American Journal of Applied Mathematics
2023; 11(1): 1-6
http://www.sciencepublishinggroup.com/j/ajam
doi: 10.11648/j.ajam.20231101.11
ISSN: 2330-0043 (Print); ISSN: 2330-006X (Online)

Qualitative Study of a Novel Nonlinear Difference Equation of a General Order

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To cite this article:
Ahmed Sayed Etman, Ahmed Essam Hammad. Qualitative Study of a Novel Nonlinear Difference Equation of a General Order. *American Journal of Applied Mathematics*. Vol. 11, No. 1, 2023, pp. 1-6. doi: 10.11648/j.ajam.20231101.11

Received: November 30, 2022; **Accepted**: December 26, 2022; **Published**: January 10, 2023

Abstract: Difference equations play a key role in analyzing many natural phenomena. Difference equations have many applications in different areas such as economic, biological physics, engineering, ecology, physiology, population dynamics and social sciences. Difference equations could also be used to simplify the dynamical systems represented by differential equations. So there exist rapid interest in investing the dynamics of the solutions of the difference equations. There exist different forms of difference equations including rational, nonlinear, max type and system of difference equations. In this paper, a novel nonlinear difference equation of general order is introduced and some qualitative properties of its solutions are studied. The parameters and the initial conditions of the difference equation are assumed to be positive real numbers. New results concerning the periodicity, semicycles, boundedness and global asymptotically stability are established. We prove that the proposed difference equation has unique positive equilibrium point. The periodic solutions with period two are studied. The semicycle analysis of the proposed difference equation is provided. The boundedness of the solutions is investigated. We give upper and lower bounds on the solutions in terms of the parameters of the proposed difference equation. Moreover, the local and global stability are investigated. Some numerical examples are provided to illustrate our results. The proposed difference equation is of general order, so the obtained results could be used for many difference equations.

Keywords: Difference Equations, Qualitative Properties, Periodicity, Semicycle, Boundedness, Global Stability

1. Introduction

Difference equations arise when discrete values of independent variables implicate. Difference equations play a crucial role in mathematical models that describe real-life situations in different fields such as physics, economy, engineering, biology, ecology, physiology, and population dynamics. So there exist rapid interest in investing the dynamics of the solutions of the difference equations. The dynamics of some nonlinear difference equations are investigated in the studies conducted by (Kocic, Ladas, Abd El-Moneam, Alotaibi, Zayed, Elsayed, Alofi, Khan, Ibrahim, Cinar, Jafar, and Saleh) [1-10]. The qualitative properties of some rational difference equations are studied by (Abd El-Moneam, Zayed, Alamoudy, Hamza, Khalaf-Allah, Kulenović, Ladas, Prokup, Zayed, Elsayed, Yingchao, Cuiping, Amleh, and Camouzis) [11-20]. Periodicity of the solutions of difference equations are investigated in the studies conducted by (Grove, Ladas, and Khalaf-Allah) [21, 22]. Solutions of some difference equations are presented by (Elsayed, Din, Bukhary, Halim, and Bayram) [23- 25]. Some Max type difference equations are studied by (Stević, Ibrahim, and Touafek) [26, 27]. Systems of difference equations are investigated in the studies conducted by (Phong, Elsayed, Khan, Qureshi, and Stević) [28-34]. Some open problems in difference equations are suggested by (Kulenović, Ladas and Camouzis) [35, 36]. Amleh et al. [37] studied the dynamic properties of the solutions of the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n} \tag{1}$$

where α is a nonnegative real number. Saleh et al. [38] extended equation (1) to the following higher order difference equation

$$y_{n+1} = A + \frac{y_{n-k}}{y_n} \tag{2}$$

Hamza et al. [39] made another extension to equation (1) and studied the dynamics of the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n^k} \tag{3}$$

where α, k and the initial values are positive real numbers. A further extension is performed by Yalcinkaya [40] who discussed the periodic solutions, bounded solutions, and the global stability of solutions of the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k} \tag{4}$$

The above studies motivated us to study the dynamics of the nonlinear difference equation

$$x_{n+1} = 1 + A\frac{x_n}{x_{n-m}^k} \tag{5}$$

with A and the initial conditions are positive numbers and $m, k \in \{2, 3, 4, \cdots\}$. We investigate the periodicity, semicycles, boundedness and global asymptotic stability.

Now, we show some important definitions and theorems which will be used in proving some of the obtained results.

Definition 1.1. A difference equation of order m+1 is an equation that has the form

$$x_{n+1} = g(x_n, x_{n-1}, \ldots\ldots, x_{n-m}) \tag{6}$$

where g is a function which maps I^{m+1}into I. The set I may be a subset of real numbers or integer numbers. A sequence $\{x_n\}_{n=-m}^{\infty}$that satisfies equation (6) for all $n \geq 0$ is called a solution for Eq. (6). If g is a continuously differentiable function, then for any set of initial conditions, equation (6) has a unique solution $\{x_n\}_{n=-m}^{\infty}$ [1].

Definition 1.2. The equilibrium point or fixed point of equation (6) is a point $\bar{x} \in I$ that satisfies

$$\bar{x} = g(\bar{x}, \bar{x}, \ldots\ldots, \bar{x})$$

The sequence $x_n = \bar{x}$ for all $n \geq -m$ is called an equilibrium solution for equation (6)

Definition 1.3. The linearized equation of equation (6) about the equilibrium point $\bar{x}$ is the difference equation

$$z_{n+1} = \sum_{i=0}^{m} \frac{\partial g(\bar{x}, \bar{x}, \ldots\ldots, \bar{x})}{\partial x_{n-i}} z_{n-i} \tag{7}$$

Definition 1.4. An equilibrium point or fixed point $(\bar{x})$ of equation (6) could be Locally stable point, locally asymptotically stable, global attractor, globally asymptotically stable or unstable according to the following conditions

(i) $\bar{x}$ is locally stable if for every $\epsilon_1 > 0$ there exists $\epsilon_2 > 0$ such that if $\{x_n\}_{n=-m}^{\infty}$ is a solution of equation (6) and $\sum_{i=0}^{m} |x_{-i} - \bar{x}| < \epsilon_2$ then $|x_n - \bar{x}| < \epsilon_1$ for all $n \geq -m$.

(ii) $\bar{x}$ is locally asymptotically stable if it is locally stable and there exist $\epsilon > 0$ such that if $\{x_n\}_{n=-m}^{\infty}$ is a solution of equation (6) with $\sum_{i=0}^{m} |x_{-i} - \bar{x}| < \epsilon$ then $\lim_{n\to\infty} x_n = \bar{x}$.

(iii) $\bar{x}$ is a global attractor if for every solution $\{x_n\}_{n=-m}^{\infty}$ of equation (6) we have $\lim_{n\to\infty} x_n = \bar{x}$.

(iv) $\bar{x}$ is a globally asymptotically stable if it satisfies both conditions in (i) and (iii).

(v) $\bar{x}$ is unstable if it is not locally stable.

Definition 1.5. A sequence $\{x_n\}_{n=-m}^{\infty}$ is called periodic with period N if $x_{n+N} = x_n$ for all $n \geq -m$.

Definition 1.6. A positive semicycle of a solution $\{x_n\}_{n=-m}^{\infty}$ of equation (6) is a subsequence $\{x_{ps}, x_{ps+1}, \ldots\ldots, x_t\}$ of the solution. All terms in this subsequence are greater than or equal to the equilibrium point $\bar{x}$, with $ps \geq -m$ and $t \leq \infty$ such that

either $ps = -m$ or $ps > -m$ and $x_{ps-1} < \bar{x}$

and

either $t = \infty$ or $t < \infty$ and $x_{t+1} < \bar{x}$

A negative semicycle of a solution $\{x_n\}_{n=-m}^{\infty}$ of equation (6) is a subsequence $\{x_{ns}, x_{ns+1}, \ldots\ldots, x_t\}$ of the solution. All terms in this subsequence are less than the equilibrium point $\bar{x}$, with $ns \geq -m$ and $t \leq \infty$ such that

either $ns = -m$ or $ns > -m$ and $x_{ns-1} \geq \bar{x}$

and

either $t = \infty$ or $t < \infty$ and $x_{t+1} \geq \bar{x}$

Theorem 1.1. Assume that $f(x)$ is a polynomial function whose number of variations in sign is v then the number of positive real zeros of $f(x)$ (counting the multiplicity) equals one of the numbers in the set $\{v, v-2, v-4, \ldots.\}$ [41].

Theorem 1.2. A sufficient condition for the asymptotic stability of the difference equation $z_{n+m} + \rho_1 z_{n+m-1} + \cdots + \rho_m z_n = 0, n = 0, 1, 2, \ldots..$ is $\sum_{i=1}^{m} |\rho_i| < 1$ [1].

Theorem 1.3. Let $g: I^{m+1} \to I$ be a continuous function that satisfies

$g(u_1, u_2, \ldots, u_{m+1})$ is non decreasing in u_1 and is non increasing in $u_2, \ldots, u_{m+1}$

If $(d, D) \in I \times I$ is a solution of the system

$D = g(D, d, d, \ldots, d)$ and $d = g(d, D, D, \ldots, D)$ then $d = D$.

Then $x_{n+1} = g(x_n, x_{n-1}, \ldots\ldots, x_{n-m})$ has unique equilibrium point $\bar{x}$ and every solution converges to it.

In section 2, we show that there exists a unique positive equilibrium point for equation (5). We investigate the periodic solutions with period two in section 3. Semicycle analysis is discussed in section 4. Section 5 is devoted to handle the boundedness of the solutions. Stability analysis is studied in section 6. Some numerical examples are provided in section 7. Finally, the paper is concluded in section 8.

2. Equilibrium Point

The equilibrium points of Equation (5) satisfy $\bar{x} = 1 + A\frac{\bar{x}}{\bar{x}^k}$. Rearranging the terms, we can get equation (8)

$$(\bar{x})^k - (\bar{x})^{k-1} - A = 0 \tag{8}$$

Equation (8) has only one change in sign. Using Theorem 1.1, equation (8) has unique positive solution. So, equation (5) has unique positive equilibrium point.

3. Periodicity of the Solutions

This section is devoted to study the periodic solutions with period two.

Theorem 3.1. equation (5) has no periodic solutions with period two unless m is even and $k \geq 3$.

Proof. Assume that there exists a periodic solution with period two for equation (5): $\ldots\ldots, a, b, a, b, a, b, \ldots\ldots$ where a, b are distinct positive real numbers. There are two cases.

Case 1: m is odd. Then from equation (5), we can write

$$a = 1 + A\frac{b}{a^k}, b = 1 + A\frac{a}{b^k} \tag{9}$$

Let $a = b\,q$ where $q \in (0, \infty) - \{1\}$ and substitute in equation (9) to get

$$bq = 1 + \frac{A}{q^k b^{k-1}}, b = 1 + \frac{Aq}{b^{k-1}} \tag{10}$$

From equation (10) we can write

$$b(q-1) = \frac{A}{b^{k-1}}\left(\frac{1-q^{k+1}}{q^k}\right) \tag{11}$$

The sides of equation (11) have different signs which leads to contradiction and so equation (5) has no periodic solutions with period two in this case.

Case 2: m is even. Then from equation (5), we can write

$$a = 1 + \frac{A}{b^{k-1}}, b = 1 + \frac{A}{a^{k-1}} \tag{12}$$

From equation (12) we can write

$$A = b^{k-1}(a-1) = a^{k-1}(b-1) \tag{13}$$

Let $a = b\,q$ where $q \in (0, \infty) - \{1\}$ and substitute in equation (13) to get

$$q^{k-1} - \frac{b}{b-1}q + \frac{1}{b-1} = 0 \tag{14}$$

For $k = 2$, equation (14) has a unique solution $(q = 1)$ and so equation (5) has no periodic solutions with period two. For $k \geq 3$, equation (14) has two positive solutions and so there exist another solution for equation (14) other than $(q = 1)$. So equation (5) has two periodic solutions for even m and $k \geq 3$.

4. Semicycle Analysis

This section is devoted to deal with the semicycle analysis of equation (5).

Theorem 4.1. If a solution of equation (5) has a semicycle of length m then it will have a semicycle of length at least m+1.

Proof. Consider first a positive semicycle. Let $\{x_n\}_{n=-m}^{\infty}$be a solution of equation (5) that has a positive semicycle with only m terms. Assume that the semicycle starts at x_{ps}. So, we have

$$x_{ps}, x_{ps+1}, \ldots\ldots, x_{ps+m-1} \geq \bar{x}$$

Also

$$x_{ps+m} = 1 + A\frac{x_{ps+m-1}}{x_{ps-1}^k} \geq \bar{x} \tag{15}$$

So, the length of the positive semicycle is at least m+1.

Then we consider a negative semicycle. Let $\{x_n\}_{n=-m}^{\infty}$be a solution of equation (5) that has a negative semicycle with only m terms. Assume that the semicycle starts at x_{ns}. So, we have

$$x_{ns}, x_{ns+1}, \ldots\ldots, x_{ns+m-1} < \bar{x}$$

Also

$$x_{ns+m} = 1 + A\frac{x_{ns+m-1}}{x_{ns-1}^k} < \bar{x} \tag{16}$$

So, the length of the negative semicycle is at least m+1. So, if a solution of equation (5) has a semicycle of length m then this semicycle will be of length at least m+1.

5. Boundedness of the Solutions

In this section, we illustrate the boundedness of the solutions of equation (5) through the following theorem.

Theorem 5.1. Every solution of Eq. (5) is bounded such that $1 < x_n < 1 + A(1 + A)^m$ for $n \geq 2m + 2$.

Proof. From equation (5), since A and x_j (for all $j \geq -m$) are positive, $x_n > 1$ (for all $n \geq 1$). On the other side, from equation (5) we have

$$x_n = 1 + A\frac{x_{n-1}}{x_{n-m-1}^k}$$

$$x_n = 1 + \frac{A}{x_{n-m-1}^{k-1}}\left(\frac{x_{n-1}}{x_{n-2}}\frac{x_{n-2}}{x_{n-3}}\frac{x_{n-3}}{x_{n-4}}\ldots\ldots\ldots.\frac{x_{n-m}}{x_{n-m-1}}\right)$$

From equation (5) and the fact $x_{n-m-1} > 1$ for all $n \geq m + 2$, the following inequality can be deduced

$$x_n < 1 + A\left(\frac{x_{n-1}}{x_{n-2}}\frac{x_{n-2}}{x_{n-3}}\frac{x_{n-3}}{x_{n-4}}\ldots\ldots\ldots.\frac{x_{n-m}}{x_{n-m-1}}\right)$$

From equation (5), we can get

$$\frac{x_i}{x_{i-1}} = \frac{1}{x_{i-1}} + \frac{A}{x_{i-m-1}^k} < 1 + A, i \geq m + 2$$

So, we obtain

$x_n < 1 + A(1 + A)^m, n \geq 2m + 2$. So, the proof of *Theorem* 5.1 is completed.

6. Stability Analysis

In this section, we study the local stability then the global stability. Let $g: I^{m+1} \to I$ be defined by

$$g(x_n, x_{n-1}, \dots\dots, x_{n-m}) = 1 + A\frac{x_n}{x_{n-m}^k}$$

so we have

$$\left.\frac{\partial g}{\partial x_n}\right|_{(x_n,x_{n-1},\dots\dots,x_{n-m})=(\bar{x},\bar{x},\dots\dots,\bar{x})} = \frac{A}{\bar{x}^k}$$

$$\left.\frac{\partial g}{\partial x_{n-1}}\right|_{(x_n,x_{n-1},\dots\dots,x_{n-m})=(\bar{x},\bar{x},\dots\dots,\bar{x})} = \left.\frac{\partial g}{\partial x_{n-2}}\right|_{(x_n,x_{n-1},\dots\dots,x_{n-m})=(\bar{x},\bar{x},\dots\dots,\bar{x})} = \dots = \left.\frac{\partial g}{\partial x_{n-m+1}}\right|_{(x_n,x_{n-1},\dots\dots,x_{n-m})=(\bar{x},\bar{x},\dots\dots,\bar{x})} = 0$$

$$\left.\frac{\partial g}{\partial x_{n-m}}\right|_{(x_n,x_{n-1},\dots\dots,x_{n-m})=(\bar{x},\bar{x},\dots\dots,\bar{x})} = \frac{-k\,A}{\bar{x}^k}$$

so, the linearized equation of equation (5) about its unique positive equilibrium point is

$$z_{n+1} - \frac{A}{\bar{x}^k} z_n + \frac{k\,A}{\bar{x}^k} z_{n-m} = 0 \tag{17}$$

The characteristic equation of Eq. (17) is

$$\lambda^{m+1} - \frac{A}{\bar{x}^k}\lambda^m + \frac{k\,A}{\bar{x}^k} = 0 \tag{18}$$

Theorem 6.1. The equilibrium point $\bar{x}$ is locally asymptotically stable if $A < \frac{(k+1)^{k-1}}{k^k}$

Proof. According to *Theorem* 1.2, the equilibrium point $\bar{x}$ is locally asymptotically stable if $\left|-\frac{A}{\bar{x}^k}\right| + \left|\frac{k\,A}{\bar{x}^k}\right| < 1$. So

$$(k+1)\,A < \bar{x}^k \tag{19}$$

using equation (8), the inequality in (19) will be

$$kA < \bar{x}^{k-1} \tag{20}$$

From equation (8), (20) we can write

$$(kA)^{\frac{k}{k-1}} - kA - A > 0$$

Which implies

$$(kA)^k < (k+1)^{k-1}A^{k-1}$$

So, we can conclude

$$A < \frac{(k+1)^{k-1}}{k^k} \tag{21}$$

Theorem 6.2. If $A < \frac{(k+1)^{k-1}}{k^k}$, the equilibrium point $\bar{x}$ is globally asymptotically stable.

Proof. Let $g(u, v) = g(x_n, x_{n-m}) = 1 + A\frac{x_n}{x_{n-m}^k}$

$g(u, v)$is nondecreasing in u and nonincreasing in v. Let (d, D) be a solution of the system

$$d = g(d, D), D = g(D, d)$$

Hence, we obtain that

$$d = 1 + A\frac{d}{D^k}, D = 1 + A\frac{D}{d^k}$$

Therefore, we have $d = D$. According to *Theorem* 1.3, every solution of equation (5) converges to $\bar{x}$. So $\bar{x}$ is globally asymptotically stable.

7. Numerical Examples

In this section, some numerical examples are presented to verify the theoretical results.

Example 7.1. Consider equation (5) with m=2, k=3 and $A = 9/2$, we get the following difference equation

$$x_{n+1} = 1 + \frac{9}{2}\frac{x_n}{x_{n-2}^3} \tag{22}$$

Let the initial conditions be $x_{-2} = 1.5, x_{-1} = 3, x_0 = 1.5$. Figure 1 shows the first 30 terms of the solution of equation (22).

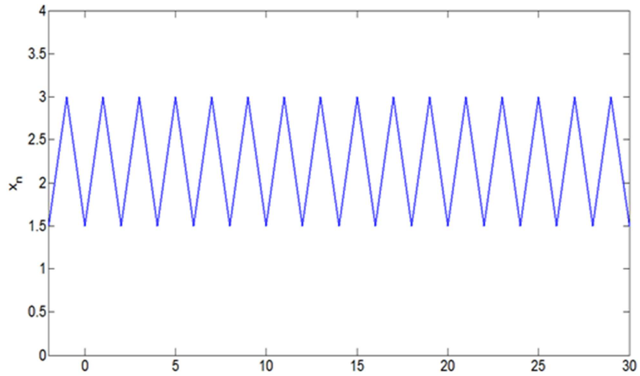

Figure 1. *Solution of equation (22).*

Figure 1 shows that equation (22) has a periodic solution with period two because m is even and $k \geq 3$ which agrees with theorem 1. We can also note that x_n is bounded between 1 and $1 + A(1 + A)^m = 137.125$ for $n \geq 2m + 2 = 6$ which agrees with *Theorem* 5.1.

Example 7.2. Consider equation (5) with m=3, k=4 and $A = 1/2$, we get the following difference equation

$$x_{n+1} = 1 + \frac{1}{2}\frac{x_n}{x_{n-3}^4} \tag{23}$$

Let the initial conditions be $x_{-3} = 3, x_{-2} = 4, x_{-1} = 5, x_0 = 2$. Figure 2 shows the first 300 terms of the solution of equation (23).

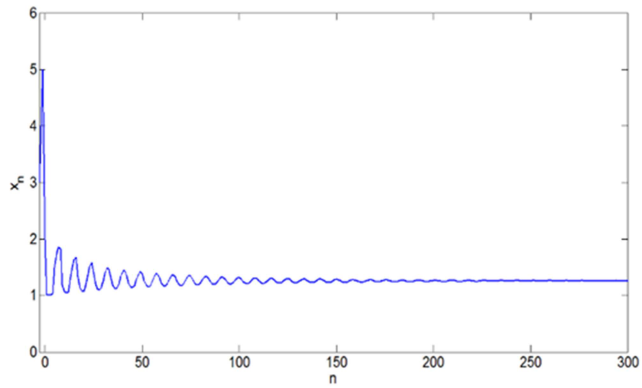

Figure 2. *Solution of equation (23).*

From Figure 2, it is clear that the solution of equation (23) converges to the equilibrium point $\bar{x} = 1.254$. We can also note that x_n is bounded between 1 and $1 + A(1 + A)^m = 2.6875$ for $n \geq 2m + 2 = 8$ which agrees with *Theorem* 5.1.

Example 7.3. Consider equation (5) with m=6, k=5 and $A = 1/3$, we get the following difference equation

$$x_{n+1} = 1 + \frac{1}{3}\frac{x_n}{x_{n-6}^5} \quad (24)$$

Let the initial conditions be $x_{-6} = 3, x_{-5} = 6, x_{-4} = 2, x_{-3} = 1, x_{-2} = 7, x_{-1} = 5, x_0 = 4$. Figure 3 shows the first 300 terms of the solution of equation (24).

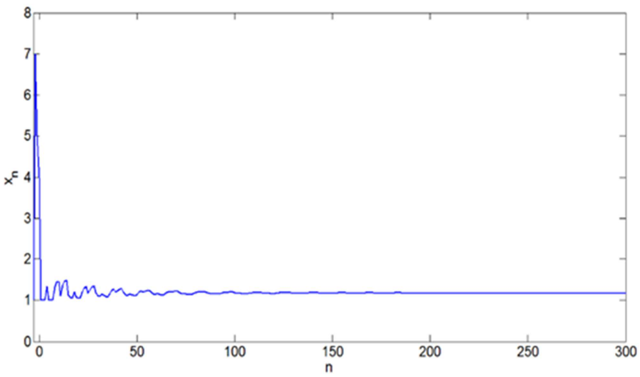

Figure 3. *Solution of equation (24).*

It is clear from Figure 3 that the solution of equation (24) converges to the equilibrium point $\bar{x} = 1.1749$. We can also note that x_n is bounded between 1 and $1 + A(1 + A)^m = 2.8729$ for $n \geq 2m + 2 = 14$ which agrees with *Theorem* 5.1.

8. Conclusion

In this paper, the qualitative properties of a novel nonlinear difference equation of general order are examined. First, we find out that it has unique positive equilibrium point. Then, we obtain the conditions that must be satisfied to make the proposed difference equation has periodic solution with period two. Then, we study the semicycles of the solutions of the difference equation. Moreover, we prove that every solution of the proposed difference equation is bounded. We further obtain the condition of the stability of the equilibrium point. Finally, the theoretical results are verified by some numerical examples. As the order of the proposed difference equation is general, our results could be used for many difference equations. Further, other qualitative properties may be investigated for the proposed difference equation in future research.

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