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Integral Inequalities for Some New Classes of Convex Functions

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Abstract: In this paper, we introduce a new class of convex functions, which is called nonconvex functions. We show that this class unifies several previously known and new classes of convex functions. We derive several new inequalities of Hermite-Hadamard type for nonconvex functions. Some special cases are also discussed. Results proved in this paper continue to hold for these special cases.

Keywords: Convex, Hermite-Hadamard's Inequalities, Convex Functions

1. Introduction

Theory of convexity has played an important role in the development of various branches of pure and applied sciences, see [2]. In recent years, the cconcepts of convex functions have been generalized in various directions using novel and innovative ideas, see[1,2,3,4,5,7,8,10,11,12,13,16]. Varosanec [11] introduced the class of h-convex functions. This class of functions unifies various classes of convex functions and is being used to discuss several concepts in a unified manners.

Definition1.1[16]. Let $h: J=[0,1] \to \mathbb{R}$ a nonnegative function. We say that $f: I \subset \mathbb{R} \to \mathbb{R}$ is h-convex function, if f is nonnegative and for all $x, y \in I$ and $t \in [0,1]$.

$$f\left(tx+(1-t)y\right) \le h(t)f(x)+h(1-t)f(y), \quad (1)$$

The class of h-convex functions along with classical convex functions unifies some other known classes of convex functions, that is, s-Breckner convex functions [1], P-convex functions [7], Godunova-Levin functions [8] and s-Godunova-Levin functions [4].

It is known that theory of convex functions and theory of inequalities are very much interrelated to each other. One of the most extensively studied inequality in the literature is Hermite-Hadmard's inequality. This inequality provides necessary and sufficient condition for a function to be convex. For some useful details on Hermite-Hadmard's inequality, readers are referred to [3,4,5,6,7,9,10,11,12,13,14,15].

Inspired by the ongoing research, we in this paper, introduce a class of (h,s)-convex functions, which is called nonconvex. This class also generalizes the class of h-convex functions. This new class of noconvex functions is quite flexible and includes several previously known and new classes as special cases. We derive some Hermite-Hadmard type inequalities for this new class, which is main motivation of this paper. The interested readers may find its applications in pure and applied sciences.

We now define a new class of convex functions, which is called as (h,s)-convex function.

Definition1.2. Let $h: J=[0,1] \to \mathbb{R}$ a nonnegative function. A function $f: I \subset \mathbb{R} \to \mathbb{R}$ is $(h,s)-$ convex function, if f is nonnegative and $f\left(tx+(1-t)y\right) \le h(t^s)f(x)+h((1-t)^s)f(y)$, $\forall x, y \in I, s \in (0,1]$, and $t \in [0,1]$.

Remark. For $s=1$ in Definition 1.2, then, we have the definition of h-convex functions, introduced and studied by Varosanec [16]. For $h(t^s)=t^s$, $h(t^s)=t^{-s}$, $h(t^s)=1$, the class of (h,s)-convex functions reduces to the class of , s-Breckner convex functions [1], s-Godunova-Levin functions [4] and P-functions [7] respectively.

In the next section, we derive some Hermite-Hadamard type inequalities for the class of (h,s)-convex functions. We also discuss some special cases.

To prove some of our main results, we need following result, which is proved using the technique of [15].

Lemma 1.4. Let f be (h,s)-convex functions.Then for any $x\in[a,b]$ and $\forall t\in[0,1]$, we have

$$f(a+b-x)\le(h(t^s)+h((1-t)^s))[f(a)+f(b)]-f(x).$$

Proof. As we know that $x\in[a,b]$, can be represented as $x=ta+(1-t)b$, $\forall t\in[0,1]$.

Thus which is the desired result.

$$f(a+b-x)$$
$$=f((1-t)a+tb)$$
$$\le h((1-t)^s)f(a)+h(t^s)f(b)$$
$$=(h(t^s)+h((1-t)^s))[f(a)+f(b)]$$
$$-[h(t^s)f(a)+h((1-t)^s)f(b)]$$
$$\le(h(t^s)+h((1-t)^s))[f(a)+f(b)]$$
$$-f(ta+(1-t)b)$$
$$=(h(t^s)+h((1-t)^s))[f(a)+f(b)]$$
$$-f(x),$$

2. Main Results

In this section we derive several integral inequalities of Hermite-Hadamard type for (h,s)-convex functions, which is the main motivation of this paper.

Theorem 2.1. Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be (h,s)-convex function. If $f\in L_1[a,b]$, then, for $h(\frac{1}{2^s})\neq 0$, we have

$$\frac{1}{2h(\frac{1}{2^s})}f(\frac{a+b}{2})\le\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\le[f(a)+f(b)]\int_0^1 h(t^s)\mathrm{d}t.$$

Proof. Let f be (h,s)-convex, then

$$f\left(\frac{a+b}{2}\right)=f\left(\frac{ta+(1-t)b+(1-t)a+tb}{2}\right)$$
$$\le h(\frac{1}{2^s})(f(ta+(1-t)b)+f((1-t)a+tb).$$

Integration of above inequality with respect to t on $[0,1]$, leads to the following step

$$\frac{1}{2h(\frac{1}{2^s})}f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x.$$

Also

$$f(ta+(1-t)b)\le h(t^s)f(a)+h((1-t)^s)f(b).$$

Integration of above inequality with respect to t on $[0,1]$, we have

$$\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\le[f(a)+f(b)]\int_0^1 h(t^s)\mathrm{d}t.$$

This completes the proof.

Special Cases.

We now discuss some special cases.

i. If $s=1$ in Theorem 2.1 then, we have Theorem 6 [14].

ii. If $h(t^s)=t^s$ in Theorem 2.1 then, we have Hermite-Hadamard type inequality for s-convex functions, see [6].

iii. If $h(t^s)=t^{-s}$ in Theorem 2.1 then, we have Hermite-Hadamard type inequality for s-Godunova-Levin convex functions, see [4].

iv. If $h(t^s)=1$ in Theorem 2.1 then, we have Hermite-Hadamard type inequality for P functions, see [7].

We need the following result, which is proved essentially using the technique of [15].

Theorem 2.2. Let f be (h,s)-convex function on $I=[a,b]$, $f\in L_1[a,b]$ and $w:[a,b]\to\mathbb{R}^n$ is non-negative, integrable and symmetric about $\frac{a+b}{2}$, then

$$\int_a^b f(x)w(x)\mathrm{d}x\le\frac{f(a)+f(b)}{2}\times\int_a^b\left(h\left((\frac{b-x}{b-a})^s\right)+h\left((\frac{x-a}{b-a})^s\right)\right)w(x)\mathrm{d}x.$$

Proof. Using the given facts, we have $\int_a^b f(x)w(x)\mathrm{d}x$

$$=\frac{1}{2}\left\{\int_a^b f(x)w(x)\mathrm{d}x+\int_a^b f(a+b-x)w(a+b-x)\mathrm{d}x\right\},$$
$$=\frac{1}{2}\left\{\int_a^b(f(x)+f(a+b-x))w(x)\mathrm{d}x\right\},$$
$$=\frac{1}{2}\int_a^b\left\{\begin{array}{l}f\left(\frac{b-x}{b-a}a+\frac{x-a}{b-a}b\right)\\+f\left(\frac{x-a}{b-a}a+\frac{b-x}{b-a}b\right)\end{array}\right\}w(x)\mathrm{d}x,$$
$$\le\frac{1}{2}\left\{(f(a)+f(b))\int_a^b h\left((\frac{b-x}{b-a})^s\right)w(x)\mathrm{d}x\right.$$
$$\left.+(f(a)+f(b))\int_a^b h\left((\frac{x-a}{b-a})^s\right)w(x)\mathrm{d}x\right\},$$
$$=\frac{f(a)+f(b)}{2}\int_a^b\left(\begin{array}{l}h\left((\frac{b-x}{b-a})^s\right)\\+h\left((\frac{x-a}{b-a})^s\right)\end{array}\right)w(x)\mathrm{d}x,$$

which is the desired result.

Note that, if s=1, then Theorem 2.2 reduces to Theorem 5 [15].

We now derive Hermite-Hadamard-Fejer version of inequality for the class of $(h,s)-$convex functions.

Theorem 2.3. Let f be (h,s) -convex functions and $a,b\in I$, $f\in L_1[a,b]$ and $w:[a,b]\to\mathbb{R}^n$ is a non-negative, integrable and symmetric about $\frac{a+b}{2}$, then, for $h(\frac{1}{2^s})\neq 0$, we have

$$\frac{1}{2h(\frac{1}{2^s})}f\left(\frac{a+b}{2}\right)\int_a^b w(x)\mathrm{d}x$$
$$\leq \int_a^b f(x)w(x)\mathrm{d}x$$
$$\leq \frac{f(a)+f(b)}{2}(h(t^s)+h((1-t)^s))\int_a^b w(x)\mathrm{d}x.$$

Proof. Using the given facts and Lemma 1.4, we have

$$\frac{1}{2h(\frac{1}{2^s})}f\left(\frac{a+b}{2}\right)\int_a^b w(x)\mathrm{d}x$$
$$=\frac{1}{2h(\frac{1}{2^s})}\int_a^b f\left(\frac{a+b-x+x}{2}\right)w(x)\mathrm{d}x$$
$$\leq \frac{1}{2h(\frac{1}{2^s})}\int_a^b\left[h(\frac{1}{2^s})(f(a+b-x)+f(x))\right]w(x)\mathrm{d}x$$
$$=\frac{1}{2}\int_a^b f(a+b-x)w(a+b-x)\mathrm{d}x+\frac{1}{2}\int_a^b f(x)w(x)\mathrm{d}x$$
$$\leq \int_a^b f(x)w(x)\mathrm{d}x.$$

To prove the other part of the inequality, we consider

$$\int_a^b f(x)w(x)\mathrm{d}x$$
$$=\frac{1}{2}\int_a^b f(a+b-x)w(a+b-x)\mathrm{d}x+\frac{1}{2}\int_a^b f(x)w(x)\mathrm{d}x$$
$$=\frac{1}{2}\int_a^b f(a+b-x)w(x)\mathrm{d}x+\frac{1}{2}\int_a^b f(x)w(x)\mathrm{d}x$$
$$\leq\frac{1}{2}\int_a^b[(h(t^s)+h((1-t)^s))[f(a)+f(b)]$$
$$-f(x)]w(x)\mathrm{d}x+\int_a^b f(x)w(x)\mathrm{d}x$$
$$=\frac{f(a)+f(b)}{2}(h(t^s)+h((1-t)^s))\int_a^b w(x)\mathrm{d}x.$$

This completes the proof.

Theorem 2.4. Let fw be the product of two (h,s) -convex functions, $a,b\in I$ and $fw\in L_1[a,b]$, then, for $h(\frac{1}{2^s})\neq 0$

$$\frac{1}{2h(\frac{1}{2^s})}(fw)\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)w(x)\mathrm{d}x$$
$$\leq\frac{1}{b-a}\int_a^b(fw)(x)\mathrm{d}x\leq[(fw)(a)+(fw)(b)]\int_0^1 h(t^s)\mathrm{d}t.$$

Proof. Since fw is (h,s) -convex function, then, for $t\in(0,1)$, we have

$$(fw)(tx+(1-t)y)\leq h(t)(fw)(x)+h(1-t)(fw)(y).$$

For $x=ta+(1-t)b,\quad y=(1-t)a+tb,\ t=\frac{1}{2},$ we have

$$(fw)\left(\frac{a+b}{2}\right)\leq h\left(\frac{1}{2^s}\right)(fw)(ta+(1-t)(b)$$
$$+h\left(\frac{1}{2^s}\right)(fw)((1-t)a+tb).$$

Integration of both sides of the above inequality with respect to t on $[0,1]$, leads to following step

$$\frac{1}{2h(\frac{1}{2^s})}(fw)\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b(fw)(x)\mathrm{d}x,$$

which is the first part of the inequality.

Using the fact that fw is (h,s) -convex function and a simple integrations gives us the second part of the inequality. This completes the proof.

Theorem 2.5. Let $f(x)\in(h_1,s)$ and $w(x)\in(h_2,s)$, $a,b\in I$, be the functions such that $fw\in L_1[a,b]$, then, for $h_1(\frac{1}{2^s})\neq 0$ and $h_2(\frac{1}{2^s})\neq 0$, we have

$$\left[\frac{1}{2h_1(\frac{1}{2^s})h_2(\frac{1}{2^s})}f\left(\frac{a+b}{2}\right)w\left(\frac{a+b}{2}\right)\right]$$
$$-\left[\begin{array}{l}M(a,b)\int_0^1 h_1(t^s)h_2((1-t)^s)\mathrm{d}t\\ +N(a,b)\int_0^1 h_1(t^s)h_2(t^s)\mathrm{d}t\end{array}\right]$$
$$\leq\left[\begin{array}{l}M(a,b)\int_0^1 h_1(t^s)h_2(t^s)\mathrm{d}t\\ +N(a,b)\int_0^1 h_1(t^s)h_2((1-t)^s)\mathrm{d}t\end{array}\right].$$

where $M(a,b)=f(a)w(a)+f(b)w(b)$ and $N(a,b)=f(a)w(b)+f(b)w(a)$.

Proof. We can write

$$f\left(\frac{a+b}{2}\right)w\left(\frac{a+b}{2}\right)$$
$$= f\left(\frac{at+(1-t)b}{2}+\frac{(1-t)a+tb}{2}\right)$$
$$\times w\left(\frac{at+(1-t)b}{2}+\frac{(1-t)a+tb}{2}\right)$$
$$\leq h_1(\frac{1}{2^s})[f(at+(1-t)b)$$
$$+f((1-t)a+tb)]h_2(\frac{1}{2^s})[w(at+(1-t)b)$$
$$+w((1-t)a+tb)]$$
$$= h_1(\frac{1}{2^s})h_2(\frac{1}{2^s})[f(at+(1-t)b)w(at+(1-t)b)$$
$$+f((1-t)a+tb)\,w((1-t)a+tb)]$$
$$+h_1(\frac{1}{2^s})h_2(\frac{1}{2^s})[f(at+(1-t)b)w((1-t)a+tb)$$
$$+f((1-t)a+tb)w(at+(1-t)b)],$$
$$\leq h_1(\frac{1}{2^s})h_2(\frac{1}{2^s})[f(at+(1-t)b)w(at+(1-t)b)$$
$$+f((1-t)a+tb)w((1-t)a+tb)]$$
$$+h_1(\frac{1}{2^s})h_2(\frac{1}{2^s})[\{h_1(t^s)f(a)+h_1((1-t)^s)f(b)\}$$
$$\{h_2((1-t)^s)w(a)+h_2(t^s)w(b)\}$$
$$+\{h_1((1-t)^s)f(a)\ \ +h_1(t^s)f(b)\}$$
$$\{h_2(t^s)w(a)+h_2((1-t)^s)w(b)\}],$$
$$= h_1(\frac{1}{2^s})h_2(\frac{1}{2^s})[f(at+(1-t)b)w(at+(1-t)b)$$
$$+f((1-t)a+tb)w((1-t)a+tb)]$$
$$+h_1(\frac{1}{2^s})h_2(\frac{1}{2^s})[\{h_1(t^s)h_2((1-t)^s)$$
$$+h_1((1-t)^s)h_2(t^s)\}$$
$$\{f(a)w(a)+f(b)w(b)\}+\{h_1(t^s)h_2(t^s)$$
$$+h_1((1-t)^s)h_2((1-t)^s)\}\{f(a)w(b)+f(b)w(a)\}]$$
$$= h_1(\frac{1}{2^s})h_2(\frac{1}{2^s})[f(at+(1-t)b)w(at+(1-t)b)$$
$$+f((1-t)a+tb)w((1-t)a+tb)]$$
$$+h_1(\frac{1}{2^s})h_2(\frac{1}{2^s})[\{h_1(t^s)h_2((1-t)^s)$$
$$+h_1((1-t)^s)h_2(t^s)\}M(a,b)+\{h_1(t^s)h_2(t^s)$$
$$+h_1((1-t)^s)h_2((1-t)^s)\}N(a,b)].$$

Integration of both sides with respect to t on $[0,1]$, leads to the following step

$$\leq h_1(\frac{1}{2^s})h_2(\frac{1}{2^s})\begin{bmatrix}\frac{1}{b-a}\int_a^b f(x)w(x)\mathrm{d}x\\ +\frac{1}{b-a}\int_a^b f(x)w(x)\mathrm{d}x\end{bmatrix}$$
$$+h_1(\frac{1}{2^s})h_2(\frac{1}{2^s})\begin{bmatrix}2M(a,b)\int_0^1 h_1(t^s)h_2((1-t)^s)\mathrm{d}t\\ +2N(a,b)\int_0^1 h_1(t^s)h_2(t^s)\mathrm{d}t\end{bmatrix}.$$

This implies that

$$\left[\frac{1}{2h_1(\frac{1}{2^s})h_2(\frac{1}{2^s})}f\left(\frac{a+b}{2}\right)w\left(\frac{a+b}{2}\right)\right]$$
$$-\begin{bmatrix}M(a,b)\int_0^1 h_1(t^s)h_2((1-t)^s)\mathrm{d}t\\ +N(a,b)\int_0^1 h_1(t^s)h_2(t^s)\mathrm{d}t\end{bmatrix}$$
$$\leq \frac{1}{b-a}\int_a^b f(x)w(x)\mathrm{d}x,$$

which is the left side of the inequality. Now we derive the other side of the inequality.

For this we have

$$f(ta+\leq[h_1(t^s)f(a)+h_1((1-t)^s)f(b)]$$
$$\times[h_2(t^s)w(a)+h_2((1-t)^s)w(b)].$$

Integration of both sides of above inequality with respect to t on $[0,1]$, leads to the following step

$$\frac{1}{b-a}\int_a^b f(x)w(x)\mathrm{d}x \leq f(a)w(a)\int_0^1 h_1(t^s)h_2(t^s)\mathrm{d}t$$
$$+f(a)w(b)\int_0^1 h_1(t^s)h_2((1-t)^s)\mathrm{d}t$$
$$+f(b)w(a)\int_0^1 h_2(t^s)h_1((1-t)^s)\mathrm{d}t$$
$$+f(b)w(b)\int_0^1 h_1((1-t)^s)h_2((1-t)^s)\mathrm{d}t$$
$$=[f(a)w(a)+f(b)w(b)]\int_0^1 h_1(t^s)h_2(t^s)\mathrm{d}t$$
$$+[f(a)w(b)+f(b)w(a))]\int_0^1 h_1(t^s)h_2((1-t)^s)\mathrm{d}t,$$

$$= \begin{bmatrix} M(a,b)\int_0^1 h_1(t^s)h_2(t^s)\mathrm{d}t \\ +N(a,b)\int_0^1 h_1(t^s)h_2((1-t)^s)\mathrm{d}t \end{bmatrix},$$

which is the desired result.

Remark. For suitable and appropriate choces of auxiliary function $h(.)$ one can obtain several new and known Hermite-Hadamard type inequalities for other classes of convex functions. We leave this to the interested readers.

3. Conclusion

We have introduced and investigated some new classes of convex functions involving and auxiliary function h. Several Hermite-Hadamard type of inequalities are obtained. Our results represents refinement and improvement of previously known results. The ideas and techniques of this paper may inspire further research in this field. The interested readers are encouraged to find the application of these new classes of convex functions in pure and applied sciences. This is an other aspect of future research.

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