right]$$

where we have used the condition that $\sum_k$, commutes with α_k and β. Expressing $\sum_1$, α_1 and α_3 in terms of Pauli matrices α_k, using the anti-commutation relations for α_k, and the relation $\sigma_k\sigma_\ell = i\sigma_m$ where k, ℓ, m are a cyclic permutation of (1, 2, 3) we find

$$\left[(\hbar/2)\textstyle\sum_1, \hat{\mathrm{H}}\right] = i\hbar c(\hat{\mathrm{p}}_2\alpha_3 - \hat{\mathrm{p}}_3\alpha_2) \tag{50}$$

Adding this to Eq. (122), we find

$$\left[\hat{\mathrm{L}}_1, \hat{\mathrm{H}}\right] + \left[(\hbar/2)\textstyle\sum_1, \hat{\mathrm{H}}\right] = \left[(\hat{\mathrm{L}}_1 + (\hbar/2)\textstyle\sum_1), \hat{\mathrm{H}}\right] = 0. \tag{51}$$

Similarly, by considering the commutators of $\sum_2$,and $\sum_3$, with $\hat{\mathrm{H}}$, we can show that

$$\left[(\hat{\mathrm{L}}_2 + (\hbar/2)\textstyle\sum_2), \hat{\mathrm{H}}\right] = 0. \tag{52}$$

$$\left[(\hat{\mathrm{L}}_3 + (\hbar/2)\textstyle\sum_3), \hat{\mathrm{H}}\right] = 0. \tag{53}$$

By adding Eqs. (51) through (53) we find that the observable $\hat{L} + \frac{\hbar}{2}\sum \equiv \hat{J}$ commutes with the Hamiltonian and, therefore, is a constant of motion. The observable $\hat{J}$ may be called the total angular momentum of the electron. Thus preserving the conservation of angular momentum Dirac equation requires the electron to possess an intrinsic angular momentum. This instrinsic angular momentum is referred to as the spin of the electron. The operator $\frac{\hbar}{2}\sum = \frac{\hbar}{2}\begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$ may be regarded as the spin operator of the electron, where $\sum = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$. Thus spin, an intrinsic property of the electron, follows naturally from the Dirac equation.

6. Separation of Angular Momentum

In the following we are going to justify the identification of the Fourier coefficients with the components of the photon wave function in the formula (30) by unifying the field picture and the photon picture. We can see that the second term in the left hand side of (30) involves complex conjugation, and this is dictated by the fact that the photon energy is always positive. Therefore, the time evolution of the wavefunction is given by the factor $e^{-i\omega_k t}$ Therefore, the reversal of the sign in the exponent requires complex conjugation and then we discuss the pulling out of the factor $\sqrt{N}$ to assure the normalization of f.

Firstly, we shall combine now the field picture and the photon picture to obtain the decomposition of the total angular momentum of the field. To this end, we use the substituting of the Fourier representation of the field into the formulas (26), (27),(28) and (29),

$$H = N\int \frac{d^3k}{hw_k} f^\dagger(k) \cdot hw_k f(k) \tag{54}$$

$$P = N\int \frac{d^3k}{hw_k} f^\dagger(k) \cdot hkf(k) \tag{55}$$

$$J = N\int \frac{d^3k}{hw_k} f^\dagger(k) \cdot [ihD \times k + h\hat{x}n_k] f(k) \tag{56}$$

$$K = N\int \frac{d^3k}{hw_k} f^\dagger(k) \cdot ihw_k Df(k) \tag{57}$$

We have to note that, the resulting expressions have the form of quantum mechanical expectation values:

$$H = N\langle f|\hat{H}|f\rangle \tag{58}$$

$$P = N\langle f|\hat{P}|f\rangle \tag{59}$$

$$J = N\langle f|\hat{J}|f\rangle \tag{60}$$

$$K = N\langle f|\hat{K}|f\rangle \tag{61}$$

As Darwin had anticipated, we can see that, these formulas exhibit a perfect agreement between the results obtained from the particle picture and from the field picture. Also, every value calculated for the total electromagnetic field is a product of the quantum mechanical average value per one photon, multiplied by N [1]. This means that, the normalization factor N is the total number of photons [1]. Now, we may unambiguously split that the total angular momentum of the electromagnetic field (56) into two parts as done in [1]. The vector J_0 whose integrand is perpendicular to the wavevector is the orbital part and the vector J_s whose integrand is parallel to the wavevector is the spin part represented by helicity:

$$J_0 = N\int \frac{d^3k}{hw_k} f^\dagger(k) \cdot [ihD \times k] f(k) \tag{62}$$

$$J_s = N\int \frac{d^3k}{hw_k} f^\dagger(k) \cdot h\hat{x}n_k f(k) = N\int \frac{d^3k}{w_k} n_k[|f_L(k)|^2 - |f_R(k)|^2] \tag{63}$$

The final step of our studding of the above analysis is the proof that the expressions for J_0 and J_s coincide with those obtained by Darwin. In [1] they employ the relation between $E(k)$ and $f(k)$ that follows from the formula(30):

$$E(k) = \sqrt{\frac{N}{2\epsilon_0}}[e(k)f_L(k) + e^*(k)f_R(k)] \quad (64)$$

where $E(k)$ is the plane –wave component of the electric field,

$$E(r,t) = \int \frac{d^3k}{(2\pi)^{3/2}}[E(k)e^{-iw_k t + ik\cdot r} + c] \quad (65)$$

Now, by using the properties of the polarization vectors (35) and (37) we get [1]:

$$-2i\epsilon_0 \int \frac{d^3}{c|k|} E^*(k) \times E(k) = -iN \int \frac{d^3}{c|k|}[e^*(k) \times e(k)|f_L(k)|^2 + e(k) \times e^*(k)|f_R(k)|^2] = J_s \quad (66)$$

Now, we have to note that, the separation of the total angular momentum into its orbital and spin parts is conserved in time since both parts are separately time independent.

7. Another Separation of Angular Momentum

Now by using the formula (13) we may again unambiguously split that the total angular momentum of electromagnetic field (56) into two parts. The vector J_{OM} whose integrand is perpendicular to the wave vector is the orbital part and the vector J_{SM} whose integrand is parallel to the wave vector is the spin part represented by helicity; then we can rewrite (56) by using (13) in the form:

$$J = N \int \frac{d^3k}{hw_k}[f_L^*(k)f_L(k) + f_R^*(k)f_R(k)].[ihD \times k + h\hat{x}n_k] \quad (67)$$

There for:

$$J_{OM} = N \int \frac{d^3k}{hw_k}[f_L^*(k)f_L(k) + f_R^*(k)f_R(k)].[ihD \times k] \quad (68)$$

$$J_{SM} = N \int \frac{d^3k}{w_k}[f_L^*(k)f_L(k) + f_R^*(k)f_R(k)].[\hat{x}n_k] \quad (69)$$

We can see that, the formulas (68) and (69) in our above analysis are coincide authors, and are separately time independent.

8. Conclusions

As a conclusion, in this paper we discussed the total angular momentum of electromagnetic field into its two parts, the orbital and spin. Our main tools, is the quantum mechanics of photons. In fact we revisit the results obtained by Darwin using Maxwell fields properties. We have also shown that our results coincide with previous results obtained vi several authors. Our last observation was that, when comparing the energy momentum and the total angular momentum of electromagnetic field, the two parts of momentum can't be expressed as an integral forms of local densities, and the formulas (68) and (69) in our above analysis are coincide authors, and are separately time independent.

References

[1] Iwo Bialynicki-Birula and Zofia Bialynicki-Birula 2011 Canonical separation of angular momentum of light into its orbital and spin parts.

[2] Lomont J S and Moses H E 1962 Simple realizations of the infinitesimal generators of the proper orthochronous inhomogeneous Lorentz group for mass zero J. Math. Phys.

[3] Bialynicki-Birula I and Bialynicka-Birula Z 1975 Quantum Electrodynamics (Oxford: Pergamon).

[4] Bialynicki-Birula I and Bialynicka-Birula Z 1987 Berry's phase in the relativistic theory of spinning particles Phys. Rev. D 35 2383.

[5] Bialynicki-Birula I 1996 Photon wavefunction Progress in Optics vol 36, ed E Wolf (Amsterdam: Elsevier).

[6] Bialynicki-Birula I and Bialynicka-Birula Z 2006 Beams of electromagnetic radiation carrying angular momentum: the Riemann–Silberstein vector and the classical-quantum correspondence Opt. Commune.

[7] Rob Van der Vorst. Introduction to differential manifolds.

[8] John M. Lee.. Introduction to smooth manifolds.

[9] M. E. Rose, Relativistic electron theory (John Wiley & Sons, New York, 1961).

[10] O. Hosten and P. Kwiat, Science 319, 787 (2008).

[11] Barnett S M 2010 Rotation of electromagnetic fields and thenature of optical angular momentum J. Mod.

[12] Li C-F 2009 Spin and orbital angular momentum of a class of nonparaxial light beams having a globally defined polarization.

[13] Allen L, Barnett S M and Padget M J 2003 Optical Angular Momentum (Bristol: Institute of Physics Publishing) This is a collection of papers with introductory material by the editors.

[14] Lenstra D and Mandel L 1982 Angular momentum of the quantized electromagnetic field with periodic boundary conditions Phys. Rev. 26 3428.

[15] J`aregui R and Hacyan S 2005 Quantum-mechanical properties of Bessel beams Phys. Rev. A 71 033411.

[16] Calvo G F, Pic`on A and Bagan E 2006 Quantum field theory of photons with orbital angular momentum Phys. Rev. A 73 013805.

[17] Aiello A, Marquardt C and Leuchs G 2009 Transverse angular momentum of photons Phys. Rev. A 81 053838.

[18] Cohen-Tannoudji C, Dupont-Roc J and Grynberg G 1989 Photons and Atoms: Introduction to Quantum Electrodynamics (New York: Wiley) chapter 1.

[19] Schweber S S 1961 An Introduction to Relativistic Quantum Field Theory (Evanston: Row, Peterson and Co) chapter 9.

[20] D. F. Walls, Nature, 301(1983)141; R. W. Henry and S. C. Glotzer, Am. J. Phys. 56, 4, (1988) 318; M. M. Nieto, in Frontiers of Non-equilibrium Statistical Mechanics, Proceedings of NATO Advanced Study Institute, ed. G. T. Moore and M. O. Scully, Plenum, NY.

[21] B. Yurke, S. L. McCall, J. R. Klauder, Phys. Rev. A33 6 (1986) 4033; M. Hillery and L. Mlodinow, Phys. Rev. A48, 2 (1993) 1548.

[22] E. P. Wigner in Group Theory and its applications to the Quantum Mechanics of Atomic Spectra, Academic Press, NY (1959); A. Vaglica and G. Vetri, Optics Communications, 51, 4 (1984) 239.

[23] J. Schwinger in Quantum theory of Angular Momentum, ed. L. Beidenharn and H. van Dam, Academic Press, NY. (1965) 22.

[24] Most general result can be found in the doctoral Thesis of Abir Bandyopadhyay, submitted on December 13, 1996, and successfully defended on November 27, 1997, at Indian Institute of Tecnology, Kanpur. Also corrected electronic copy is available on request from him through email.

[25] G. K. Campbell, A. E. Leanhardt, J. Mun, M. Boyd, E. W. Streed, W. Ketterle, and D. E. Pritchard, Phys. Rev. Lett. 94, 170403 (2005).

[26] S. M. Barnett, Phys. Rev. Lett. 104, 070401 (2010).

[27] T. Ramos, G. F. Rubilar, and Y. N. Obukhov, Phys. Lett. A 375, 1703 (2011).

[28] F. W. Hehl and Y. N. Obukhov, Foundations of Classical Electrodynamics: Charge, Flux, and Metric (Birkh¨auser, Boston, 2003).

[29] Cf. Bendixson (1901, p. 11 and next); Coddington and Levinson (1955, p. 391 and next); Hirsch et al. (2004, p. 225 and next).

[30] Cf. Guckenheimer and Holmes (1983, p. 141); Wiggins (1990, p. 212); Dang-Vu (2000, p. 47).