

On the Solution Procedure of Partial Differential Equation (PDE) with the Method of Lines (MOL) Using Crank-Nicholson Method (CNM)

Mohammad Roknujjaman¹, Mohammad Asaduzzaman^{2,*}

¹Department of Computer Science and Engineering, Northern College Bangladesh, Dhaka, Bangladesh

²Department of Mathematics, Islamic University (IU), Kushtia, Bangladesh

Email address:

rkripon.mathiu@gmail.com (M. Roknujjaman), asad@math.iu.ac.bd (M. Asaduzzaman)

*Corresponding author

To cite this article:

Mohammad Roknujjaman, Mohammad Asaduzzaman. On the Solution Procedure of Partial Differential Equation (PDE) with the Method of Lines (MOL) Using Crank-Nicholson Method (CNM). *American Journal of Applied Mathematics*. Vol. 6, No. 1, 2018, pp. 1-7.

doi: 10.11648/j.ajam.20180601.11

Received: December 28, 2017; **Accepted:** January 16, 2018; **Published:** February 12, 2018

Abstract: In this paper we used the method of lines (MOL) as a solution procedure for solving partial differential equation (PDE). The range of applications of the MOL has increased dramatically in the last few years; nevertheless, there is no introductory to initiate a beginner to the method. This Paper illustrates the application of the MOL using Crank-Nicholson method (CNM) for numerical solution of PDE together with initial condition and Dirichlet’s Boundary Condition. The implementation of this solutions is done using Microsoft office excel worksheet or spreadsheet, Matlab programming language. Finally, here we analysis the particular solution and numerical solution of Laplace equation obtained by MOL along with CNM.

Keywords: Dirichlet’s Boundary Condition, Laplace Equation, MOL, PDE, CNM

1. Introduction

Method of lines (MOL) is an alternative computational procedure which involves making an approximation to the space derivatives and reducing the problem to a system of ordinary differential equations (ODEs) in the variable time and then a proper initial value problem solver can be used to solve this system of ordinary differential equations. Usually the method is used on equations involving a time variable t and one or more space variables x_1, x_2, \dots, x_m . They are solved by converting them to a system of ODEs. The part of the equations involving the space variables is discretized, giving us a system of ODEs approximating the PDE. This system can then be integrated directly with a standard ODE code. It might be more efficient to solve a PDE by a method specially constructed to suit the problem, but the MOL usually enables us to solve quite general and complicated PDEs relatively easily and with acceptable efficiency. The MOL is also attractive since we can use the theoretical knowledge from ODEs to solve PDEs, and powerful ODE

solvers are readily available; see for instance [1-11].

The connection between partial and ordinary differential equation was already known to Lagrange in 1759 [12]. There he obtained a system of ordinary differential equations

$$\left. \begin{aligned} y_1'' &= K^2(-2y_1 + y_2) \\ y_2'' &= K^2(y_1 - 2y_2 + y_3) \\ &\vdots \\ y_n'' &= K^2(y_{n-1} - 2y_n) \end{aligned} \right\} \quad (1)$$

Lagrange observed that by modeling D’Alembert’s equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

with a finite number of mass points attached to a light string, the same equations as like (1) are found. It is also noted that Fourier in 1807 [12] motivated by the problem of heat conduction and he obtained the following equation

$$y_i' = K^2(y_{i-1} - 2y_i + y_{i+1}), \quad i = 1, 2, \dots, N \quad (2)$$

By taking N larger and larger in (2) he obtained the heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

using the inverse of the MOL. But the MOL as a numerical method was first applied by Rothe in 1930 [12]. He used the method on the parabolic equation

$$u_{xx} = R(x, t)u_t + S(x, t, u) \quad (3)$$

With $R > 0, 0 \leq x \leq 1, 0 \leq t \leq T$. He discretised the time variables in (3) and approximated the equation by the scheme

$$u_{n+1}'' = R(x, t_{n+1}) \frac{u_{n+1} - u_n}{h} + S(x, t_{n+1}, u_n), t_n = nh. \quad (4)$$

The equation (4) is the integration of the ODEs along lines parallel to the x -axis, and this is called a transversal scheme. That is, longitudinal schemes lead to initial value problems, while transversal schemes lead to boundary value problems. In this paper we established a solution procedure of PDE by the help of MOL along with CNM and finally studied the particular and numerical solution of Laplace equation by that solution procedure obtained by MOL along with CNM.

2. Conversion of PDE to the System of ODE

The basic idea of the MOL is to replace the spatial (boundary value) derivatives in the PDE with algebraic approximations. Once this is done, the spatial derivatives are no longer stated explicitly in terms of the spatial independent variables. Thus, in effect only the initial value variable, typically time in a physical problem, remains, we have a system of ODEs that approximate the original PDE. The challenge, then, is to formulate the approximating system of ODEs. Once this is done, we can apply any integration algorithm for initial value ODEs to compute an approximate numerical solution to the PDE. Thus, one of the salient features of the MOL is the use of existing, and generally well established, numerical methods for ODEs.

To illustrate this procedure, we consider the MOL solution of

$$u_t + v u_x = 0 \quad (5)$$

First we need to replace the spatial derivative u_x with an algebraic approximation. In this case we will use a finite difference (FD) such as

$$u_x \approx \frac{u_i - u_{i-1}}{\Delta x} \quad (6)$$

Then the MOL approximation of equation (5) is

$$\frac{du_i}{dt} = -v \frac{u_i - u_{i-1}}{\Delta x}, 1 \leq i \leq M \quad (7)$$

Note that the equation (7) represents a system of ODEs. This transformation of a PDE equation (5) to a system of ODEs equation (7) is so that the solution of a system of ODEs approximates the solution of the original PDE. Since equation (5) is first order, it requires one IC and one BC.

These will be as follows, which are collected from [21]:

$$u(x, t = 0) = f(x) \quad (8)$$

$$u(x = 0, t) = g(x) \quad (9)$$

Since equation (7) constitute M initial value ODEs, M initial conditions are required and from equation (8), these are

$$u(x_i, t = 0) = f(x_i) \quad 1 \leq i \leq M \quad (10)$$

Also, application of BC (9) gives for grid point $i=l$

$$u(x_l, t) = g(t), t \geq 0 \quad (11)$$

The solution of this ODE system is as follows which are collected from [16]:

$$u_1(t), u_2(t), u_3(t), \dots, u_{M-1}(t), u_M(t) \quad (12)$$

That is, an approximation to $u(x, t)$ at the grid points $i = l, 2, \dots, M$

Before we go on to consider the numerical integration of the approximating ODEs, in this case equation (7), we briefly consider further the FD approximation of equation (12), which can be written as

$$u_x \approx \frac{u_i - u_{i-1}}{\Delta x} + O(\Delta x) \quad (13)$$

where, $O(\Delta x)$ denotes of order Δx .

3. Laplace Equation [13-16] and Problem Definition

Consider the PDE of the following form

$$\Delta u = 0 \quad (14)$$

where the Laplacian Δ is defined in Cartesian coordinates by

$$\Delta u = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_n^2}$$

This is known as Laplace's equation. It is also called a harmonic function. Equation (14) is Elliptic second order linear PDE. The initial condition is

$$u(x, 0) = f(x) \quad a < x < b \quad (15)$$

And the Dirichlet Boundary conditions are

$$u(a, t) = c_1, u(b, t) = c_2, 0 \leq t \leq d \quad (16)$$

Equation (14), (15) and (16) are called initial boundary

condition. To compute the numerical solution of equation (14) together equation (15), (16). we divide the interval into n equal parts with step length $h = \Delta x = \frac{a}{N+1}$. we consider a plane region defined $R = \{(x, y): 0 \leq x \leq 0.5 \text{ and } 0 \leq y \leq 0.5\}$ we will also impose the Dirichlet boundary condition $u(0, y) = 0, u(x, 0) = 0; u(x, 0.5) = 200x, u(0.5, y) = 200y$

Consider a two-dimensional solution shown in Figure 1 [4]

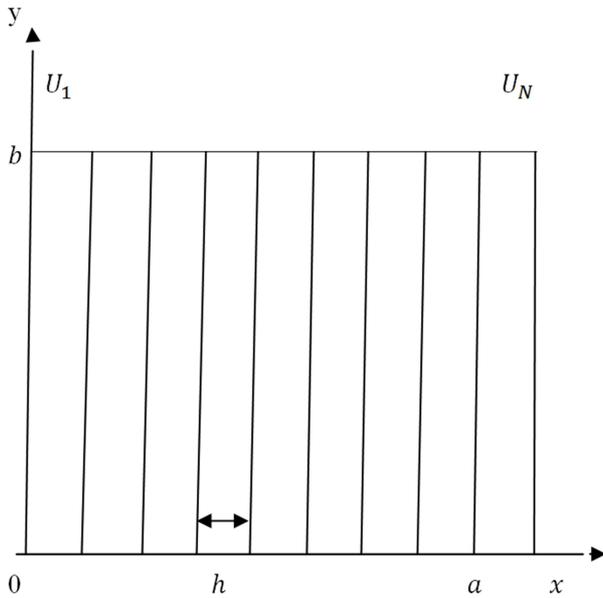


Figure 1. Illustration of discretization in the x-direction.

The first step is discretization of the x -variable. The region is divided into strips by N dividing straight lines (hence the name method of lines) parallel to the y -axis. Since we are discretizing along x , we replace the second derivative with respect to x with its finite difference equivalent. We apply the three-point central difference scheme,

$$\frac{\partial^2 U_i}{\partial x^2} = \frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} \tag{17}$$

Where h is the spacing between discretised lines, that is,

$$h = \Delta x = \frac{a}{N+1} \tag{18}$$

Replacing the derivative with respect to x by its finite difference equivalent equation (14) becomes

$$\frac{\partial^2 U_i}{\partial y^2} + \frac{1}{h^2} [U_{i-1}(y) - 2U_i(y) + U_{i+1}(y)] = 0 \tag{19}$$

Thus the potential U in equation (14) can be replaced by a vector of size N namely

$$[U] = [U_1, U_2, \dots, \dots, U_N]^t$$

Where,

$$U_i(y) = U(x_i, y), i = 1, 2, \dots, N$$

and $x_i = i\Delta x$.

Substituting equations (18) and (19) into (14) yield

$$\frac{\partial^2 U(y)}{\partial y^2} - \frac{1}{h^2} [P][U(y)] = 0 \tag{20}$$

where $[P]$ is an $N \times N$ tridiagonal matrix representing the discretised form of the second derivative with respect to x .

$$[P] = \begin{bmatrix} p_l - 1 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & \dots & 0 \\ & & \ddots & \ddots & & & \\ 0 & \dots & -1 & 2 & -1 & & \\ 0 & \dots & 0 & -1 & p_r & & \end{bmatrix}$$

Since all the elements of matrix $[P]$ are zeros except the tridiagonal terms; the elements of the first and the last row of $[P]$ depends on the boundary conditions at $x = 0$ and $x = a$. $p_l = 2$ for the Dirichlet boundary condition and $p_l = 1$ for the Neumann boundary condition. The same is true of p_r . The next step is to solve the resulting equations analytically along the y coordinate. To solve equation (20) analytically, we need to obtain a system of uncoupled ordinary differential equations from the coupled equations (20). To achieve this, we define the transformed potential $[\bar{U}]$ by letting

$$[U] = [T][\bar{U}] \tag{21}$$

And requiring that

$$[P][T] = [\lambda^2] \tag{22}$$

where, $[\lambda^2]$ is a diagonal matrix and $[T]^t$ is the transpose of $[T]$. $[\lambda^2]$ and $[T]$ are eigenvalue and eigenvector matrices belonging to $[P]$. The transformation matrix $[T]$ and the eigenvalue matrix $[\lambda^2]$ depend on the boundary conditions and are given in Table 1 for various combinations of boundaries. It should be noted that the eigenvector matrix $[T]$ has the following property:

$$[T]^{-1} = [T]^t \tag{23}$$

And

$$[T][T]^t = [T]^t[T] = [I] \tag{24}$$

Where, $[I]$ is an identity matrix. Substituting equation (21) into equation (20) gives

$$\frac{\partial^2 [T][\bar{U}]}{\partial y^2} - \frac{1}{h^2} [P][T][\bar{U}] = 0 \tag{25}$$

Multiplying through by $[T]^{-1} = [T]^t$ yields

$$\left(\frac{\partial^2}{\partial y^2} - \frac{1}{h^2} [\lambda^2]\right) [\bar{U}] = 0 \tag{26}$$

Table 1. Elements of Transformation matrix [T] and Eigen values [19-20].

Left boundary	Right boundary	T_{ii}	λ_i
Dirichlet	Dirichlet	$\sqrt{\frac{2}{N+1} \sin \frac{ij\pi}{N+1}}, [T_{DD}]$	$2 \sin \frac{i\pi}{2(N+1)}$
Dirichlet	Neumann	$\sqrt{\frac{2}{N+0.5} \sin \frac{i(j-0.5)\pi}{N+0.5}}, [T_{DN}]$	$2 \sin \frac{(i-0.5)\pi}{2N+1}$
Neumann	Dirichlet	$\sqrt{\frac{2}{N+0.5} \cos \frac{(i-0.5)(j-0.5)\pi}{N+0.5}}, [T_{ND}]$	$2 \sin \frac{(i-0.5)\pi}{2N+1}$
Neumann	Neumann	$\sqrt{\frac{2}{N+0.5} \cos \frac{(i-0.5)(j-1)\pi}{N+0.5}},$ $j > 1, [T_{NN}] \sqrt{\frac{1}{N}}, j = 1$	$2 \sin \frac{(i-1)\pi}{2N}$

Note: where $i, j = 1, 2, N$ and subscripts D and N are for Dirichlet and Neumann conditions respectively. This is an ordinary differential equation with solution.

$$\bar{U}_i = A_i \cosh \alpha_i y + B_i \sinh \alpha_i y \tag{27}$$

where, $\alpha_i = \lambda_i/h$.

4. A Particular Solution of Laplace 2-D Equation

In this section we obtained a particular solution of Laplace 2-D equation by applying MOL.

Example 1.

For the rectangular region in Figure 1, let

$$U(0, y) = U(a, y) = U(x, 0) = 0, U(x, b) = 100$$

and $a = b = 1$.

Find the potential at

$$(0.0625, 0.9375), (0.125, 0.875), (0.1875, 0.8125), (0.25, 0.75), (0.3125, 0.6875), (0.375, 0.625),$$

$$(0.4375, 0.5625), (0.5, 0.5), (0.5625, 0.4375), (0.625, 0.375), (0.6875, 0.3125), (0.75, 0.25),$$

$$(0.8125, 0.1875), (0.875, 0.125), (0.9375, 0.625).$$

Solution of Example 1.

In this case, we have Dirichlet boundaries at $x = 0$ and $x = 1$, which are already indirectly taken care of in the solution in (27). Hence, from Table 1,

$$\lambda_i = 2 \sin \frac{i\pi}{2(N+1)} \tag{28}$$

and

$$T_{ij} = \sqrt{\frac{2}{N+1}} \sin \frac{ij\pi}{N+1} \tag{29}$$

Let $N=15$ So then $h = \Delta x = 1/16$ and

$$x = 0.0625, 0.125, 0.1875, 0.25, 0.3125, 0.375, 0.4375, 0.5, 0.5625,$$

0.625, 0.6875, 0.75, 0.8125, 0.875, 0.9375 will correspond to $i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15$ respectively.

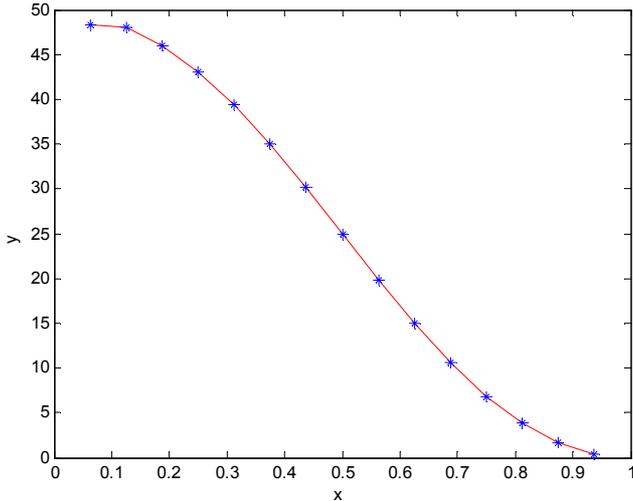


Figure 2. Graphical Representation of the particular solutions of Laplace 2-D Equation by the MOL for Example 4.1.

5. Crank-Nicholson Algorithm [17-18]

This note provides a brief introduction to finite difference methods for solving partial differential equations. We focus on the case of a PDE in one state variable plus time. Suppose one wishes to find the function $u(x, t)$ satisfying the PDE

$$u_{i,n+1} = \left(\frac{k}{h^2} a + \frac{k}{2h} b\right)u_{i+1,n} + (1 + kc - \frac{2k}{h^2} a)u_{i,n} + \left(\frac{k}{h^2} a - \frac{k}{2h} b\right)u_{i-1,n} \tag{32}$$

The result of the equation (32) is called an explicit finite difference solution for u . Unfortunately the numerical solution is unstable unless the ratio k/h^2 is sufficiently small. The recommended method for most problems in the Crank-Nicholson algorithm, Thus the expressions for u , u_x and u_{xx} are averages of what we had in (31) for times n and $n+1$

$$\begin{aligned} u &= \frac{u_{i,n} + u_{i,n+1}}{2} \\ \frac{\partial u}{\partial t} &= \frac{u_{i,n+1} - u_{i,n}}{k} \\ \frac{\partial u}{\partial x} &= \frac{u_{i+1,n} - u_{i-1,n} + u_{i+1,n+1} - u_{i-1,n+1}}{4h} \\ \frac{\partial^2 u}{\partial x^2} &= \frac{u_{i+1,n} - 2u_{i,n} + u_{i-1,n} + u_{i+1,n+1} - 2u_{i,n+1} + u_{i-1,n+1}}{2h^2} \end{aligned} \tag{33}$$

6. A Numerical Solution of Laplace 2-D Equation

In this section we obtained a numerical solution of Laplace 2-D equation by applying MOL

Example 2.

Solve the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ with boundary condition,

$$au_{xx} + bu_x + cu - u_t = 0 \tag{30}$$

subject to the initial condition $u(x, 0) = f(x)$ and other possible boundary conditions. Explaining the negative u_t term in (30), focusing on an arbitrary internal grid point in one could approximate the partial derivatives at that point by the following:

$$\begin{aligned} u &= u_{i,n} \\ \frac{\partial u}{\partial t} &= \frac{u_{i,n+1} - u_{i,n}}{k} \\ \frac{\partial u}{\partial x} &= \frac{u_{i+1,n} - u_{i-1,n}}{2h} \\ \frac{\partial^2 u}{\partial x^2} &= \frac{u_{i+1,n} - 2u_{i,n} + u_{i-1,n}}{h^2} \end{aligned} \tag{31}$$

The differences in the x , direction have been centered on the point in to give ‘second order’ accuracy to the approximation. These expressions could then be substituted into the PDE Solving the resulting equation for $u_{i,n+1}$ gives the explicit solution

$$R = \langle (x,y) / 0 < x < 0.5, 0 < y < 0.5 \rangle \text{ and}$$

$$u(0, y) = 0 ; u(x, 0) = 0$$

$$u(x, 0.5) = 200x \text{ and } u(0.5, y) = 200y$$

using CNM and MOL.

Solution of Example 2.

Given that,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{34}$$

Now we applying the CNM

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2h^2} [u_{i+1,j} - 2u_{i,j} + u_{i-1,j} + u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}]$$

Where, h is the spacing between discreted lines [4]

$$h = \Delta x = \frac{a}{N+1}$$

Equation (34) becomes,

$$\frac{\partial^2 u_{(i,j)}}{\partial y^2} + \frac{1}{2h^2} [u_{i+1,j} - 2u_{i,j} + u_{i-1,j} + u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}] = 0$$

$$\frac{\partial^2 u_{(i,j)}}{\partial y^2} = \frac{1}{2h^2} [-u_{i+1,j} + 2u_{i,j} - u_{i-1,j} - u_{i+1,j+1} + 2u_{i,j+1} - u_{i-1,j+1}] \tag{35}$$

Now using the Figure 3 which domain is divided into squares of 0.125 unit size as illustrated below,

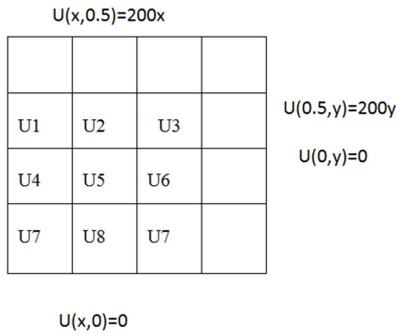


Figure 3. Domain division into squares of 0.125 unit size

The boundary condition implies that,

$$u_{1,0} = u_{2,0} = u_{3,0} = u_{0,1} = u_{0,2} = u_{0,3} = 0$$

$$u_{1,4} = u_{4,1} = 25, u_{2,4} = u_{4,2} = 50 \text{ and } u_{3,4} = u_{4,3} = 75$$

for each $i, j=1, 2, 3 \dots$ and so on.

Now we use the boundary condition and finite difference method to convert PDE to ODE and then apply the ODE solver to solve the system of ODE. Then we use Matlab code to find the value of ODE and to find the numerical solution of the PDE by applying the CNM.

Again, we have from CNM

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{2h^2} [u_{i,j+1} - 2u_{i,j} + u_{i,j-1} + u_{i+1,j+1} - 2u_{i+1,j} + u_{i+1,j-1}]$$

Similarly, $\frac{\partial^2 u}{\partial x^2} = \frac{1}{2h^2} [u_{i+1,j} - 2u_{i,j} + u_{i-1,j} + u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}]$

Then the CNM for the Laplace equation is as follows

$$\frac{1}{2h^2} [u_{i+1,j} - 2u_{i,j} + u_{i-1,j} + u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1} + u_{i,j+1} - 2u_{i,j} + u_{i,j-1} + u_{i+1,j+1} - 2u_{i+1,j} + u_{i+1,j-1}] = 0$$

$$\therefore [-u_{i+1,j} - 4u_{i,j} + u_{i-1,j} + 2u_{i+1,j+1} - u_{i,j+1} + u_{i-1,j+1} + u_{i,j-1} + u_{i+1,j-1}] = 0$$

Again for $i, j=1, 2, 3 \dots$ and use Matlab code for solving the numerical solution of Laplace equation using CNM.

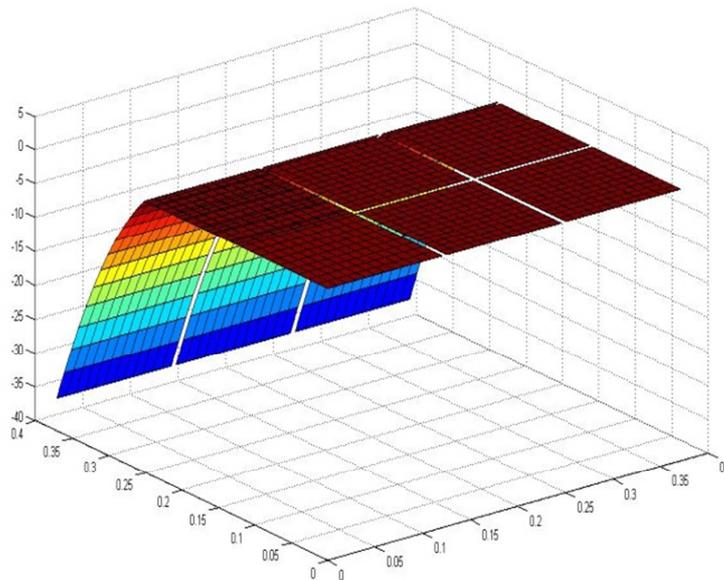


Figure 4. Graphical Representation of the numerical solution of Laplace Equation by the MOL for Example 2.

7. Results and Discussion

Here we give a simple introduction on MOL and on CNM with illustrative examples. The graph of Example 1 presented in Figure 2 represents the particular solution of Laplace equation and the graph of Example 2 presented in Figure 4 represents the numerical solution of Laplace equation.

8. Conclusion

The MOL is generally recognized as a comprehensive and powerful approach to approximate the numerical solution of PDEs. Here we used this method in two separate steps. First we discretized the PDE using crank-Nicholson method to obtain a system of ordinary differential equations (ODEs). Then we solved the system of ODEs using ODE solvers. The success of this method is explained by the availability of high-quality numerical algorithms for the solution of stiff systems of ODEs. Here we investigated MOL approach for solving the two dimensional Elliptic equation with boundary condition. The computational results confirmed the efficiency, reliability and accuracy of this procedure and this superior performance is achieved with very little increased computational effort. Thus, we conclude that the use of CNM in the MOL solution for the Laplace equation and for other partial differential equations is very effective and appropriate procedure.

Conflict of Interests

The authors declare that they have no any conflict of interests.

References

- [1] I. Ayranci and N. Selçuk, "MOL solution of DOM for transient radiative transfer in 3-D scattering media", *J. Quant. Spectrosc. Radiat. Transfer* 84, 409–422 (2004).
- [2] A. G. Bratsos, "The solution of the two-dimensional sine-Gordon equation using the method of lines", *J. Comput. Appl. Math.* 206, 251–277 (2007).
- [3] A. G. Bratsos, "The solution of the Boussinesq equation using the method of lines", *Comput. Methods Appl. Mech. Eng.* 157, 33–44 (1998).
- [4] A. G. Bratsos and E. H. Twizell, "The solution of the sine-Gordon equation using the method of lines", *Int. J. Comput. Math.* 61, 271–292 (1996).
- [5] V. D. Egorov, "The method of lines for condensation kinetics problem solving", *J. Aerosol Sci.* 30 (Suppl. I), S247–S248 (1999).
- [6] A. Erdem and S. Pamuk, "The method of lines for the numerical solution of a mathematical model for capillary formation: the role of tumor angiogenic factor in the extracellular matrix", *Appl. Math. Comput.* 186, 891–897 (2007).
- [7] G. Hall and J. M. Watt, "Modern Numerical Methods for Ordinary Differential Equations, Clarendon", Oxford (1976).
- [8] S. Hamdi, W. H. Enright, Y. Ouellet and W. E. Schiesser, "Method of lines solutions of the extended Boussinesq equations", *J. Comput. Appl. Math.* 183, 327–342 (2005).
- [9] H. Han and Z. Huang, "The direct method of lines for the numerical solution of interface problem", *Comput. Methods Appl. Mech. Eng.* 171, 61–75 (1999).
- [10] T. Koto, "Method of lines approximations of delay differential equations", *Comput. Math. Appl.* 48, 45–59 (2004).
- [11] S. Pamuk and A. Erdem, "The method of lines for the numerical solution of a mathematical model for capillary formation: the role of endothelial cells in the capillary", *Appl. Math. Comput.* 186, 831–835 (2007).
- [12] W. E. Schiesser, "The Numerical Method of Lines", Academic Press, San Diego.
- [13] D. J. Jones et al, "On the numerical solution of elliptic partial differential equations by the method of lines", *J. Comput. Phys.*, 9, 496–527 (1972).
- [14] J. G. Ma and Z. Chen, "Application of the method of lines to the Laplace equation", *Micro. Opt. Tech. Lett.*, 14 (6), 330–333 (1997).
- [15] A. A. Sharaf and H. O. Bakodah, "A Good Spatial Discretization in the Method of Lines", *Applied Mathematics and Computation* (2005).
- [16] Z. A. Vlasova, "A Numerical Realization of the Method of Reduction to Ordinary Differential Equations", *Sibirsk. Math. Za.* 4, 475-479 (1963).
- [17] W. E. Schiesser, and G. W. Griffiths, "A Compendium of Partial Differential Equation Models: Method of Lines Analysis with Matlab", Cambridge University Press (2009).
- [18] G. D. Smith, "Numerical Solution of Partial Differential Equations (Crank-Nicholson Methods)", 3rd Edition, Oxford University Press, Oxford (1985).
- [19] S. G. Mikhlin, and K. L. Smolitskii, "Approximate Methods for Solving Differential and Integral Equations", Moscow 1965, pp 329-335 (1965).
- [20] H. O. Kreiss and J. Lorenz, "Initial-Boundary Value Problems and the Navier-Stokes Equations", SIAM, Philadelphia (2004).