Strong Matching Preclusion for Augmented Butterfly Networks

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Abstract: The matching preclusion number of a graph is the minimum number of edges whose deletion results in a graph that has neither perfect matchings nor almost-perfect matchings. The strong matching preclusion number (or simply, SMP number) *smp(G)* of a graph *G* is the minimum number of vertices and/or edges whose deletion results in a graph that has neither perfect matchings nor almost-perfect matchings. This is an extension of the matching preclusion problem and has been introduced by Park and Ihm. Butterfly Networks are interconnection networks which form the back bone of distributed memory parallel architecture. One of the current interests of researchers is Butterfly graphs, because they are studied as a topology of parallel machine architecture. Butterfly network has many weaknesses. It is non-Hamiltonian, not pancyclic and its toughness is less than one. But augmented butterfly network retains most of the favorable properties of the butterfly network. In this paper, we determine the strong matching preclusion number of the Augmented Butterfly networks.

keywords: Matching, Strong Matching Preclusion, Augmented Butterfly Networks

1. Introduction

Let G be a connected graph, $V(G)$ and $E(G)$ are its vertex set and edge set respectively. A perfect matching is a set of edges such that every vertex is incident with exactly one edge in this set. An almost-perfect matching in a graph is a set of edges such that every vertex except one is incident with exactly one edge in this set, and the exceptional vertex is incident to none. In this paper, we say that such exceptional vertex isn't matched under the almost perfect matching.

All graphs considered in this paper are undirected, finite and simple. It is referred to the book [3] for graph theoretical notation and terminology not described here. Let X be a set of vertices and edges. Let $G - X$ denote the subgraph of G obtained by removing all the vertices of X together with the edges incident with them from G as well as removing all the edges of X from G. If $X = v$ and $X = e$ where v is a vertex and e is an edge, simply wrote by $G - v$ and $G - e$ for $G - v$ and $G - e$, respectively. The degree of a vertex v in a graph G, denoted by $deg_G(v)$, is the number of edges of G incident with v. Let $\delta(G)$ be the minimum degree of the vertices of G,

respectively. A component of a graph is odd or even according to whether it has an odd or even number of vertices.

A set of edges F is called a matching preclusion se t if $G - F$ has no perfect matching, and it is called an optimal matching preclusion set if F is one with the smallest size. The matching preclusion number of an even graph G, denoted by $mp(G)$, is the minimum number of edges whose deletion leaves the resulting graph without a perfect matching. If G has no perfect matchings, then $mp(G) = 0$. The concept of matching preclusion was introduced by Brigham et al. [2], as a measure of robustness in the event of edge failure in interconnection networks. An interconnection network with a larger MP number may be considered as more robust in the event of link failures.

Proposition 1-1

Let G be a graph with an even number of vertices. Then $mp(G) \leq \delta(G)$, where $\delta(G)$ is the minimum degree of G

The concept of strong matching preclusion was proposed by Park and Ihm [2]. A set F of vertices and edges in a graph G is called a strong matching preclusion set SMP set for short if $G - F$ has neither a perfect matching nor an almost perfect matching. The strong matching preclusion number of G,

denoted by $smp(G)$, is defined to be the minimum size of all possible SMP sets of G. Strong matching preclusion that additionally permits more destructive vertex faults in a graph is a more extensive form of the original matching preclusion that assumes only edge faults. According to the definition of $mp(G)$ and $smp(G)$, we have that $smp(G) \leq mp(G) \leq \delta(G)$.

Park and Ihm [7] established the SMP number and all possible minimum strong matching preclusion sets for complete graphs, regular bipartite graphs, restricted HL-graphs, and recursive circulant graphs. Park and Ihm [8] also studied the problem of strong matching preclusion under the condition that no isolated vertex is created as a result of faults, and established the conditional strong matching preclusion number for the class of restricted hypercube-like graphs, which include most non bipartite hypercube-like networks found in the literature. SMP numbers of augmented cubes, arrangement graphs, alternating group graphs and split-star, pancake graphs, 2-matching composition networks, k-ary n cubes, n-dimensional torus networks, k-composition networks are also investigated; see [1, 9-14].

In this paper, the strong matching preclusion number of the Augmented Butterfly networks is investigated.

2. Augmented Butterfly Networks and Some Lemmas

Butterfly network has many weakness. It is non-Hamiltonian, not pancyclic and its toughness is less than one. But augmented butterfly network retains most of the favorable properties of the butterfly network.
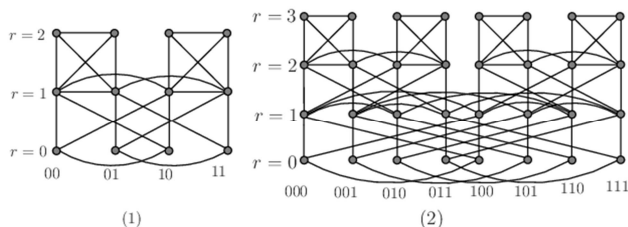

Figure 1. AB_2 and AB_3.

Let $n \geq 1$ be an integer. The vertices of the n-dimensional augmented butterfly network are the pairs (r, x) where r is a non-negative integer $0 \leq r \leq n$ called the level, and $x = (x_1 x_2 \cdots x_n)$ is a binary string of length n. In AB_n the vertex $(r, x), 0 \leq r \leq n-1$, is adjacent to the vertices $(r+1, x)$, $(r+1, x_1 x_2 \cdots x_r \overline{x_{r+1}} x_{r+2} \cdots x_n), (r, x_1 x_2 \cdots x_{r-1} \overline{x_r} x_{r+1} \cdots x_n)$ and $(r, x_1 x_2 \cdots x_r \overline{x_{r+1}} x_{r+2} \cdots x_n)$. In particular, when $r = 0$, the vertex $(0, x_1 x_2 x_3 \cdots x_n)$ is adjacent to the vertices $(1, x_1 x_2 x_3 \cdots x_n), (1, \overline{x_1} x_2 x_3 \cdots x_n)$ and $(0, \overline{x_1} x_2 x_3 \cdots x_n)$. Also when $r = n$, $(n, x_1 x_2 x_3 \cdots x_n)$ is adjacent to the vertices $(n, x_1 x_2 x_3 \cdots \overline{x_n}), (n-1, x_1 x_2 x_3 \cdots x_n)$ and $(n-1, x_1 x_2 x_3 \cdots \overline{x_n})$.

Clearly AB_n has $(n+1)2^n$ vertices and $3n \times 2^n$ edges [4]. AB_2 and AB_3 are shown in Figure 1(1) and Figure 1(2) respectively.

Let $r \in \{0, 1, 2, \ldots, n\}$, we define V_r be the set of vertices of the form (r, x), that is, the vertices in layer r. Let $i \in \{0, 1\}$. Define AB_n^i be the subgraph of AB_n induced by vertices of the form (r, ix') where $1 \leq r \leq n$ and x' is a binary string of length $n-1$. Note that AB_n^i is isomorphic to AB_{n-1}. Thus for any $AB_n^i (i = 0, 1)$, let $j \in \{0, 1\}$, the subgraph induced by the vertices of the form (r, ijx'') where $2 \leq r \leq n$ and x'' is a binary string of length $n-2$, denoted by AB_n^{ij}, which is isomorphic to AB_{n-2}.

The edges between (r, x) and $(r, x_1 x_2 \cdots x_{r-1} \overline{x_r} x_{r+1} \cdots x_n), 0 \leq r \leq n$ and between (r, x) and $(r, x_1 x_2 \cdots x_r \overline{x_{r+1}} \cdots x_n), 0 \leq r \leq n$ are called level edges. The edges between (r, x) and $(r+1, x)$ are called straight edges while the edges between (r, x) and $(r+1, x_1 x_2 \cdots x_r \overline{x_{r+1}} \cdots x_n)$ are called cross edges.

For complete graphs, Park and Ihm [7] derived the following results.

Lemma 2-1 [7]

For complete graph K_n, $smp(K_n) = n-1$.

The conditional matching preclusion number of a graph G, denoted by $mp_1(G)$, with n vertices is the minimum number of edges whose deletion results in graph without an isolated vertex and does not have a perfect matching if n is even, or almost-perfect matching if n is odd. The conditional matching preclusion number for the Augmented Butterfly Network is given in a study [5].

Lemma 2-2

Let G be the Augmented Butterfly Network AB_n, then $mp_1(G) = 3$.

3. Strong Matching Preclusion of Augmented Butterfly Network

Lemma 3-1

For any two vertices of AB_2, say u, v, $AB_2 - \{u, v\}$ has a perfect matching if and only if $\{u, v\} \neq \{(1, 00), (1, 11)\}$ and $\{(1, 01), (1, 10)\}$.

Proof. We can assume, without loss of generality, that $\{u, v\} = \{(1, 00), (1, 11)\}$, then $AB_2 - \{u, v\}$ has two components with odd vertices, clearly $AB_2 - \{u, v\}$ has no perfect matching. It suffices to prove that $AB_2 - \{u, v\}$ has a perfect matching if $\{u, v\} \neq \{(1, 00), (1, 11)\}$ and $\{(1, 01), (1, 10)\}$. By the construction of AB_2, we can check the result easily, thus we omit the remainder proof here.

From Lemma3-1, we know that for any vertex of AB_2, say u, and any vertex in layer 0, say v, there is a perfect matching of $AB_2 - \{u, v\}$, say M, then $M \cup \{v\}$ is an almost perfect matching of $AB_2 - u$. That means there exists an almost perfect matching of $AB_2 - u$ such that any vertex in layer 0 can not matched.

Let $F = F_V \cup F_E$, $F_V \subseteq V(AB_n)$ and $F_E \subseteq E(AB_n)$.

Theorem 3-1

$smp(AB_2) = 2$.

Proof. From Lemma 3-1, $AB_2 - \{(1, 00), (1, 11)\}$ or $AB_2 - \{(1, 01), (1, 10)\}$ has no perfect matching, then $smp(AB_2) \leq 2$. We need to show $smp(AB_2) \geq 2$. It suffices to prove that for every F and $|F| = 1$, $AB_2 - F$ has a perfect matching or an almost perfect matching. If $F = \{v\}$, as AB_2 has a perfect matching, $AB_2 - F$ must have an almost perfect matching. If $F = \{e\}$, from Lemma 2-1, we know $AB_2 - F$ must have a perfect matching. Thus $smp(AB_2) \geq 2$, as desired.

Theorem 3-2

For AB_3, $smp(AB_3) = 3$.

Proof As $\delta(AB_n) = 3$, $smp(AB_3) \leq 3$. We need to show $smp(AB_3) \geq 3$. It suffices prove that for each F and $|F| = 2$, $AB_3 - F$ has a perfect matching or an almost perfect matching. If F consists of two edges or one vertex and one edge, from Lemma 2-2, $AB_3 - F$ has a perfect matching or an almost perfect matching. So we only need to consider that F consists of two vertices, say $F = \{v_1, v_2\}$, and complete the proof by distinguishing the following four cases.

Case 1. Both v_1 and v_2 are in the same $AB_3^i(i = 0, 1)$.

Without loss of generality, suppose $v_1, v_2 \in V(AB_3^0)$. From Lemma 3-1, we know that if $\{v_1, v_2\} \neq \{(2, 000), (2, 011)\}$ and $\{v_1, v_2\} \neq \{(2, 001), (2, 010)\}$, then $AB_3^0 - \{v_1, v_2\}$ has a perfect matching, say M^0, and AB_3^1 has a perfect matching, say M^1, then $M = M^0 \cup M^1 \cup (0, 0x_2x_3)(0, 1x_2x_3)$ (x_2x_3 is a binary string of length 2) is a perfect matching of $AB_3 - F$. Now suppose $\{v_1, v_2\} = \{(2, 000), (2, 011)\}$, as well as the proof of $\{v_1, v_2\} = \{(2, 001), (2, 010)\}$. From Lemma 3-1, $AB_3 - \{(1, 100), (1, 111)\}$ has a perfect matching, say M^1. Let $M^0 = (3, 000)(3, 001) \cup (3, 010)(3, 011)$, and M^2 be the set of all level edges in layer 0. Then $M = M^0 \cup M^1 \cup M^2 \cup (2, 011)(1, 001) \cup (2, 010)(1, 010) \cup (1, 000)(1, 100) \cup (1, 011)(1, 111)$ is a perfect matching of $AB_3 - F$.

Case 2. v_1 and v_2 are in different $AB_3^i(i = 0, 1)$.

Without loss of generality, suppose $v_1 \in V(AB_3^0)$ and $v_2 \in V(AB_3^1)$. We can choose a vertex u, $u \in V_1$, $u \in V(AB_3^0), u \neq v_1$ such that u is not adjacent to v_2, let u'

be the neighbor of u in V_1 and $u' \in AB_3^1$. By Lemma 3-1, $AB_3^0 - \{v_1, u\}$ has a perfect matching, say M^0, and $AB_3^1 - \{v_2, u'\}$ has a perfect matching, say M^1. All level edges of layer 0 induce a perfect matching of V_0, denoted by M^2. Then $M = M^0 \cup M^1 \cup M^2 \cup (uu')$ is a perfect matching of $AB_3 - \{v_1, v_2\}$.

Case 3. $v_1 \in V(AB_3^i)(i = 0, 1)$ and v_2 is in layer 0, that is $v_2 \in V_0$.

Suppose $v_1 \in V(AB_3^0)$, $v_2 = (0, x_1x_2x_3)$, $(x_1x_2x_3)$ is a binary string of length 3. Without loss of generality, assume $x_1 = 0$, and the proof can be obtained similarly if $x_1 = 1$. If $v_1 = (1, x_1x_2x_3)$ or $(1, \overline{x_1}x_2x_3)$, this means v_1 is adjacent to v_2, say $v_1 = (1, x_1x_2x_3)$. All straight edges between layer 2 and layer 3 induce a perfect matching of $V_2 \cup V_3$, denoted by M_0. And $M_1 = \{(0, x_1x_2x_3)(1, x_1x_2x_3)\} \setminus (v_1v_2)$, thus $M = M_0 \cup M_1$ is a perfect matching of $AB_3 - \{v_1, v_2\}$.

If $v_1 \neq (1, x_1x_2x_3)$ and $(1, \overline{x_1}x_2x_3)$, this means v_1 is not adjacent to v_2. We know that $(0, \overline{x_1}x_2x_3)$ is adjacent to v_2 in layer 0, and $(1, x_1x_2x_3)$ is adjacent to both $(0, \overline{x_1}x_2x_3)$ and v_2. From Theorem 3-1, $AB_3^0 - \{v_1, (1, x_1x_2x_3)\}$ has a perfect matching, say M^0, and AB_3^1 has a perfect matching, say M^1. Let M^2 be a perfect matching of $V_0 - \{v_2, (0, \overline{x_1}x_2x_3)\}$ induced by all level edges of layer 0 except the edge $(v_2, (0, \overline{x_1}x_2x_3))$, thus $M = M^0 \cup M^1 \cup M^2 \cup ((0, \overline{x_1}x_2x_3)(1, x_1x_2x_3))$ is a perfect matching of $AB_3 - \{v_1, v_2\}$.

Case 4. Both v_1 and v_2 are in layer 0.

Let $v_1 = (0, x_1x_2x_3)$, $v_2 = (0, y_1y_2y_3)$, $(x_1x_2x_3)$ and $(y_1y_2y_3)$ is a binary string of length 3. If $(y_1y_2y_3) = (\overline{x_1}x_2x_3)$, implying that v_1 is adjacent to v_2. As AB_3^0 and AB_3^1 have a perfect matching respectively, say M^0 and M^1. The level edges of layer 0 except (v_1v_2) induce a perfect of $V_0 - \{v_1, v_2\}$, denoted by M^2. Then $M = M^0 \cup M^1 \cup M^2 \cup (v_1v_2)$ is a perfect matching of $AB_3 - \{v_1, v_2\}$.

If $(y_1y_2y_3) \neq (\overline{x_1}x_2x_3)$, that is v_1 isn't adjacent to v_2. Let u_1 and u_2 be the neighbors of v_1 and v_2 in layer 0 respectively, and u_1^1, u_2^1 be the neighbors of u_1 and u_2 of AB_3^1 in layer 1 respectively. And from Theorem 3-1, $AB_3^1 - \{u_1^1, u_2^1\}$ has a perfect matching, say M^1, AB_3^0 has a perfect matching, say M^0. The level edges of layer 0 except $\{(v_1u_1), (v_2u_2)\}$ induce a perfect matching of

$V_0 - \{v_1, v_2, u_1, u_2\}$, denoted by M^2.

Thus $M = M^0 \cup M^1 \cup M^2 \cup \{(u_1 u_1^1), (u_2 u_2^1)\}$ is a perfect matching of $AB_3 - \{v_1, v_2\}$, completing the proof.

Theorem 3-3

If $n \geq 4$, $smp(AB_n) = 3$.

Proof. We complete the proof by induction on n. The result is true for $n = 3$ by Theorem 3-2. Assume that the result is true for AB_{n-1}. As $\delta(AB_n) = 3$, $smp(AB_n) \leq 3$. We need to show $smp(AB_n) \geq 3$. It suffices to prove that for each F and $|F| = 2$, $AB_n - F$ has a perfect matching or an almost perfect matching. If F contains at least one edge, that is F contains two edges or one edge and one vertex, then by Lemma 2-2, $AB_n - F$ has a perfect matching or an almost perfect matching. Now we only need to consider that F consists of two vertices, say $F = \{v_1, v_2\}$, and complete the proof by the following cases.

Case 1. v_1 and v_2 are in different $AB_n^i (i = 0, 1)$.

Without loss of generality, let $v_1 \in V(AB_n^0)$ and $v_2 \in V(AB_n^1)$. We can choose a vertex of V_1, say u, $u \in V(AB_n^0), u \neq v_1$ and u is not adjacent to v_2, let u' be the vertex adjacent to u in layer 1. By induction hypothesis, $AB_{n-1}^0 - \{v_1, u\}$ has a perfect matching, say M^0, and $AB_{n-1}^1 - \{v_2, u'\}$ has a perfect matching, say M^1. Denote the perfect matching of V_0 induced by all the level edges in layer 0 by M^2. Then $M = M^0 \cup M^1 \cup M^1 \cup M^2 \cup (uu')$ is a perfect matching of $AB_n - \{v_1, v_2\}$.

Case 2. $v_1 \in V(AB_n^i)(i = 0, 1)$ and v_2 is in layer 0.

Suppose $v_1 \in V(AB_n^0)$, $v_2 = (0, x_1 x_2 \cdots x_n)$. If v_1 is adjacent to v_2. Denote a perfect matching of AB_n^{ij} by $M^{ij}(i, j = 0, 1)$, and let M^2 be the perfect matching of $V_0 \cup V_1 - \{v_1, v_2\}$ induced by the straight edges between layer 0 and layer 1 except the edge $(v_1 v_2)$. Then $M = M^{00} \cup M^{01} \cup M^{10} \cup M^{11} \cup M^2$ is a perfect matching of $AB_n - \{v_1, v_2\}$.

If v_1 is not adjacent to v_2. There is a vertex in layer 0 which is adjacent to v_2, say v_2', and the vertex of AB_n^0 in layer 1 which is adjacent to v_2, denoted by v_2''. By induction hypothesis, $AB_n^0 - \{v_1, v_2''\}$ has a perfect matching, say M^0. Choose a perfect matching of AB_n^1, say M^1. Let M^2 be the perfect matching of $V_0 - \{v_2, v_2'\}$ induced by the level edges in layer 0 except $(v_2 v_2')$. Then $M = M^0 \cup M^1 \cup M^2 \cup (v_2' v_2'')$ is a perfect matching of $AB_n - \{v_1, v_2\}$.

Case 3. Both v_1 and v_2 are in layer 0.

Suppose $v_1 = (0, x_1 x_2 \cdots x_n)$ and $v_2 = (0, y_1 y_2 \cdots y_n)$. If $(y_1 y_2 \cdots y_n) = (\overline{x_1} x_2 \cdots x_n)$, that means v_1 is adjacent to v_2, then $AB_n^i (i = 0, 1)$ has a perfect matching, say M^i. All level edges in layer 0 except $(v_1 v_2)$ induce a perfect matching of $V_0 - \{v_1, v_2\}$, denoted by M^2. Thus $M = M^0 \cup M^1 \cup M^2$ is a perfect matching of $AB_n - \{v_1, v_2\}$.

If $(y_1 y_2 \cdots y_n) \neq (\overline{x_1} x_2 \cdots x_n)$, that means v_1 isn't adjacent to v_2. Let $v_1' = (0, \overline{x_1} x_2 \cdots x_n)$ and $v_2' = (0, \overline{y_1} y_2 \cdots y_n)$, and $v_i''(i = 1, 2)$ is the vertex of AB_n^0 in layer 1 which is adjacent to $v_i'(i = 1, 2)$. From induction hypothesis, $AB_n^0 - \{v_1'', v_2''\}$ has a perfect matching, say M^0, and AB_n^1 has a perfect matching, say M^1. Let M^2 be the perfect matching of $V_0 - \{v_1, v_2, v_1', v_2'\}$ induced by the level edges of layer 0 except $(v_1 v_1')$ and $(v_2 v_2')$. Thus $M = M^0 \cup M^1 \cup M^2 \cup (v_1' v_1'') \cup (v_2' v_2'')$ is a perfect matching of $AB_n - \{v_1, v_2\}$, completing the proof.

4. Conclusions

The problem of investigating the strong matching preclusion of interconnection networks is very difficult as strong matching preclusion that additionally permits more destructive vertex faults in a graph is a more extensive form of the original matching preclusion that assumes only edge faults. Augmented Butterfly networks is an important network, in this paper, the strong matching preclusion number for n-dimension Augmented Butterfly networks is obtained by induction hypothesis. Furthermore, determining the strong matching preclusion number for other interconnection networks are under study.

Acknowledgements

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