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A Comparative Study of Numerical Methods for Solving Initial Value Problems (IVP) of Ordinary Differential Equations (ODE)

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Abstract: Numerical methods for solving Ordinary Differential Equations differ in accuracy, performance, and applicability. This paper presents a comparative study of numerical methods, mainly Euler's method, the Runge-Kutta method of order 4^{th} & 6^{th} and the Adams-Bashforth-Moulton method for solving initial value problems in ordinary differential equations. Our aim in this paper is to show that which method gives better accuracy for the initial value problem in numerical methods. Comparisons are made among the direct method, Euler's method, Runge-Kutta fourth and sixth order and the Adams-Bashforth-Moulton method for solving the initial value problem. The comparisons with error analysis are also shown in the graphical and tabular form. MATHEMATICA 5.2 software is used for programming code and solving the particular problems numerically. It is found that the calculated results for a particular problem using the Runge-Kutta fourth order give good agreement with the exact solution, whereas the Runge-Kutta sixth order defers slightly for a particular problem. Approximate solution using the Adams-Bashforth method with error estimation is also investigated. Moreover, we are also investigated of the Euler methods, the Runge-Kutta methods of order 4^{th} & 6^{th} and the Adams-Bashforth method for solving a particular initial value problem. Finally, it is found that the Adams-Bashforth method gives a better approximation result among the others mentioned methods for solving initial value problems in ordinary differential equations.

Keywords: Euler's Method (EM), Runge-Kutta Method of Order Four (RK-4), Runge-Kutta Method of Order Six (RK-6), Adamsh-Bashforth Moulton Method (ABMM)

1. Introduction

The subject of differential equations constitutes a large and very important branch of modern mathematics. From the early days of the calculus the subject has been an area of great theoretical research and practical application, and it continues to be so in our day. Differential equations can describe nearly all systems of undergoing change. They are widespread in science and engineering, as well as economics, social science, biology, business etc. [4, 16]. The analytical methods of solution, with which the reader is assumed to be familiar, can be applied to solve only a selected class of differential equations. Those equations which govern physical systems do not possess, in general closed-form solutions, and hence recourse must be made to numerical methods for solving such differential equations [14]. Differential equations are very important part of many areas of mathematics, from fluid dynamics to celestial mechanics. They are used by applied mathematician, physicists and engineers to help in the designing of everything from bridge to ballistic missiles [12].

Many mathematicians studied the nature of these equations and many complicated systems can be described quite precisely with compact mathematical expressions. However, many systems involving differential equations are so complex or the systems that they describe are so large that a purely analytical solution to the equation is not tractable. Really when we can't solve the mathematical problems by using various algebraic methods and can't get the exact solution or output then we use the concept of the numerical methods [2, 5-9]. Numerical methods are extremely powerful problem-solving tools [15]. It is in this system where computer simulations and numerical approximations are useful [16]. The techniques for solving differential equations based on numerical approximations were developed before programmable computers existed [10]. A differential equation involving ordinary derivatives of one or more dependent variables with respect to a single independent variable is called an ordinary differential equation [13]. The order of the equation is determined by the order of the highest derivative. For example, if the first derivative is the only derivative, the equation is called a first-order ordinary differential equation. In the same way, if the highest derivative is second order, the equation is called a second-order ordinary differential equation [2]. The problem of solving ordinary differential equations is classified into two namely initial value problems and boundary value problems, depending on the conditions at the end points of the domain are specified. Numerical methods for the solution of initial value problems in ordinary differential equations made enormous progress for several reasons [11]. All the conditions of initial value problem are specified at the initial point. There are numerous methods that produce numerical approximations to solution of initial value problem in ordinary differential equations.

This paper presents a comparative study of numerical methods, mainly Euler's method, the Runge-Kutta method of order 4^{th} & 6^{th} and the Adams-Bashforth-Moulton method for solving initial value problems in ordinary differential equations. Our aim in this paper is to show that which method gives better accuracy for the initial value problem in numerical methods.

2. Materials and Methods

Euler's Method, Runge-Kutta Fourth Order Method, Runge-Kutta Sixth Order Method, and Adams-Bashforth-Moulton Method have been applied to analyze the behavior of the solution of first-order ordinary differential equations for initial value problems. Finally, a comparative study has been made among these methods.

2.1. Basic Idea of Euler's Method

Euler's method is the most elementary approximation technique for solving initial-value problems [1]. The object of Euler's method is to obtain approximations to the well-posed initial-value problem

$$\frac{dy}{dx} = f(t,y), a \le t \le b, y(a) = \alpha \quad (1)$$

A continuous approximation to the solution $y(t)$ will not be obtained; instead, approximation to y will be generated at various values, called mesh points, in the interval $[a,b]$. Once the approximate solution is obtained at the points, the approximate solution at other points in the interval can be found by interpolation.

We first make the stipulation that the mesh points are equally distributed throughout the interval $[a,b]$, This condition is ensured by choosing a positive integer N and selecting the mesh points

$$t_i = a + ih, for\ each\ i = 0,1,2,3 \dots, N.$$

The common distance between the points $h = \frac{b-a}{N} = t_{i+1} - t_i$ is called the step size. We use Taylor's Theorem to derive Euler's method. Suppose that $y(t)$, the unique solution to (1), has two continuous derivative on $[a,b]$, so that for each $i = 0,1,2, \dots N-1$,

$$y(t_{i+1}) = y(t_i) + (t_{i+1} - t_i)y'(t_i) + \frac{(t_{i+1}-t_i)^2}{2} y''(\xi_i),$$

$$For\ some\ number\ \xi_i\ in\ (t_i, t_{i+1}). Because\ h = t_{i+1} - t_i, we\ have$$

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2} y''(\xi_i),$$

And, because $y(t)$ satisfies the differential equation (1)

$$y(t_{i+1}) = y(t_i) + hf\big(t_i, y(t_i)\big) + \frac{h^2}{2} y''(\xi_i)$$

Euler's method constructs $\omega_i \approx y(t_i)$, for each $i = 1,2, \dots, N$, by deleting the remainder term.

Thus Euler's method is

$$\omega_0 = \alpha,$$

$$\omega_{i+1} = \omega_i + hf(t_i, \omega_i), for\ each\ i = 0,1,2, \dots, N-1.$$

2.2. Concept of Runge-Kutta Method

Probably one of the more popular as well as most accurate, numerical procedures used in approximate solutions to a first order initial–value problem $y' = f(x,y)$, $y(x_0) = y_0$, is the fourth-order Runge-kutta method [3]. As the name suggests, there are Runge-Kutta methods of different orders.

Fundamentally, all Runge-Kutta methods are generalizations of the basic Euler formula $y_{n+1} = y_n + hf(x_n, y_n)$, in that the slope function f is replaced by a weighted average of slopes over the interval defined by $x_n \le x \le x_{n+1}$. That is

$$y_{n+1} = y_n + h(w_1 k_1 + w_2 k_2 + \dots + w_m k_m) \quad (2)$$

Here the weights w_i , i =1,2,…,m are constant that generally satisfy $w_1 + w_2 + \dots + w_m = 1$ and each k_i, i=1,2,…, m is the function f evaluated at a selected point (x,y) for which $x_n \le x \le x_{n+1}$. We shall see that the k_i are defined recursively. The number m is called the order of the method. observe that by taking $m = 1, w_1 = 1, and\ k_1 = f(x_n, y_n)$ we get the familiar Euler formula $y_{n+1} = y_n + hf(x_n, y_n)$. Hence, Euler's method is said to be a first-order Runge-Kutta method. The average is not formed willy-nilly, but parameters are chosen so that (2) agrees with a Taylor polynomial of degree m. As we have seen in the last section if a function $y(x)$ processes $k+1$ derivatives that are continuous on an open interval containing a and x, then we can write

$$y(x) = y(a) + y'(a)\frac{x-a}{1!} + y''(a)\frac{(x-a)^2}{2!} + \cdots + y^{(k+1)}(c)\frac{(x-a)^{k+1}}{(k+1)!},$$

Where c is some number between $a\ and\ b$. If we replace a by $x_{n+1} = x_n + h$, then the foregoing formula becomes

$$y(x_{n+1}) = y(x_n + h) = y(x_n) + hy'(x_n) + \frac{h^2}{2!}y''(x_n) + \ldots + \frac{h^{k+1}}{(k+1)!}\, y^{(k+1)}(c)$$

Where c is now some number between x_n and x_{n+1}. When $y(x)$ is a solution of $y'=f(x,y)$, in the case $k=1$ and the remainder $\frac{1}{2}hy''(c)$ is small, we see that a Taylor Polynomial $y(x_{n+1)} = y(x_n) + hy'(x_n)$ of degree 1 agrees with the approximation formula of Euler's method.

$$y_{n+1} = y_n + hy'_n = y_n + hf(x_{n,}\, y_n).$$

2.3. A Second- Order Runge-Kutta Method

To further illustrate (2), we consider now a second-order Runge-Kutta method. This consists of finding constants, or parameters, w_1, w_2, α and β so that the form

$$y_{n+1} = y_n + h(w_1 k_1 + w_2 k_2), \tag{3}$$

Where $k_1 = f\big(x_{n,}\, y_n\big)$,

$$k_2 = f(x_n + \alpha h, y_n + \beta h k_1)$$

agrees with a Taylor polynomial of degree 2. For our purposes it suffices to say that this can be done whenever the constant satisfy

$$w_1 + w_2 = 1,\ w_2\alpha = \frac{1}{2},\ \text{and}\ w_2\beta = \frac{1}{2}. \tag{4}$$

This is an algebraic system of three equations in four unknown and has infinitely many solutions:

$$w_1 = 1 - w_2,\ \alpha = \frac{1}{2w_2},\ \text{and}\ \beta = \frac{1}{2w_2}, \tag{5}$$

Where $w_2 \neq 0$. For example, the choice $w_2 = \frac{1}{2}$ yields $w_1 = \frac{1}{2}$, α=1, β=1 and so (3) become

$$y_{n+1} = y_n + \frac{h}{2}(k_1 + k_2),$$

Where $k_1 = f(x_{n,}\, y_n)$, and $k_2 = f(x_n + h, y_n + hk_1)$.

Since $x_{n+h} = x_{n+1}$ and $y_n + hk_1 = y_n + hf(x_{n,}\, y_n)$ the foregoing result is recognized to be the improved Euler's method that is summarized in (4) and (5) of Euler's method. In view of the fact that $w_2 \neq 0$ can be chosen arbitrarily in (5), there are many possible second order Runge-Kutta methods [3].

2.4. A Fourth-Order Runge- Kutta Method

A Fourth–Order Runge-Kutta procedure consists of finding parameters so that the formula

$$y_{n+1} = y_n + h(w_1 k_1 + w_2 k_2 + w_3 k_3 + w_4 k_4), \tag{6}$$

Where $k_1 = f(x_n, y_n)$

$$k_2 = f(x_n + \alpha_1 h, y_n + \beta_1 h k_1)$$

$$k_3 = f(x_n + \alpha_2 h, y_n + \beta_2 h k_1 + \beta_3 h k_2)$$

$$k_4 = f(x_n + \alpha_3 h, y_n + \beta_4 h k_1 + \beta_5 h k_2 + \beta_6 h k_3)$$

agrees with a Taylor polynomial of degree 4. This result is a system of 11 equations in 13 unknowns. The most commonly used set of values for the parameters yields the following result:

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

$$k_1 = f(x_n, y_n),$$

$$k_2 = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1) \tag{7}$$

$$k_3 = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_2)$$

$$k_4 = f(x_n + h, y_n + hk_3)$$

While other fourth- order formula are easily derived, the algorithm summarized in (7) is so widely used and recognized as a valuable computational toll it is often referred to as the fourth-order Runge-Kutta method or the classic Runge-Kutta method [3]. Note that k_2 depends on k_1, k_3 depends on k_2, and k_4 depends on k_3. Also, k_2 and k_3 involve approximations to the slope at the midpoint $x_n + \frac{1}{2}h$ of the interval $[x_n, x_{n+1}]$.

2.5. Summary of the Runge-Kutta Method of Order Six (n=6)

$$y_{n+1} = y_n + (9k_1 + 64k_3 + 49k_6 + 9k_7)/180 \tag{8}$$

Where,

$$k_1 = hf(x_{n,}\, y_n)$$

$$k_2 = hf(x_n + h, y_n + k_1)$$

$$k_3 = hf(x_n + \frac{h}{2}, y_n + (3k_1 + k_2)/8)$$

$$k_4 = hf(x_n + \frac{2h}{3}, y_n + 8k_1 + 2k_2 + 8k_3)/27)$$

$$k_5 = hf\left[\left(x_n + \frac{(7-\sqrt{21})\times h}{14}\right), \left(y_n + \frac{3(3\sqrt{21}-7)k_1 - 8(7-\sqrt{21})k_2 + 48(7-\sqrt{21})k_3 - 3(21-\sqrt{21})k_4}{392}\right)\right]$$

$$k_6 = hf[(x_n + \frac{(7-\sqrt{21})\times h}{14}),$$

$$(y_n + \frac{-5(123+51\sqrt{21})k_1 - 40(7+\sqrt{21})k_2 - 320\sqrt{21}k_3 + 3(21+121\sqrt{21})k_4 + 392((6+\sqrt{21})k_5}{1960})]$$

$$k_7 = hf[(x_n + h),$$

$$(y_n + \frac{15(22+7\sqrt{21})k_1 - 120k_2 + 40(7\sqrt{21}-5)k_3 - 63(3\sqrt{21}-2)k_4 - 14(49+9\sqrt{21})k_5 + 70(7-\sqrt{21})k_6}{180})]$$

2.6. Concept of Adams-Bashforth-Moulton Method

The multistep method discussed in this section is called the fourth-order Adams-Bashforth-Moulton method, or a bit more awkwardly, the Adams-Bashforth/Adams-Moulton method. Like the improved Euler's method it is a predictor-corrector method, that is, one formula is used to predict a value y^*_{n+1}, which in turn is used to obtain a corrected value y_{n+1}. The predictor in this method is the Adams-Bashforth formula.

$$y^*_{n+1} = y_n + \frac{h}{24}(55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3}), \quad (9)$$

$$y'_n = f(x_n, y_n)$$

$$y'_{n-1} = f(x_{n-1}, y_{n-1})$$

$$y'_{n-2} = f(x_{n-2}, y_{n-2})$$

$$y'_{n-3} = f(x_{n-3}, y_{n-3})$$

For $n \geq 3$. The value of y^*_{n+1} is then substituted into the Adams-Moulton corrector

$$y_{n+1} = y_n + \frac{h}{24}(9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2}), \quad (10)$$

$$y'_{n+1} = f(x_{n+1}, y^*_{n+1}).$$

Notice that formula (9) requires that we know the values of y_0, y_1, y_2 and y_3 in order to obtain y_4. The value of y_0 is, of course, the given initial condition. Since the local truncation error of the Adams-Bashforth/Adams-Moulton method is $O(h^5)$, the value of y_1, y_2 and y_3 are generally computed by a method with the same error property, such as the fourth-order Runge-Kutta formula [3].

3. Results and Discussion

In this section, we will solve the following initial value problems of Ordinary Differential Equations by using Euler's Method, the Runge-Kutta 4th Order Method, the Runge-Kutta 6th Order Method, and the Adams-Bashforth Moulton Method.

Example 3.1

Consider the initial value problem:

$$y' = 3 - 2t - 0.5y, y(0) = 1, 0 \leq t \leq 1, with\ h = 0.2$$

Find approximate solution using Euler's method.

Solution: For this problem

$$f(t, y) = 3 - 2t - 0.5y$$

$$So\ \omega_0 = y(0) = 1$$

$$\omega_1 = \omega_0 + (0.2)(3 - 0 - 0.5\omega_0)$$

$$= 1 + (0.2)(3 - 0 - 0.5) = 1.5$$

$$\omega_2 = \omega_1 + (0.2)(3 - 2(0.2) - 0.5\omega_1)$$

$$= 1.5 + (0.2)\big(3 - 2(0.2) - 0.5(1.5)\big)$$

$$\approx 1.87$$

$$\omega_3 = \omega_2 + (0.2)(3 - 2(0.4) - 0.5\omega_2)$$

$$\omega_3 = 1.87 + (0.2)\big(3 - 2(0.4) - 0.5(1.87)\big)$$

$$\approx 2.123$$

$$\omega_4 = \omega_3 + (0.2)(3 - 2(0.6) - 0.5\omega_3)$$

$$\omega_4 = 2.123 + (0.2)\big(3 - 2(0.6) - 0.5(2.123)\big)$$

$$\approx 2.2707$$

and

$$y(1) \approx \omega_5 = \omega_4 + (0.2)(3 - 2(0.8) - 0.5\omega_4)$$

$$\omega_5 = 2.2707 + (0.2)\big(3 - 2(0.8) - 0.5(2.2707)\big)$$

$$\approx 2.32363$$

Example 3.2
Consider the initial value problem:
$x'(t) = t^2 - e^x sin(t),\ x(0) = 1$ with $h = 0.25$, Find approximate solution by using Runge-Kutta 4th Order method.
Solution: Here the interval is [0,1], so $a = 0,\ b = 1$. Since $h = \frac{b-a}{n} = 0.25$ we have $n = 4$ and we need to compute $x_1,\ x_2,\ x_3,\ x_4$ starting with $x_0 = x(0) = 1,\ t_0 = 0$
x_1:

$$k_1 = hf(t_0, x_0) = 0.25 * f(0,1) = 0$$

$$k_2 = hf\left(t_0 + \tfrac{1}{2}h, x_0 + \tfrac{1}{2}k_1\right) = 0.25 * f(0.125,1) = -0.08081901556$$

$$k_3 = hf\left(t_0 + \tfrac{1}{2}h, x_0 + \tfrac{1}{2}k_2\right)$$

$$= 0.25 * f(0.125, 0.9595904922)$$

$$= -0.07746356198$$

$$k_4 = hf(t_0 + h, x_0 + k_3)$$

$$= 0.25 * f(0.25,0.922536438) = -0.1399712545$$

$$x_1 = x_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_{4)}$$

$$= 1 + \frac{1}{6}(0 + 2 * (-0.08081901556) + 2 * (-0.07746356198) - 01399712545)$$

$$= 0.923911$$

$$t_1 = t_0 + h = 0 + 0.25 = 0.25$$

x_2:

$$k_1 = hf(t_1, x_1) = 0.25 * f(0.25,0.923911) = -0.1401852783$$

$$k_2 = hf\left(t_1 + \frac{1}{2}h, x_1 + \frac{1}{2}k_1\right)$$

$$= 0.25 * f(0.375,0.8538183609) = -0.1799004407$$

$$k_3 = hf\left(t_1 + \frac{1}{2}h, x_1 + \frac{1}{2}k_2\right)$$

$$= 0.25 * f(0.375,0.8339607796) = -0.1756720567$$

$$k_4 = hf(t_1 + h, x_1 + k_3)$$

$$= 0.25 * f(0.5,0.7482389433) = -0.1907895181$$

$$x_2 = x_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_{4)}$$

$$= 0.923911 + \frac{1}{6}(-0.1401852783 + 2 * (-0.1799004407) + 2 * (-0.1756720567) - 0.1907895181)$$

$$= 0.750224$$

$$t_2 = t_1 + h = 0.25 + 0.25 = 0.5$$

x_3:

$$k_1 = hf(t_2, x_2) = 0.25 * f(0.5, 0.750224) = -0.1912978874$$

$$k_2 = hf\left(t_2 + \frac{1}{2}h, x_2 + \frac{1}{2}k_1\right)$$

$$= 0.25 * f(0.625, 0.6545750563) = -0.183823021$$

$$k_3 = hf\left(t_2 + \frac{1}{2}h, x_2 + \frac{1}{2}k_2\right)$$

$$= 0.25 * f(0.625, 0.6583124895) = -0.1848769993$$

$$k_4 = hf(t_2 + h, x_2 + k_3)$$

$$= 0.25 * (0.75, 0.5653470007) = -0.1593060095$$

$$x_3 = x_2 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 0.750224 + \frac{1}{6}(-0.1912978874 + 2 * (-0.183823021) + 2 * (-0.1848769993) - 0.1593060095)$$

$$= 0.5688900104$$

$$t_3 = t_2 + h = 0.5 + 0.25 = 0.75$$

x_4:

$$k_1 = hf(t_3, x_3) = 0.25 * f(0.75, 0.5688900104) = -0.1603705527$$

$$k_2 = hf\left(t_3 + \frac{1}{2}h, x_3 + \frac{1}{2}k_1\right)$$

$$= 0.25 * f(0.875, 0.4887047341) = -0.1214067385$$

$$k_3 = hf\left(t_3 + \frac{1}{2}h, x_3 + \frac{1}{2}k_2\right)$$

$$= 0.25 * f(0.875, 0.5081866411) = -0.1275606827$$

$$k_4 = hf(t_3 + h, x_3 + k_3)$$

$$= 0.25 * (1.0, 0.4413293277) = -0.07707401776$$

$$x_4 = x_3 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 0.5688900104 + \frac{1}{6}(-0.1603705527 + 2 * (-0.1214067385) + 2 * (-0.1275606827) - 0.07707401776)$$

$$= 0.4463267749$$

So the approximation to $x(1)$ is $x(1) \approx x_4 = 0.446327$.
Example 3.3
Consider the initial value problem:

$$y' = -2x^3 + 12x^2 - 20x + 8.5, 0 \le x \le 4, y(0) = 1$$

Using Runge-Kutta 4th Order method, find approximation and compare with exact.
Solution: Given equation is $y' = -2x^3 + 12x^2 - 20x + 8.5, 0 \le x \le 4, y(0) = 1$
Therefore the analytic solution is $y(x) = 1 + 8.5x - 10x^2 + 4x^3 - 0.5x^4$
Now the complete set of approximation result can be listed in the following Table 1 with exact solution.

Table 1. *Comparison Between Runge-kutta-4th order method with exact solution.*

x	Exact	RK-4	Error RK-4
0.0	1.0	1.0	0.0
0.5	3.21875	3.21875	0.0
1.0	3.0	3.0	0.0
1.5	2.21875	2.21875	0.0
2.0	2.0	2.0	0.0
2.5	2.71875	2.71875	0.0
3.0	4.0	4.0	0.0
3.5	4.71875	4.71875	0.0
4.0	3.0	3.0	0.0

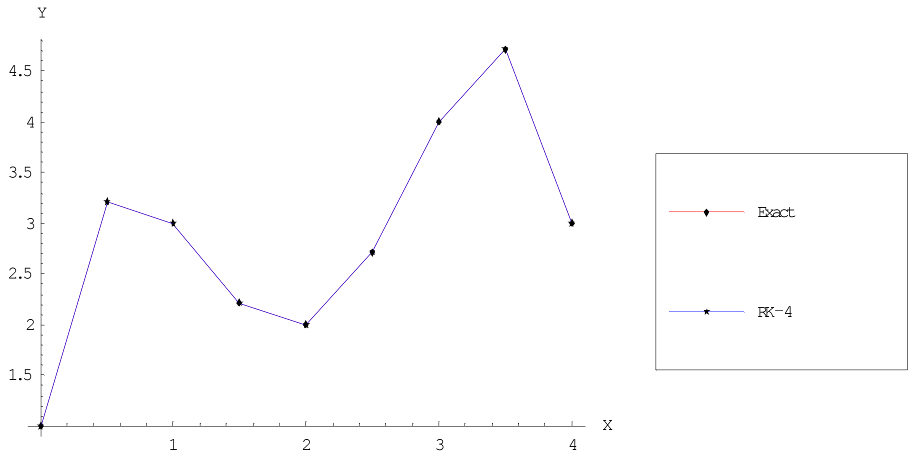

Figure 1. *Compare the approximate solution with exact solution.*

Example 3.4

Consider the initial value problem:

$y' = y - x^2 + 1, 0 \leq x \leq 2, y(0) = 0.5$ Using Runge-Kutta $6th$ Order method, find approximation and compare with exact.

Solution: Given equation is $\frac{dy}{dx} = y - x^2 + 1;\ 0 \leq x \leq 2,$ $y(0) = 0.5$

Therefore the analytic solution is $y(x) = (x+1)^2 - \ 0.5e^x$

Now the complete set of approximation result can be listed in the following Table 2 with exact solution. The approximate solution using Runge-Kutta 6th Order with error finding.

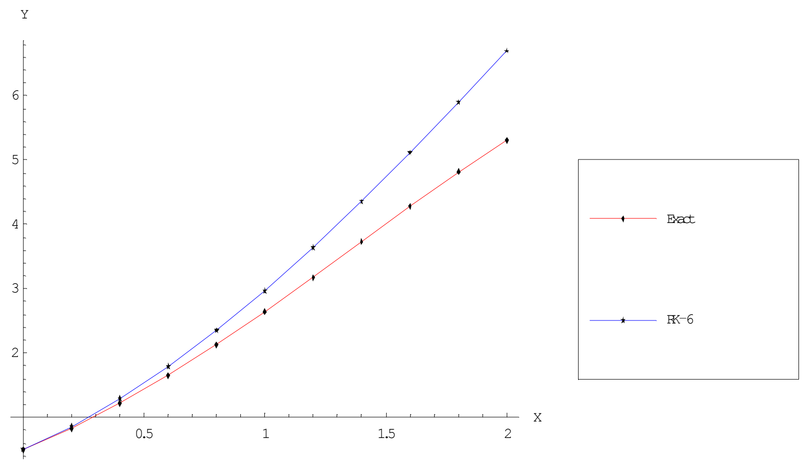

Figure 2. *Compare the approximate solution with exact solution.*

Table 2. *Comparison Between Runge-kutta-6th order method with exact solution.*

x	Exact	RK-6	Error RK-6
0.0	0.5	0.5	0.0
0.2	0.829299	0.861827	0.0325287
0.4	1.21409	1.29246	0.078368
0.6	1.64894	1.78916	0.14022
0.8	2.12723	2.34855	0.221324

x	Exact	RK-6	Error RK-6
1.0	2.64086	2.96642	0.325565
1.2	3.17994	3.63753	0.457593
1.4	3.7324	4.35537	0.622975
1.6	4.28348	5.11185	0.828367
1.8	4.81518	5.8969	1.08172
2.0	5.30547	6.698	1.39253

Example 3.5

Consider the initial value problem:

$$y'(x) = 30 - 5y \text{ with } y(0) = 1, 0 \leq x \leq \frac{13}{5},$$

$$n = 13, h = 0.2$$

Using Adams-Bashforth-Moulton method, find approximation and compare with exact.

Solution: Given equation is $\frac{dy}{dx} = 30 - 5y$ with $y(0) = 1, 0 \leq x \leq \frac{13}{5}$, $n = 13, h = 0.2$

Therefore the analytic solution is $y(x) = 6 - 5e^{-5x}$

Now we generate the approximations listed in the following Table 3 with exact solution. Approximate solution using Adams-Bashforth method and error estimation.

Table 3. *Comparison Between Adams-Bashforth-Moulton method with exact solution.*

x	Exact	Adams-Bashforth	Err. Adams-Bashforth
0.0	1.0	1.0	0.0
0.2	4.1606	4.125	0.0356028
0.4	5.32332	5.29688	0.0264486
0.6	5.75106	5.73633	0.0147365
0.8	5.90842	6.01633	0.107905
1.0	5.96631	6.08593	0.119615
1.2	5.98761	6.0053	0.017693
1.4	5.99544	5.9882	0.00724349
1.6	5.99832	6.03184	0.0335195
1.8	5.99938	6.02124	0.0218558
2.0	5.99977	5.98491	0.0148584
2.2	5.99992	5.99314	0.00677924
2.4	5.99997	6.01293	0.0129568
2.6	5.99999	6.00277	0.00278277

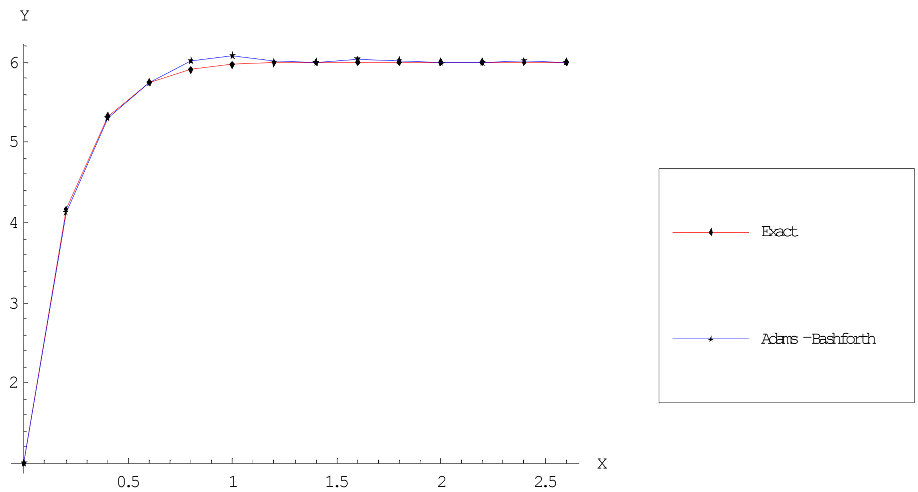

Figure 3. *Comparison curve Adams-Bashforth with Exact solution.*

4. A Comparative Study on Different Numerical Methods

In this section, we will determine the exact and approximate results that will be shown in tabular as well as graphical form. And finally, we will show that a comparison table and a comparison figure among Euler's method, the Runge-Kutta 4th & 6th Order methods, and the Adams-Bashforth-Moulton method.

Example 4.1

Consider the initial-value problem

$$y' = -2x^3 + 12x^2 - 20x + 8.5,$$

$$0 \leq x \leq 4, y(0) = 1, \qquad h = 0.5$$

Using Euler's method and Runge-Kutta 6th Order method, find approximations and compare with exact solution.

Solution: Given that,

$$y' = -2x^3 + 12x^2 - 20x + 8.5, 0 \leq x \leq 4,$$

$$y(0) = 1, h = 0.5$$

Therefore the analytic solution is

$$y(x) = 1 + 8.5x - 10x^2 + 4x^3 - 0.5x^4$$

Now the following Table 4 shows the results obtained by using Euler's method & Runge-kutta-6th Order methods on the interval [0,4] with step size $h = 0.5$. Approximations using Euler's & Runge-kutta-6th Order methods with error estimation.

***Table 4.** Comparison Between Euler's Method and Runge-kutta 6th Order Method.*

x	Exact	Euler's	RK-6	Err. Euler's	Err. RK-6
0.0	1.0	1.0	1.0	0	0.0
0.5	3.21875	5.25	3.21875	2.03125	0.0
1.0	3.0	5.875	3.0	2.875	0.0
1.5	2.21875	5.125	2.21875	2.90625	0.0
2.0	2.0	4.5	2.0	2.5	3.33067×10^{-14}
2.5	2.71875	4.75	2.71875	2.03125	3.26686×10^{-14}
3.0	4.0	5.875	4.0	1.875	0.0
3.5	4.71875	7.125	4.71875	2.40625	9.41116×10^{-14}
4.0	3.0	7.0	3.0	4.0	2.07242×10^{-13}

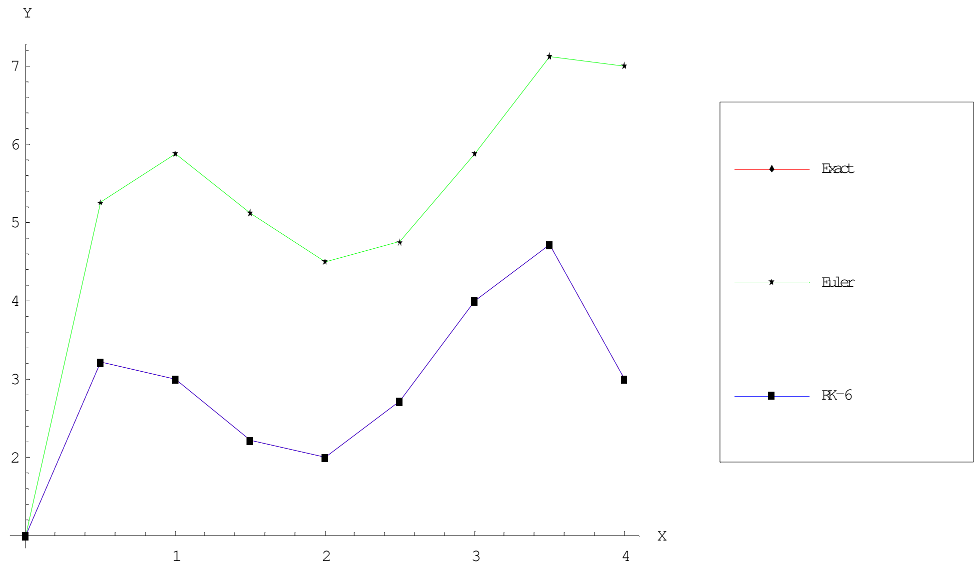

***Figure 4.** Comparison curves between Euler's method and Runge-kutta 6th Order method with exact solution.*

Example 4.2

Consider the initial-value problem

$$y'(x) = -2x - y \text{ with } y(0) = -1, 0 \leq x \leq 2, h = 0.1$$

Using Runge-kutta-4th & Runge-kutta-6th Order methods, find approximations and compare with exact solution.

Solution: Given equation is

$$\frac{dy}{dx} = -2x - y \text{ with } y(0) = -1, 0 \leq x \leq 2, h = 0.1$$

Therefore the analytic solution is

$$y(x) = 2 - 3e^{-x} - 2x$$

Now the following Table 5 shows the results obtained by using Runge-kutta-4th & Runge-kutta-6th Order methods on the interval [0,2] with step size $h = 0.1$.

***Table 5.** Comparison Between Runge-kutta-4th & Runge-kutta-6th Order Methods.*

x	Exact	Rk-4	RK-6	Error RK-4	Error RK-6
0.0	-1.0	-1.0	-1.0	0.0	0.0
0.1	-0.914512	-0.915413	-0.918566	0.0000268878	0.00443227
0.2	-0.856192	-0.856193	-0.862623	0.0000519725	0.00751118

x	Exact	Rk-4	RK-6	Error RK-4	Error RK-6
0.3	-0.822455	-0.822455	-0.829864	0.0000734336	0.00900871
0.4	-0.81096	-0.810961	-0.818188	0.0000898497	0.00891231
0.5	-0.819592	-0.819593	-0.825685	0.000100554	0.007434
0.6	-0.846435	-0.846436	-0.850619	0.000105719	0.00494283
0.7	-0.889756	-0.889757	-0.89141	0.000106168	0.00185888
0.8	-0.947987	-0.947988	-0.946622	0.000103045	0.00143977
0.9	-1.01971	-1.01971	-1.01495	0.0000975158	0.00466809
1.0	-1.10364	-1.10364	-1.0952	0.0000905842	0.00764358
1.1	-1.19861	-1.19861	-1.1863	0.0000830163	0.0102706
1.2	-1.30358	-1.30358	-1.28727	0.0000753465	0.012516
1.3	-1.4176	-1.4176	-1.3972	0.0000679176	0.0143861
1.4	-1.53979	-1.53979	-1.51529	0.0000609296	0.0159089
1.5	-1.66939	-1.66939	-1.64081	0.0000544836	0.0171225
1.6	-1.80569	-1.80569	-1.77307	0.0000486161	0.0180675
1.7	-1.94805	-1.94805	-1.91146	0.0000433234	0.0187831
1.8	-2.0959	-2.0959	-2.05544	0.0000385786	0.0193051
1.9	-2.24871	-2.24871	-2.20448	0.0000343428	0.0196651
2.0	-2.40601	-2.40601	-2.35815	0.0000305716	0.0198904

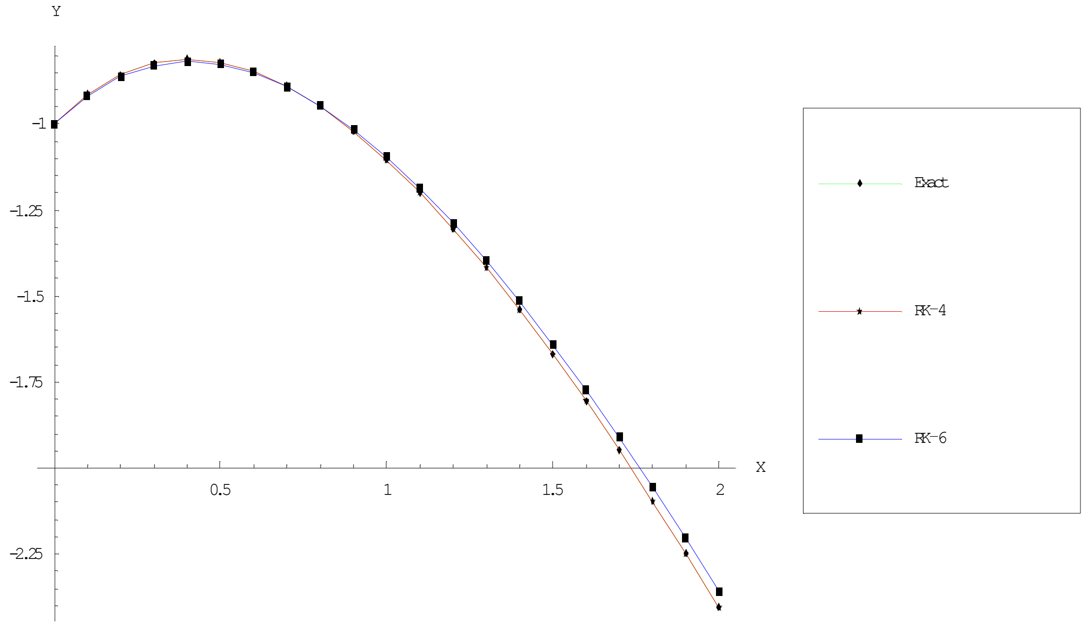

Figure 5. *Comparison curves between Runge-kutta 4th & 6th Order methods with exact solution.*

Example 4.3

Let be considered the initial value problem

$$y' = y - x^2 + 1, 0 \leq x \leq 2, y(0) = 0.5$$

We have to find the approximate solution using Euler's method, Runge-kutta 4th, 6th Order methods, & Adams-Bashforth method and also show the error of these methods.

Solution: Given that,

$$y' = y - x^2 + 1, 0 \leq x \leq 2, y(0) = 0.5$$

Therefore the analytic solution is

$$y(x) = (x+1)^2 - 0.5e^x$$

Now we generate the approximations listed in the following Table 6. Approximations using Euler's method, Runge-kutta 4th, 6th Order, & Adams-Bashforth method with error estimation.

Table 6. *Comparison Among Euler's Method, Runge-kutta 4th, 6th Order Methods, & Adams-Bashforth-Moulton Method.*

x	$Exact$	$Euler's$	$RK-4$	$RK-6$	$Adams-Bashforth$	$Err.Euler's$	$Err.RK-4$	$Err.RK-6$	$Err.Adams-Bashforth$
0.0	0.500000	0.50000	0.500000	0.500000	0.500000	0.0000000	0	0.0000000	0
0.2	0.829299	0.80000	0.829293	0.861827	0.829293	0.0292986	6.37597×10^{-6}	0.0392244	5.28759×10^{-6}
0.4	1.214090	1.15200	1.214080	1.292460	1.214080	0.0620877	9.42314×10^{-6}	0.0645489	1.14405×10^{-5}
0.6	1.648940	1.55040	1.648920	1.789160	1.648920	0.0985406	1.12695×10^{-5}	0.0850362	1.85828×10^{-5}
0.8	2.127230	1.98848	2.127200	2.348550	2.127210	0.1387500	1.26224×10^{-5}	0.1040430	2.39033×10^{-5}

x	$Exact$	$Euler's$	$RK-4$	$RK-6$	$Adams-Bashforth$	$Err.Euler's$	$Err.RK-4$	$Err.RK-6$	$Err.Adams-Bashforth$
1.0	2.640860	2.45818	2.640820	2.966420	2.640830	0.1826830	1.37808×10^{-5}	0.1232800	3.04898×10^{-5}
1.2	3.179940	2.94981	3.179890	3.637530	3.179900	0.2301300	1.4896×10^{-5}	0.1429000	3.89032×10^{-5}
1.4	3.732400	3.45177	3.732340	4.355370	3.732350	0.2806270	1.60604×10^{-4}	0.1669100	4.9535×10^{-5}
1.6	4.283480	3.95013	4.283410	5.111850	4.283420	0.3333560	1.73432×10^{-5}	0.1933860	6.29643×10^{-5}
1.8	4.815180	4.42815	4.815090	5.896500	4.815100	0.3870230	1.88099×10^{-5}	0.2246480	7.99125×10^{-5}
2.0	5.305470	4.86578	5.305360	6.698000	5.305370	0.4396870	2.05354×10^{-5}	0.2624710	1.10279×10^{-4}

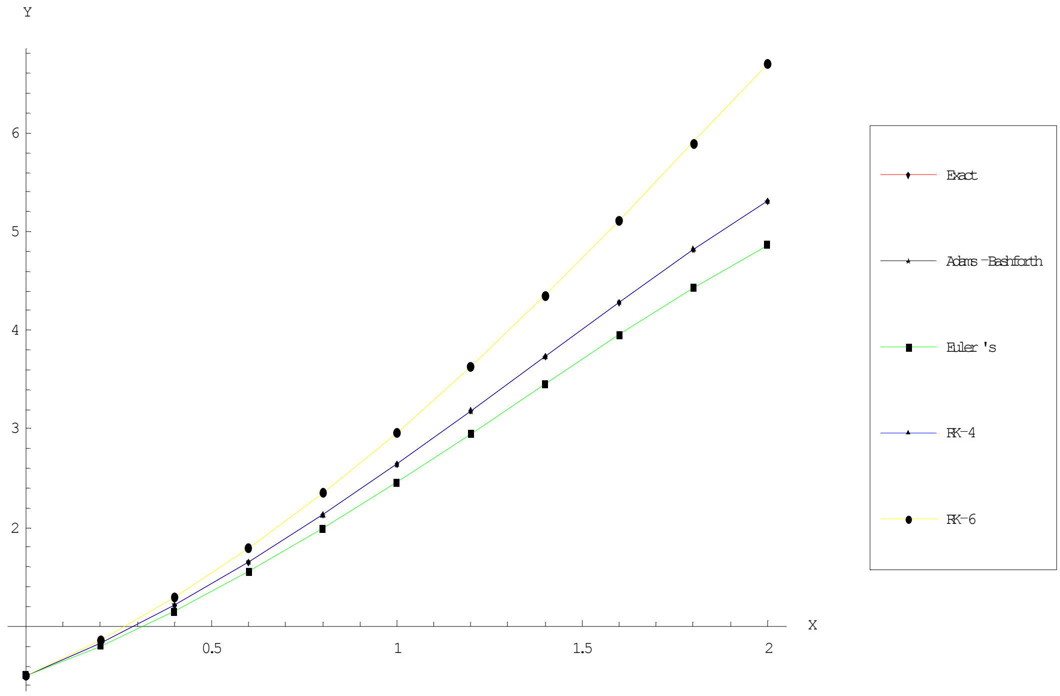

Figure 6. *Comparison curves of different methods.*

Example 4.4

Let be considered the initial value problem

$$y' = -2x^3 + 12x^2 - 20x + 8.5,$$

$$from\ x = 0\ to\ x = 4$$
$$using\ a\ step\ size\ of\ 0.5, and\ the\ initial\ condition$$

$$at\ x = 0\ is\ y = 1$$

We have to find the approximate solution using Euler's method, Runge-kutta 4th, 6th Order methods, & Adams-Bashforth method and also show the error of these methods.

Solution: Given that, $y' = -2x^3 + 12x^2 - 20x + 8.5$, $0 \le x \le 4, y(0) = 1$

Therefore the analytic solution is

$$y(x) = 1 + 8.5x - 10x^2 + 4x^3 - 0.5x^4$$

Now we generate the approximations listed in the following Table 7. Approximations using Euler's method, Runge-kutta 4th, 6th Order, & Adams-Bashforth method with error estimation.

Table 7. *Comparison Among Euler's Method, Runge-Kutta 4th, 6th Order Methods, & Adams-Bashforth-Moulton Method.*

x	Exact	Euler's	RK-4	RK-6	Adams-Bashforth	Err.Euler's	Err.RK-4	Err.RK-6	Err. Adams-Bashforth
0.0	1.0	1.0	1.0	1.0	1.0	0	0.0	0.0	0.0
0.5	3.21875	5.25	3.21875	3.21875	3.21875	2.03125	0.0	0.0	0.0
1.0	3.0	5.875	3.0	3.0	3.0	2.875	0.0	0.0	0.0
1.5	2.21875	5.125	2.21875	2.21875	2.21875	2.90625	0.0	0.0	0.0
2.0	2.0	4.5	2.0	2.0	2.0	2.5	0.0	3.33067×10^{-14}	0.0
2.5	2.71875	4.75	2.71875	2.71875	2.71875	2.03125	0.0	3.26686×10^{-14}	0.0
3.0	4.0	5.875	4.0	4.0	4.0	1.875	0.0	0.0	0.0
3.5	4.71875	7.125	4.71875	4.71875	4.71875	2.40625	0.0	9.41116×10^{-14}	0.0
4.0	3.0	7.0	3.0	3.0	3.0	4.0	0.0	2.07242×10^{-13}	0.0

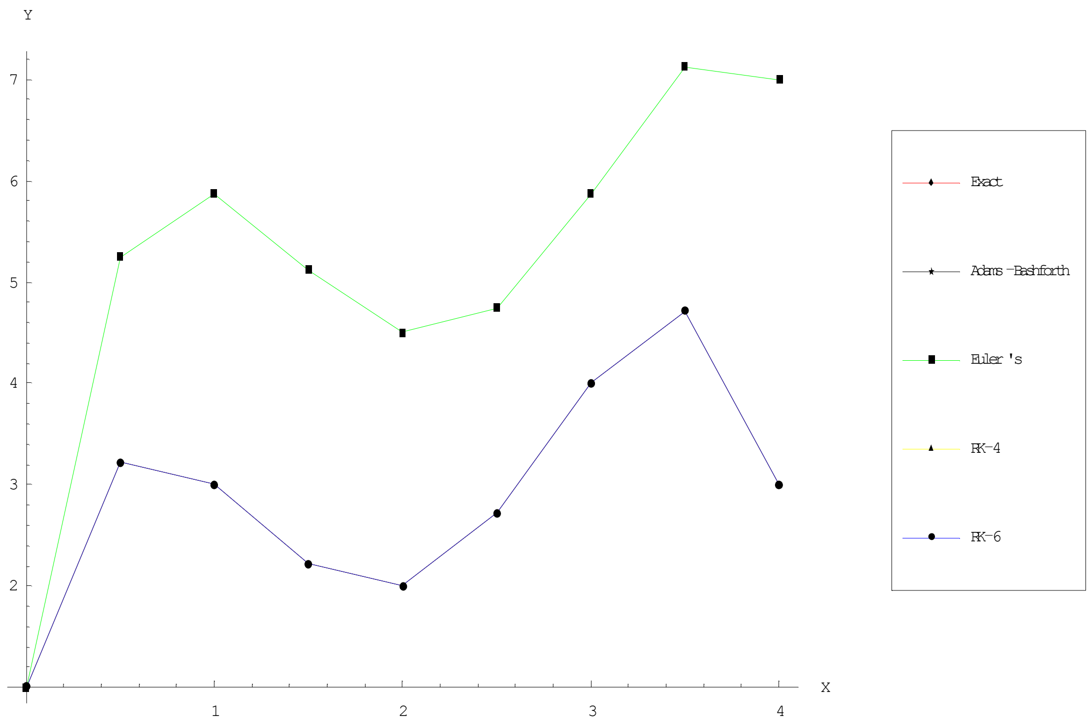

Figure 7. *Comparison curves of different methods.*

Example 4.5

Let be considered the initial value problem

$$y' = -2y + 4e^{-x} \textit{ over the range } x = 0 \textit{ to } 1$$

$$\textit{using a step size of } 0.2 \textit{ with } y(0) = 2.$$

We have to find the approximate solution using Euler's method, Runge-kutta 4^{th}, 6^{th} Order methods, & Adams-Bashforth method and also show the error of these methods.

Solution: Given that, $y' = -2y + 4e^{-x}\ 0 \le x \le 1, y(0) = 2$

Therefore the analytic solution is

$$y(x) = 2e^{-2x}(-1 + 2e^{x})$$

Now we generate the approximations listed in the following Table 8. Approximations using Euler's method, Runge-kutta 4^{th}, 6^{th} Order, & Adams-Bashforth method with error estimation.

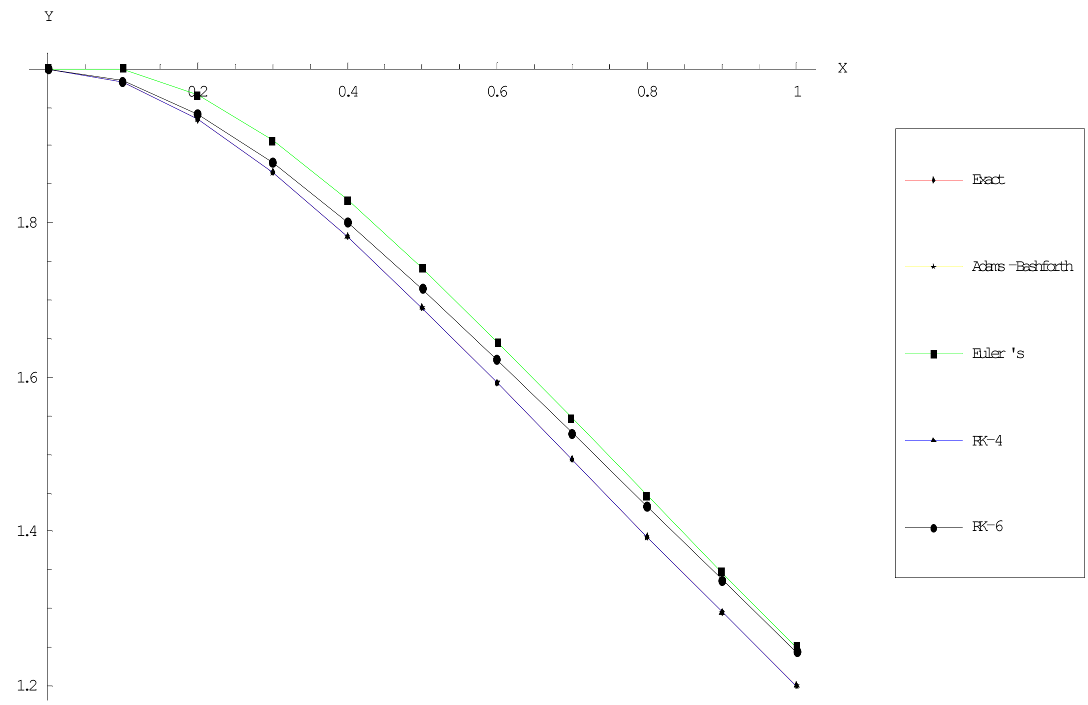

Figure 8. *Comparison curves of different methods.*

Table 8. *Comparison Among Euler's Method, Runge-Kutta 4th, 6th Order Methods, & Adams-Bashforth-Moulton Method.*

x	Exact	Euler's	RK-4	RK-6	Adams-Bashforth	Err.Euler's	Err.RK-4	Err.RK-6	Err. Adams-Bashforth
0.0	2	2	2	2	2	0	0	0	0
0.1	1.98189	2	1.98189	1.98387	1.98189	0.0181118	0.000116236	0.100137	2.30367×10^{-6}
0.2	1.93428	1.96193	1.93428	1.94072	1.93428	0.027652	0.0001823	0.332566	3.52619×10^{-6}
0.3	1.86565	1.89704	1.86565	1.87778	1.86565	0.0313907	0.00021479	0.650461	4.00722×10^{-6}
0.4	1.78262	1.81396	1.78262	1.80088	1.78264	0.0313373	0.000224198	0.010241	0.0000163388
0.5	1.69036	1.7193	1.69036	1.7146	1.69039	0.0289319	0.000217463	0.0143373	0.000028933
0.6	1.59286	1.61805	1.59285	1.62258	1.59289	0.0251906	0.000199345	0.0186572	0.000036106
0.7	1.49315	1.51396	1.49314	1.52764	1.49319	0.0208164	0.000173202	0.023104	0.0000397616
0.8	1.39352	1.40981	1.39352	1.43199	1.39356	0.0162822	0.00014146	0.0276076	0.0000407979
0.9	1.29568	1.30758	1.29568	1.33729	1.29572	0.0118948	0.000105904	0.032116	0.0000400549
1.0	1.20085	1.20869	1.20085	1.24479	1.20089	0.00784117	0.0000678664	0.0365903	0.0000381565

5. Conclusion

From this paper, it is obvious to say that this mentioned different numerical methods are very powerful tools for solving the initial value problems of ordinary differential equations. In Figure 2, we conclude that graphically approximate solution of Runge-Kutta 6th Order moves slightly from the exact results. And we have observed the nature of the Adams-Bashforth method with the exact solution. The approximation obtained by the Adams-Bashforth method is the closest to the exact solution shown in Figure 3. From Table 5, we realized that the Runge-Kutta 4th order method gives better accuracy. In our closest investigation, we have investigated of the Euler method, the Runge-Kutta methods of order 4th & 6th and the Adams-Bashforth method, after we have observed that the 4th order Runge-Kutta method gives better accuracy than other methods besides Adams-Bashforth method is the best of all in Table 6 and Figure 6. Finally, we concluded that the Adams-Bashforth method gives the best approximation.

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