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Generalized Quasi-Variational Inequalities for Pseudo-Monotone Type III and Strongly Pseudo-Monotone Type III Operators on Non-Compact Sets

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Abstract: In this paper, the authors prove some existence results of solutions for a new class of generalized quasi-variational inequalities (GQVI) for pseudo-monotone type III operators and strongly pseudo-monotone type III operators defined on non-compact sets in locally convex Hausdorff topological vector spaces. In obtaining these results on GQVI for pseudo-monotone type III operators, we shall use Chowdhury and Tan's generalized version [1] of Ky Fan's minimax inequality [2] as the main tool.

Keywords: Generalized Quasi-Variational Inequalities, Pseudo-Monotone Type III Operators, Locally Convex Topological Vector Spaces

1. Introduction

Let X be a non-empty set, and 2^X be the family of all non-empty subsets of X. Let E be a topological vector space. We shall denote by E^* the continuous dual of E, by $\langle w,x\rangle$ the pairing between E^* and E for $w\in E^*$ and $x\in E$ and by $Re\langle w,x\rangle$ the real part of $\langle w,x\rangle$. Given the maps $S: X\to 2^X$ and $T: X\to 2^{E^*}$, the generalized quasi-variational inequality problem (GQVI) is to find a point $\hat{y}\in S(\hat{y})$ and a point $\hat{w}\in T(\hat{y})$ such that $Re\langle \hat{w},\hat{y}-x\rangle\le 0$ for all $x\in S(\hat{y})$. The GQVI was introduced by Chan and Pang [3] in 1982 when E is finite dimensional and by Shih and Tan [4] in 1985 when E is infinite dimensional.

In [5] we established some existence theorems of generalized variational inequalities and generalized complementarity problems in topological vector spaces for pseudo-monotone type III operators defined as follows:

Definition 1.1. *Let E be a topological vector space, X a non-empty subset of E and $T: X\to 2^{E^*}$ a map. If $h: X\to\mathbb{R}$, then T is said to be an h-pseudo-monotone (respectively, a strongly h-pseudo-monotone) type III operator if for each $x,y\in X$ and every net $\{y_\alpha\}_{\alpha\in\Gamma}$ in X converging to y (respectively, weakly to y) with*

$$\limsup_{\alpha\in\Gamma}[\inf_{u\in T(y_\alpha)} Re\langle u,y_\alpha-x\rangle+h(y_\alpha)-h(x)]\le 0,$$

we have

$$\limsup_{\alpha\in\Gamma}[\inf_{u\in T(y_\alpha)} Re\langle u,y_\alpha-y\rangle+h(y_\alpha)-h(y)]\ge \inf_{w\in T(y)} Re\langle w,y-x\rangle+h(y)-h(x).$$

T is said to be a pseudo-monotone (respectively, a strongly pseudo-monotone) type III operator if T is an h-pseudo-monotone type III (respectively, a strongly h-pseudo-monotone type III) operator with $h\equiv 0$.

The above operators were originally named h-hemi-continuous (respectively, strong h-hemi-continuous) operators in [5]. Later, in [6], we re-named these operators pseudo-monotone type III operators.

The following result in [5] justified the validity of a set-valued pseudo-monotone (respectively, strongly pseudo-monotone) type III operator.

Proposition 1.1. Let X be a non-empty compact subset of a topological vector space E and $T: X \to 2^{E^*}$ an upper semi-continuous mapping from the relative weak topology on X to the strong topology on E^*, such that each $T(x)$ is a strongly compact subset of E^*. Then T is both a pseudo-monotone and a strongly pseudo-monotone type III operator.

If T is single-valued and continuous, the compactness of X is not required and the following result was obtained in [5]:

Proposition 1.2. Let X be a non-empty bounded subset of a topological vector space E and $T: X \to E^*$ a continuous mapping from the relative weak topology on X to the strong topology on E^*. Then T is both a pseudo-monotone and a strongly pseudo-monotone type III operator.

In this paper, we shall first obtain some general theorems on solutions for a new class of generalized quasi-variational inequalities for pseudo-monotone type III operators and strongly pseudo-monotone type III operators defined on non-compact sets in topological vector spaces. In obtaining these results, we shall mainly use the following generalized version of Ky Fan's minimax inequality [2] due to M.S.R. Chowdhury and K.-K Tan [1].

Theorem 1.3. Let E be a topological vector space, X be a non-empty convex subset of E, $h: X \to \mathbb{R}$ be lower semi-continuous on $co(A)$ for each $A \in \mathcal{F}(X)$, and $f: X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ be such that

(a) for each $A \in \mathcal{F}(X)$ and each fixed $x \in co(A)$, $y \mapsto f(x, y)$ is lower semi-continuous on $co(A)$;

(b) for each $A \in \mathcal{F}(X)$ and each $y \in co(A)$, $\min_{x \in A}[f(x, y) + h(y) - h(x)] \leq 0$;

(c) for each $A \in \mathcal{F}(X)$ and each $x, y \in co(A)$, every net $\{y_\alpha\}_{\alpha \in \Gamma}$ in X converging to y with $f(tx + (1-t)y, y_\alpha) + h(y_\alpha) - h(tx + (1-t)y) \leq 0$ for all $\alpha \in \Gamma$ and all $t \in [0,1]$, we have $f(x, y) + h(y) - h(x) \leq 0$;

(d) there exist a non-empty closed and compact subset K of X and $x_0 \in K$ such that $f(x_0, y) + h(y) - h(x_0) > 0$ for all $y \in X \backslash K$.

Then there exists $\hat{y} \in K$ such that $f(x, \hat{y}) + h(\hat{y}) - h(x) \leq 0$ for all $x \in X$.

2. Preliminaries

Let E be a topological vector space over Φ. Then, for each $x_0 \in E$, each non-empty subset A of E and each $\epsilon > 0$, let $W(x_0; \epsilon) := \{y \in E^*: |\langle y, x_0 \rangle| < \epsilon\}$ and $U(A; \epsilon) := \{y \in E^*: \sup_{x \in A} |\langle y, x \rangle| < \epsilon\}$.

Let $\sigma\langle E^*, E\rangle$ be the topology on E^* generated by the family $\{W(x; \epsilon): x \in E \text{ and } \epsilon > 0\}$ as a subbase for the neighborhood system at 0 and $\delta\langle E^*, E\rangle$ be the topology on E^* generated by the family $\{U(A; \epsilon): A$ is a non-empty bounded subset of E and $\epsilon > 0\}$ as a base for the neighborhood system at 0. We note that E^*, when equipped with the topology $\sigma\langle E^*, E\rangle$ or the topology $\delta\langle E^*, E\rangle$, becomes a locally convex Hausdorff topological vector space. Furthermore, for a net $\{y_\alpha\}_{\alpha \in \Gamma}$ in E^* and for $y \in E^*$, (i) $y_\alpha \to y$ in $\sigma\langle E^*, E\rangle$ if and only if $\langle y_\alpha, x\rangle \to \langle y, x\rangle$ for each $x \in E$ and (ii) $y_\alpha \to y$ in $\delta\langle E^*, E\rangle$ if and only if $\langle y_\alpha, x\rangle \to \langle y, x\rangle$ uniformly for $x \in A$ for each non-empty bounded subset A of E. The topology $\sigma\langle E^*, E\rangle$ (respectively, $\delta\langle E^*, E\rangle$) is called the weak*-topology (respectively, the strong topology) on E^*.

If X is a topological space and $\{U_\alpha: \alpha \in \mathcal{A}\}$ is an open cover for X, then a partition of unity subordinated to the open cover $\{U_\alpha: \alpha \in \mathcal{A}\}$ is a family $\{\beta_\alpha: \alpha \in \mathcal{A}\}$ of continuous real-valued functions $\beta_\alpha: X \to [0,1]$ such that

(a) $\beta_\alpha(y) = 0$ for all $y \in X \backslash U_\alpha$,

(b) $\{\text{support } \beta_\alpha: \alpha \in \mathcal{A}\}$ is locally finite and

(c) $\sum_{\alpha \in \mathcal{A}} \beta_\alpha(y) = 1$ for each $y \in X$.

We shall first state the following result which is Lemma 1 of Shih and Tan in [4, pp.334-335]:

Lemma 2.1. *Let X be a non-empty subset of a Hausdorff topological vector space E and $S: X \to 2^E$ be an upper semi-continuous map such that $S(x)$ is a bounded subset of E for each $x \in X$. Then for each continuous linear functional p on E, the map $f_p: X \to \mathbb{R}$ defined by $f_p(y) = \sup_{x \in S(y)} Re\langle p, x\rangle$ is upper semi-continuous; i.e. for each $\lambda \in \mathbb{R}$, the set $\{y \in X: f_p(y) = \sup_{x \in S(y)} Re\langle p, x\rangle < \lambda\}$ is open in X.*

The following result is Lemma 3 of Takahashi in [7, p.177] (see also Lemma 3 in [8, pp.68-85]):

Lemma 2.2. *Let X and Y be topological spaces, $f: X \to \mathbb{R}$ be non-negative and continuous and $g: Y \to \mathbb{R}$ be lower semi-continuous. Then the map $F: X \times Y \to \mathbb{R}$, defined by $F(x, y) = f(x)g(y)$ for all $(x, y) \in X \times Y$, is lower semi-continuous.*

We shall need the following Kneser's minimax theorem in [9, pp.2418-2420] (see also [10, pp.40-41]):

Theorem 2.3. *Let X be a non-empty convex subset of a vector space and Y be a non-empty compact convex subset of a Hausdorff topological vector space. Suppose that f is a real-valued function on $X \times Y$ such that for each fixed $x \in X$, the map $y \mapsto f(x, y)$, i.e. $f(x, \cdot)$, is lower semi-continuous and convex on Y and for each fixed $y \in Y$, the map $x \mapsto f(x, y)$, i.e. $f(\cdot, y)$ is concave on X. Then*

$$\min_{y \in Y} \sup_{x \in X} f(x, y) = \sup_{x \in X} \min_{y \in Y} f(x, y).$$

The following result is Lemma 3 in [1]:

Lemma 2.4. *Let E be a Hausdorff topological vector space, $A \in \mathcal{F}(E)$, $X = co(A)$, and $T: X \to 2^{E^*}$ be upper semi-continuous from X to the weak*-topology on E^* such that $T(x)$ is weak*-compact. Let $f: X \times X \to \mathbb{R}$ be defined by $f(x, y) = \inf_{w \in T(y)} Re\langle w, y - x\rangle$ for all $x, y \in X$. Then for each fixed $x \in X$, $y \mapsto f(x, y)$ is lower semi-continuous on X.*

3. Generalized Quasi-Variational Inequalities of Pseudo-Monotone Type III and Strongly Pseudo-Monotone Type III Operators

In this section, we shall obtain some general existence theorems for the solutions to the generalized quasi-variational inequalities for pseudo-monotone type III operators and strongly pseudo-monotone type III operators on non-compact sets.

We shall first establish the following result:

Theorem 3.1. *Let E be a locally convex Hausdorff* *topological vector space, X be a non-empty paracompact convex and bounded subset of E and $h: E \to \mathbb{R}$ be convex with $h(X)$ bounded. Let $S: X \to 2^X$ be upper semi-continuous such that each $S(x)$ is compact convex and $T: X \to 2^{E^*}$ be an h-pseudo-monotone type III (respectively, strongly h-pseudo-monotone type III) operator and be upper semi-continuous from $co(A)$ to the weak*-topology on E^* for each $A \in \mathcal{F}(X)$ and $T(X)$ is strongly bounded. Also, for each $x \in X$, $T(x)$ is weak*-compact convex. Suppose that the set*

$$\Sigma = \{y \in X: \sup_{x\in S(y)} [\inf_{w\in T(y)} Re\langle w, y-x\rangle + h(y) - h(x)] > 0\}$$

is open in X and the following conditions are satisfied:

(a) for each $A \in \mathcal{F}(X)$ and each $x, y \in co(A)$ and any net $\{y_\alpha\}_{\alpha\in\Gamma}$ in X converging to y, we have $\lim\sup_\alpha[\inf_{u\in T(y_\alpha)} Re\langle u, y_\alpha - x\rangle + h(y_\alpha) - h(x)] \le 0$ whenever $\lim\sup_\alpha[\inf_{u\in T(y_\alpha)} Re\langle u, y_\alpha - y\rangle + h(y_\alpha) - h(y)] \le 0$, and

(b) $\lim\sup_\alpha[\inf_{w\in T(y_\alpha)} Re\langle w, y_\alpha - x\rangle + h(y_\alpha) - h(x)] \ge \inf_{w\in T(y)} Re\langle w, y-x\rangle + h(y) - h(x)$ whenever $\lim\sup_\alpha[\inf_{w\in T(y_\alpha)} Re\langle w, y_\alpha - y\rangle + h(y_\alpha) - h(y)] \ge \inf_{w\in T(y)} Re\langle w, y-x\rangle + h(y) - h(x)$.

Suppose further that there exists a non-empty compact subset K of X and a point $x_0 \in X$ such that $x_0 \in K \cap S(y)$ and $\inf_{w\in T(y)} Re\langle w, y - x_0\rangle + h(y) - h(x_0) > 0$ for all $y \in X\setminus K$. Then there exists a point $\hat{y} \in K$ such that

(i) $\hat{y} \in S(\hat{y})$ and

(ii) there exists a point $\hat{w} \in T(\hat{y})$ with $Re\langle \hat{w}, \hat{y} - x\rangle \le h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

Proof. We shall complete the proof in three steps as follows:

Step 1. There exists a point $\hat{y} \in X$ such that $\hat{y} \in S(\hat{y})$ and

$$\sup_{x\in S(\hat{y})}[\inf_{w\in T(\hat{y})} Re\langle w, \hat{y} - x\rangle + h(\hat{y}) - h(x)] \le 0.$$

Suppose the contrary. Then for each $y \in X$, either $y \notin S(y)$ or there exists $x \in S(y)$ such that $\inf_{w\in T(y)} Re\langle w, y - x\rangle + h(y) - h(x) > 0$; that is, for each $y \in X$, either $y \notin S(y)$ or $y \in \Sigma$. If $y \notin S(y)$, then by a separation theorem for convex sets in locally convex Hausdorff topological vector spaces, there exists $p \in E^*$ such that $Re\langle p, y\rangle - \sup_{x\in S(y)} Re\langle p, x\rangle > 0$. For each $y \in X$, set

$$\gamma(y) := \sup_{x\in S(y)}[\inf_{w\in T(y)} Re\langle w, y-x\rangle + h(y) - h(x)].$$

Let $V_0 := \{y \in X | \gamma(y) > 0\} = \Sigma$ and for each $p \in E^*$, set

$$V_p := \{y \in X: Re\langle p, y\rangle - \sup_{x\in S(y)} Re\langle p, x\rangle > 0.$$

Then $X = V_0 \cup \bigcup_{p\in E^*} V_p$. Since each V_p is open in X by Lemma 2.1 and V_0 is open in X by hypothesis, $\{V_0, V_p: p \in E^*\}$ is an open covering for X. Since X is paracompact, there is a continuous partition of unity $\{\beta_0, \beta_p: p \in E^*\}$ for X subordinated to the open cover $\{V_0, V_p: p \in E^*\}$ (see, for example, Theorem VIII.4.2 of Dugundji in [11]), i.e. for each $p \in E^*$, $\beta_p: X \to [0,1]$ and $\beta_0: X \to [0,1]$ are continuous functions such that for each $p \in E^*$, $\beta_p(y) = 0$ for all $y \in X\setminus V_p$ and $\beta_0(y) = 0$ for all $y \in X\setminus V_0$ and $\{\text{support}\,\beta_0, \text{support}\,\beta_p: p \in E^*\}$ is locally finite and $\beta_0(y) + \sum_{p\in E^*}\beta_p(y) = 1$ for each $y \in X$. Note that for each $A \in \mathcal{F}(X)$, h is continuous on $co(A)$ (see e.g. [12, Corollary 10.1.1, p.83]). Define $\phi: X \times X \to \mathbb{R}$ by

$$\phi(x,y) = \beta_0(y)\left[\min_{w\in T(y)} Re\langle w, y-x\rangle + h(y) - h(x)\right] + \sum_{p\in E^*}\beta_p(y) Re\langle p, y-x\rangle$$

for each $x, y \in X$. Then we have the following:

(i) Since E is Hausdorff, for each $A \in \mathcal{F}(X)$ and each fixed $x \in co(A)$, the map

$$y \mapsto \min_{w\in T(y)} Re\langle w, y-x\rangle + h(y) - h(x)$$

is continuous on $co(A)$ by Lemma 2.3 and the fact that h is continuous on $co(A)$ and therefore the map

$$y \mapsto \beta_0(y)\left[\min_{w\in T(y)} Re\langle w, y-x\rangle + h(y) - h(x)\right]$$

is lower semi-continuous on $co(A)$ by Lemma 2.2. Also, for each fixed $x \in X$,

$$y \mapsto \sum_{p\in E^*}\beta_p(y) Re\langle p, y-x\rangle$$

is continuous on X. Hence, for each $A \in \mathcal{F}(X)$ and each fixed $x \in co(A)$, the map $y \mapsto \phi(x,y)$ is lower semi-continuous on $co(A)$.

(ii) For each $A \in \mathcal{F}(X)$ and for each $y \in co(A)$, $\min_{x\in A}\phi(x,y) \le 0$. If this were false, then there exists some $A = \{x_1, \ldots, x_n\} \in \mathcal{F}(X)$ and some $y \in co(A)$, say $y = \sum_{i=1}^n \lambda_i x_i$ with $\sum_{i=1}^n \lambda_i = 1$, such that $\min_{1\le i\le n}\phi(x_i, y) > 0$. Then for each $i = 1, \ldots, n$,

$$\beta_0(y)\left[\min_{w\in T(y)} Re\langle w, y - x_i\rangle + h(y) - h(x_i)\right] + \sum_{p\in E^*}\beta_p(y) Re\langle p, y - x_i\rangle > 0,$$

so that

$$\begin{aligned} 0 = \phi(y,y) &= \beta_0(y)\left[\min_{w\in T(y)} Re\,\langle w, y - \sum_{i=1}^n \lambda_i x_i\rangle + h(y) - h\left(\sum_{i=1}^n \lambda_i x_i\right)\right] \\ &\quad + \sum_{p\in E^*}\beta_p(y) Re\,\langle p, y - \sum_{i=1}^n \lambda_i x_i\rangle \\ &\ge \sum_{i=1}^n \lambda_i\left(\beta_0(y)\left[\min_{w\in T(y)} Re\langle w, y - x_i\rangle + h(y) - h(x_i)\right] + \sum_{p\in E^*}\beta_p(y) Re\langle p, y - x_i\rangle\right) > 0, \end{aligned}$$

which is a contradiction.

(iii) Suppose that $A \in \mathcal{F}(X)$, and $\{y_\alpha\}_{\alpha\in\Gamma}$ is a net in X converging to y with $\phi(tx + (1-t)y, y_\alpha) \le 0$ for all $\alpha \in \Gamma$ and all $t \in [0,1]$.

Case 1: $\beta_0(y) = 0$.

Since β_0 is continuous and $y_\alpha \to y$, we have $\beta_0(y_\alpha) \to \beta_0(y) = 0$. Note that $\beta_0(y_\alpha) \ge 0$ for each $\alpha \in \Gamma$ and $\beta_0(y_\alpha) \to 0$. Since $T(X)$ is strongly bounded and $\{y_\alpha\}_{\alpha\in\Gamma}$ is a bounded set, it follows that

$$\limsup_\alpha \left[\beta_0(y_\alpha)\left[\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - x\rangle + h(y_\alpha) - h(x)\right]\right] = 0. \quad (2.1)$$

Also, we have

$$\beta_0(y)\left[\min_{w\in T(y)} Re\langle w, y - x\rangle + h(y) - h(x)\right] = 0.$$

Thus it follows that

$$\limsup_\alpha \left[\beta_0(y_\alpha)\left[\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - x\rangle + h(y_\alpha) - h(x)\right]\right] + \sum_{p\in E^*} \beta_p(y) Re\langle p, y - x\rangle = \beta_0(y)\left[\min_{w\in T(y)} Re\langle w, y - x\rangle + h(y) - h(x)\right] + \sum_{p\in E^*} \beta_p(y) Re\langle p, y - x\rangle. \quad (2.2)$$

When $t = 1$, we have $\phi(x, y_\alpha) \le 0$ for all $\alpha \in \Gamma$, i.e.

$$\beta_0(y_\alpha)\left[\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - x\rangle + h(y_\alpha) - h(x)\right] + \sum_{p\in E^*} \beta_p(y_\alpha) Re\langle p, y_\alpha - x\rangle \le 0 \quad (2.3)$$

for all $\alpha \in \Gamma$. Therefore, by (2.3), we have

$$\begin{aligned}\limsup_\alpha &\left[\beta_0(y_\alpha)\left[\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - x\rangle + h(y_\alpha) - h(x)\right]\right] \\ &+ \liminf_\alpha \left[\sum_{p\in E^*} \beta_p(y_\alpha) Re\langle p, y_\alpha - x\rangle\right] \\ &\le \limsup_\alpha \left[\beta_0(y_\alpha)\left[\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - x\rangle \right.\right.\\ &\left.\left. + h(y_\alpha) - h(x)\right]\right] \\ &+ \sum_{p\in E^*} \beta_p(y_\alpha) Re\langle p, y_\alpha - x\rangle \le 0\end{aligned}$$

and so

$$\limsup_\alpha \left[\beta_0(y_\alpha)\left[\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - x\rangle + h(y_\alpha) - h(x)\right]\right] + \sum_{p\in E^*} \beta_p(y) Re\langle p, y - x\rangle \le 0. \quad (2.4)$$

Hence, by (2.2) and (2.4), we have $\phi(x, y) \le 0$.

Case 2. $\beta_0(y) > 0$.

Since $\beta_0(y_\alpha) \to \beta_0(y)$, there exists $\lambda \in \Gamma$ such that $\beta_0(y_\alpha) > 0$ for all $\alpha \ge \lambda$.

When $t = 0$, we have $\phi(y, y_\alpha) \le 0$ for all $\alpha \in \Gamma$, i.e.

$$\beta_0(y_\alpha)\left[\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - y\rangle + h(y_\alpha) - h(y)\right] + \sum_{p\in E^*} \beta_p(y_\alpha) Re\langle p, y_\alpha - y\rangle \le 0$$

for all $\alpha \in \Gamma$.

Thus

$$\limsup_\alpha \left[\beta_0(y_\alpha)\left[\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - y\rangle + h(y_\alpha) - h(y)\right] + \sum_{p\in E^*} \beta_p(y_\alpha) Re\langle p, y_\alpha - y\rangle\right] \le 0. \quad (2.5)$$

Hence

$$\limsup_\alpha \left[\beta_0(y_\alpha)\left[\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - y\rangle + h(y_\alpha) - h(y)\right]\right] + \liminf_\alpha\left[\sum_{p\in E^*} \beta_p(y_\alpha) Re\langle p, y_\alpha - y\rangle\right] \le \limsup_\alpha \left[\beta_0(y_\alpha)\left[\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - y\rangle + h(y_\alpha) - h(y)\right] + \sum_{p\in E^*} \beta_p(y_\alpha) Re\langle p, y_\alpha - y\rangle\right] \le 0 \ (\text{by } (2.5)).$$

Since $\liminf_\alpha\left[\sum_{p\in E^*} \beta_p(y_\alpha) Re\langle p, y_\alpha - y\rangle\right] = 0$, we have

$$\limsup_\alpha \left[\beta_0(y_\alpha)\left[\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - y\rangle + h(y_\alpha) - h(y)\right]\right] \le 0. \quad (2.6)$$

Since $\beta_0(y_\alpha) > 0$ for all $\alpha \ge \lambda$, it follows that

$$\beta_0(y) \limsup_\alpha \left[\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - y\rangle + h(y_\alpha) - h(y)\right] = \limsup_\alpha \left[\beta_0(y_\alpha)\left[\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - y\rangle + h(y_\alpha) - h(y)\right]\right]. \quad (2.7)$$

Since $\beta_0(y) > 0$, by (2.6) and (2.7) we have

$$\limsup_\alpha \left[\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - y\rangle + h(y_\alpha) - h(y)\right] \le 0.$$

Then, by hypothesis (a), we have

$$\limsup_\alpha \left[\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - x\rangle + h(y_\alpha) - h(x)\right] \le 0.$$

Since T is a pseudo-monotone type III operator, we have

$$\limsup_\alpha \left[\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - y\rangle + h(y_\alpha) - h(y)\right] \ge \min_{w\in T(y)} Re\langle w, y - x\rangle + h(y) - h(x).$$

Then, by hypothesis (b), we have

$$\limsup_\alpha \left[\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - x\rangle + h(y_\alpha) - h(x)\right] \ge \min_{w\in T(y)} Re\langle w, y - x\rangle + h(y) - h(x).$$

Since $\beta_0(y) > 0$, we have

$$\beta_0(y) \limsup_\alpha \left[\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - x\rangle + h(y_\alpha) - h(x)\right] \ge \beta_0(y)\left[\min_{w\in T(y)} Re\langle w, y - x\rangle + h(y) - h(x)\right] \quad (2.8)$$

Thus,

$$\beta_0(y)\limsup_\alpha\left[\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - x\rangle + h(y_\alpha) - h(x)\right] + \sum_{p\in E^*}\beta_p(y)Re\langle p, y - x\rangle \geq \beta_0(y)\left[\min_{w\in T(y)} Re\langle w, y - x\rangle + h(y) - h(x)\right] + \sum_{p\in E^*}\beta_p(y)Re\langle p, y - x\rangle.$$

Again, when $t = 1$, we have $\phi(x, y_\alpha) \leq 0$ for all $\alpha \in \Gamma$, i.e.

$$\beta_0(y_\alpha)\left[\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - x\rangle + h(y_\alpha) - h(x)\right] + \sum_{p\in E^*}\beta_p(y_\alpha)Re\langle p, y_\alpha - x\rangle \leq 0$$

for all $\alpha \in \Gamma$.

Thus

$$0 \geq \limsup_\alpha\left[\beta_0(y_\alpha)\left[\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - x\rangle + h(y_\alpha) - h(x)\right] + \sum_{p\in E^*}\beta_p(y_\alpha)Re\langle p, y_\alpha - x\rangle\right] \geq \limsup_\alpha\left[\beta_0(y_\alpha)\left[\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - x\rangle + h(y_\alpha) - h(x)\right]\right] + \liminf_\alpha\left[\sum_{p\in E^*}\beta_p(y_\alpha)Re\langle p, y_\alpha - x\rangle\right] = \beta_0(y)\left[\limsup_\alpha\left[\min_{w\in T(y_\alpha)} Re\langle w, y_\alpha - x\rangle + h(y_\alpha) - h(x)\right]\right] + \sum_{p\in E^*}\beta_p(y)Re\langle p, y - x\rangle \geq \beta_0(y)\left[\min_{w\in T(y)} Re\langle w, y - x\rangle + h(y) - h(x)\right] + \sum_{p\in E^*}\beta_p(y)Re\langle p, y - x\rangle \text{ (by (2.8))}. \tag{2.9}$$

Hence, we have $\phi(x, y) \leq 0$.

(iv) By hypothesis, there exists a non-empty compact (and therefore closed) subset K of X and a point $x_0 \in X$ such that $x_0 \in K \cap S(y)$ and $\inf_{w\in T(y)} Re\langle w, y - x_0\rangle + h(y) - h(x_0) > 0$ for all $y \in X\backslash K$.

Thus, for each $y \in X\backslash K$, $\sup_{x\in S(y)}\inf_{w\in T(y)}[Re\langle w, y - x\rangle + h(y) - h(x)] > 0$. Hence, $y \in V_0$ and $\beta_0(y)\left[\inf_{w\in T(y)} Re\langle w, y - x_0\rangle + h(y) - h(x_0)\right] > 0$ for all $y \in X\backslash K$; also, $Re\langle p, y - x_0\rangle > 0$ whenever $\beta_p(y) > 0$ for $p \in E^*$.

Consequently,

$$\phi(x_0, y) = \beta_0(y)\left[\inf_{w\in T(y)} Re\langle w, y - x_0\rangle + h(y) - h(x_0)\right] + \sum_{p\in E^*}\beta_p(y)Re\langle p, y - x_0\rangle > 0$$

for all $y \in X\backslash K$.

Thus, the hypothesis of *(d)* of Theorem 1.3 is satisfied trivially. (If T is a strongly h-quasi-pseudo-monotone type III operator, we equip E with the weak topology.) Thus ϕ satisfies all the hypotheses of Theorem 1.3. Hence, by Theorem 1.3, there exists a point $\hat{y} \in K$ such that $\phi(x, \hat{y}) \leq 0$ for all $x \in X$, i.e.

$$\beta_0(\hat{y})\left[\min_{w\in T(\hat{y})} Re\langle w, \hat{y} - x\rangle + h(\hat{y}) - h(x)\right] + \sum_{p\in E^*}\beta_p(\hat{y})Re\langle p, \hat{y} - x\rangle \leq 0 \tag{2.10}$$

for all $x \in X$.

If $\beta_0(\hat{y}) > 0$, then $\hat{y} \in V_0 = \Sigma$, so that $\gamma(\hat{y}) > 0$. Choose $\hat{x} \in S(\hat{y}) \subset X$ such that

$$\inf_{w\in T(\hat{y})} Re\langle w, \hat{y} - \hat{x}\rangle + h(\hat{y}) - h(\hat{x}) \geq \frac{\gamma(\hat{y})}{2} > 0.$$

Then it follows that

$$\beta_0(\hat{y})\left[\inf_{w\in T(\hat{y})} Re\langle w, \hat{y} - \hat{x}\rangle + h(\hat{y}) - h(\hat{x})\right] > 0.$$

If $\beta_p(\hat{y}) > 0$ for some $p \in E^*$, then $\hat{y} \in V_p$ and hence

$$Re\langle p, \hat{y}\rangle > \sup_{x\in S(\hat{y})} Re\langle p, x\rangle \geq Re\langle p, \hat{x}\rangle$$

and so $Re\langle p, \hat{y} - \hat{x}\rangle > 0$. Then we see that $\beta_p(\hat{y})Re\langle p, \hat{y} - \hat{x}\rangle > 0$ whenever $\beta_p(\hat{y}) > 0$ for $p \in E^*$. Since $\beta_0(\hat{y}) > 0$ or $\beta_p(\hat{y}) > 0$ for some $p \in E^*$, it follows that

$$\phi(\hat{x}, \hat{y}) = \beta_0(\hat{y})\left[\inf_{w\in T(\hat{y})} Re\langle w, \hat{y} - \hat{x}\rangle + h(\hat{y}) - h(\hat{x})\right] + \sum_{p\in E^*}\beta_p(\hat{y})Re\langle p, \hat{y} - \hat{x}\rangle > 0,$$

which contradicts (2.10). This contradiction proves Step 1. Hence we have shown that there exists a point $\hat{y} \in X$ such that $\hat{y} \in S(\hat{y})$ and

$$\sup_{x\in S(\hat{y})}\left[\inf_{w\in T(\hat{y})} Re\langle w, \hat{y} - x\rangle + h(\hat{y}) - h(x)\right] \leq 0.$$

Step 2. We need to show that there exists a point $\hat{w} \in T(\hat{y})$ such that $Re\langle \hat{w}, \hat{y} - x\rangle + h(\hat{y}) - h(x) \leq 0$ for all $x \in S(\hat{y})$.

From Step 1, we have

$$\sup_{x\in S(\hat{y})}\left[\inf_{w\in T(\hat{y})} Re\langle w, \hat{y} - x\rangle + h(\hat{y}) - h(x)\right] \leq 0, \tag{2.11}$$

where $T(\hat{y})$ is a weak*-compact convex subset of the Hausdorff topological vector space E^* and $S(\hat{y})$ is a convex subset of X.

Now, we define $f: S(\hat{y}) \times T(\hat{y}) \to \mathbb{R}$ by $f(x.w) = Re\langle w, \hat{y} - x\rangle + h(\hat{y}) - h(x)$ for each $x \in S(\hat{y})$ and $w \in T(\hat{y})$. Then, for each fixed $x \in S(\hat{y})$, the mapping $w \mapsto f(x, w)$ is convex and continuous on $T(\hat{y})$ and, for each fixed $w \in T(\hat{y})$, the mapping $x \mapsto f(x, w)$ is concave on $S(\hat{y})$. So, we can apply Kneser's Minimax Theorem (Theorem 2.3) and obtain the following:

$$\min_{w\in T(\hat{y})}\sup_{x\in S(\hat{y})}[Re\langle w, \hat{y} - x\rangle + h(\hat{y}) - h(x)] = \sup_{x\in S(\hat{y})}\left[\min_{w\in T(\hat{y})}[Re\langle w, \hat{y} - x\rangle + h(\hat{y}) - h(x)]\right].$$

Hence, by (2.11), we obtain

$$\min_{w\in T(\hat{y})}\sup_{x\in S(\hat{y})}[Re\langle w, \hat{y} - x\rangle + h(\hat{y}) - h(x)] \leq 0.$$

Since $T(\hat{y})$ is compact, there exists $\hat{w} \in T(\hat{y})$ such that

$$Re\langle \hat{w}, \hat{y} - x\rangle + h(\hat{y}) - h(x) \leq 0$$

for all $x \in S(\hat{y})$. This completes the proof. ■

When X is compact, we obtain the following immediate consequence of Theorem 3.1:

Theorem 3.2. Let E be a locally convex Hausdorff topological vector space, X be a non-empty compact convex subset of E and $h: E \to \mathbb{R}$ be convex with $h(X)$ bounded. Let $S: X \to 2^X$ be upper semi-continuous such that each $S(x)$ is closed convex and $T: X \to 2^{E^*}$ be an h-pseudo-monotone type III (respectively, a strongly h-pseudo-monotone type III) operator and be upper semi-continuous from $co(A)$ to the weak*-topology on E^* for each $A \in \mathcal{F}(X)$ and $T(X)$ is strongly bounded. Also, for each $x \in X$, $T(x)$ is weak*-compact convex. Suppose that the set

$$\Sigma = \{y \in X: \sup_{x\in S(y)} [\inf_{w\in T(y)} Re\langle w, y - x\rangle + h(y) - h(x)] > 0\}$$

is open in X and the following conditions are satisfied:

(a) For each $A \in \mathcal{F}(X)$, each $x, y \in co(A)$, and any net $\{y_\alpha\}_{\alpha\in\Gamma}$ in X converging to y, we have

(b) $\limsup_\alpha \left[\inf_{u\in T(y_\alpha)} Re\langle u, y_\alpha - x\rangle + h(y_\alpha) - h(x)\right] \le 0$, whenever $\limsup_\alpha \left[\inf_{u\in T(y_\alpha)} Re\langle u, y_\alpha - y\rangle + h(y_\alpha) - h(y)\right] \le 0$, and

(c) $\limsup_\alpha \left[\inf_{w\in T(y_\alpha)} Re\langle w, y_\alpha - x\rangle + h(y_\alpha) - h(x)\right] \ge \inf_{w\in T(y)} Re\langle w, y - x\rangle + h(y) - h(x)$, whenever $\limsup_\alpha \left[\inf_{w\in T(y_\alpha)} Re\langle w, y_\alpha - y\rangle + h(y_\alpha) - h(y)\right] \ge \inf_{w\in T(y)} Re\langle w, y - x\rangle + h(y) - h(x)$.

Then there exists a point $\hat{y} \in X$ such that

(i) $\hat{y} \in S(\hat{y})$ and

(ii) there exists a point $\hat{w} \in T(\hat{y})$ with $Re\langle \hat{w}, \hat{y} - x\rangle \le h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

Note that if the map $S: X \to 2^X$ is, in addition, lower semi-continuous and for each $y \in \Sigma$, T is upper semi-continuous at y in X, then the set Σ in Theorem 3.1 is always open in X and we obtain the following theorem:

Theorem 3.3. *Let E be a locally convex Hausdorff topological vector space, X be a non-empty paracompact convex and bounded subset of E and $h: E \to \mathbb{R}$ be convex with $h(X)$ bounded. Let $S: X \to 2^X$ be continuous such that each $S(x)$ is compact convex, $T: X \to 2^{E^*}$ be an h-pseudo-monotone type III (respectively, strongly h-pseudo-monotone type III) operator which is upper semi-continuous from $co(A)$ to the weak*-topology on E^* for each $A \in \mathcal{F}(X)$, with $T(X)$ strongly bounded. Also, for each $x \in X$, $T(x)$ is weak*-compact convex. Suppose that for each $y \in \Sigma = \{y \in X: \sup_{x\in S(y)}[\inf_{w\in T(y)} Re\langle w, y - x\rangle + h(y) - h(x)] > 0\}$, T is upper semi-continuous at y from the relative topology on X to the strong topology on E^* and the following conditions are satisfied:*

(a) For each $A \in \mathcal{F}(X)$, each $x, y \in co(A)$, and any net $\{y_\alpha\}_{\alpha\in\Gamma}$ in X converging to y, we have $\limsup_\alpha \left[\inf_{u\in T(y_\alpha)} Re\langle u, y_\alpha - x\rangle + h(y_\alpha) - h(x)\right] \le 0$, whenever $\limsup_\alpha \left[\inf_{u\in T(y_\alpha)} Re\langle u, y_\alpha - y\rangle + h(y_\alpha) - h(y)\right] \le 0$, and

(b) $\limsup_\alpha \left[\inf_{w\in T(y_\alpha)} Re\langle w, y_\alpha - x\rangle + h(y_\alpha) - h(x)\right] \ge \inf_{w\in T(y)} Re\langle w, y - x\rangle + h(y) - h(x)$, whenever $\limsup_\alpha \left[\inf_{w\in T(y_\alpha)} Re\langle w, y_\alpha - y\rangle + h(y_\alpha) - h(y)\right] \ge \inf_{w\in T(y)} Re\langle w, y - x\rangle + h(y) - h(x)$.

Suppose further that there exists a non-empty compact subset K of X and a point $x_0 \in X$ such that $x_0 \in K \cap S(y)$ and $\inf_{w\in T(y)} Re\langle w, y - x_0\rangle + h(y) - h(x_0) > 0$ for all $y \in X\backslash K$.

Then there exists a point $\hat{y} \in K$ such that

(i) $\hat{y} \in S(\hat{y})$ and

(ii) there exists a point $\hat{w} \in T(\hat{y})$ with $Re\langle \hat{w}, \hat{y} - x\rangle \le h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

The proof is similar to the proof of Theorem 3.1 in [13]. But for completeness, we shall include the detailed proof here.

Proof. The proof will follow from Theorem 3.1 if we can show that the set

$$\Sigma = \{y \in X: \sup_{x\in S(y)} [\inf_{w\in T(y)} Re\langle w, y - x\rangle + h(y) - h(x)] > 0\}$$

is open in X. To show that Σ is open in X, we start as follows:

Let $y_0 \in \Sigma$ be an arbitrary point. We show that there exists an open neighborhood N_0 of y_0 in X such that $N_0 \subset \Sigma$. Now, by definition of Σ, there exists a point x_0 in $S(y_0)$ with

$$\inf_{w\in T(y_0)} Re\langle w, y_0 - x_0\rangle + h(y_0) - h(x_0) > 0.$$

Let

$$\alpha := \inf_{w\in T(y_0)} Re\langle w, y_0 - x_0\rangle + h(y_0) - h(x_0).$$

Thus, $\alpha > 0$. Again, let

$$W := \{w \in E^*: \sup_{z_1,z_2\in X}|\langle w, z_1 - z_2\rangle| < \alpha/6\}.$$

Then W is a strongly open neighborhood of 0 in E^* and so $U_1 := T(y_0) + W$ is an open neighborhood of $T(y_0)$ in E^*. Since T is upper semi-continuous at y_0, there exists an open neighborhood N_1 of y_0 in X such that $T(y) \subset U_1$ for all $y \in N_1$. Since the mapping $x \mapsto \inf_{w\in T(y_0)} Re\langle w, x_0 - x\rangle + h(x_0) - h(x)$ is continuous at x_0, there exists an open neighborhood V_1 of x_0 in X such that

$$\left|\inf_{w\in T(y_0)} Re\langle w, x_0 - x\rangle + h(x_0) - h(x)\right| < \alpha/6$$

for all $x \in V_1$.

Since $x_0 \in V_1 \cap S(y_0) \neq \emptyset$ and S is lower semi-continuous at y_0, there exists an open neighborhood N_2 of y_0 in X such that $S(y) \cap V_1 \neq \emptyset$ for all $y \in N_2$. Since the mapping $y \mapsto \inf_{w\in T(y_0)} Re\langle w, y - y_0\rangle + h(y) - h(y_0)$ is continuous at y_0, there exists an open neighborhood N_3 of y_0 in X such that

$$\left|\inf_{w\in T(y_0)} Re\langle w, y - y_0\rangle + h(y) - h(y_0)\right| < \alpha/6$$

for all $y \in N_3$.

Let $N_0 := N_1 \cap N_2 \cap N_3$. Then N_0 is an open neighborhood of y_0 in X such that for each $y_1 \in N_0$, we have the following:

(a) $S(y_1) \cap V_1 \neq \emptyset$ as $y_1 \in N_2$; so, we can choose any $x_1 \in S(y_1) \cap V_1$.

(b) $\left|\inf_{w \in T(y_0)} Re\langle w, y_1 - y_0 \rangle + h(y_1) - h(y_0)\right| < \alpha/6$ as $y_1 \in N_3$.

(c) $T(y_1) \subset U_1 = T(y_0) + W$ as $y_1 \in N_1$.

(d) $\left|\inf_{w \in T(y_0)} Re\langle w, x_0 - x_1 \rangle + h(x_0) - h(x_1)\right| < \alpha/6$ as $x_1 \in V_1$.

Hence, we can obtain the following by omitting the details:

$$\begin{aligned}
&\inf_{w \in T(y_1)} Re\langle w, y_1 - x_1 \rangle + h(y_1) - h(x_1) \\
&\geq \inf_{w \in T(y_0)+W} Re\langle w, y_1 - x_1 \rangle + h(y_1) - h(x_1) \text{ by } (c) \\
&\geq \inf_{w \in T(y_0)} Re\langle w, y_1 - x_1 \rangle + h(y_1) - h(x_1) \\
&\qquad + \inf_{w \in W} Re\langle w, y_1 - x_1 \rangle \\
&\geq \inf_{w \in T(y_0)} Re\langle w, y_1 - y_0 \rangle + h(y_1) - h(y_0) \\
&+ \inf_{w \in T(y_0)} Re\langle w, y_0 - x_0 \rangle + h(y_0) - h(x_0) \\
&+ \inf_{w \in T(y_0)} Re\langle w, x_0 - x_1 \rangle + h(x_0) - h(x_1) \\
&+ \inf_{w \in W} Re\langle w, y_1 - x_1 \rangle \geq -\frac{\alpha}{6} + \alpha - \frac{\alpha}{6} - \frac{\alpha}{6} = \frac{\alpha}{2} > 0.
\end{aligned}$$

Consequently, we have

$$\sup_{x \in S(y_1)} [\inf_{w \in T(y_1)} Re\langle w, y_1 - x \rangle + h(y_1) - h(x)] > 0$$

since $x_1 \in S(y_1)$. Hence, $y_1 \in \Sigma$ for all $y_1 \in N_0$. Therefore, $y_0 \in N_0 \subset \Sigma$. But y_0 was arbitrary. Consequently, Σ is open in X. Thus, all the hypotheses of Theorem 3.1 are satisfied. Hence, the conclusion follows from Theorem 3.1. This completes the proof. ■

When X is compact, we obtain the following theorem:

Theorem 3.4. *Let E be a locally convex Hausdorff topological vector space, X be a non-empty compact convex subset of E and $h: E \to \mathbb{R}$ be convex with $h(X)$ bounded. Let $S: X \to 2^X$ be continuous such that each $S(x)$ is closed convex, $T: X \to 2^{E^*}$ be an h-pseudo-monotone type III (respectively, strongly h-pseudo-monotone type III) operator which is upper semi-continuous from $co(A)$ to the weak*-topology on E^* for each $A \in \mathcal{F}(X)$, with $T(X)$ strongly bounded. Also, for each $x \in X$, $T(x)$ is weak*-compact convex. Suppose that for each $y \in \Sigma = \{y \in X: \sup_{x \in S(y)}[\inf_{w \in T(y)} Re\langle w, y - x \rangle + h(y) - h(x)] > 0\}$, T is upper semi-continuous at y from the relative topology on X to the strong topology on E^* and the following conditions are satisfied:*

(a) For each $A \in \mathcal{F}(X)$, each $x, y \in co(A)$, and any net $\{y_\alpha\}_{\alpha \in \Gamma}$ in X converging to y, we have $\limsup_\alpha \left[\inf_{u \in T(y_\alpha)} Re\langle u, y_\alpha - x \rangle + h(y_\alpha) - h(x)\right] \leq 0$ whenever $\limsup_\alpha \left[\inf_{u \in T(y_\alpha)} Re\langle u, y_\alpha - y \rangle + h(y_\alpha) - h(y)\right] \leq 0$, and

(b) $\limsup_\alpha \left[\inf_{w \in T(y_\alpha)} Re\langle w, y_\alpha - x \rangle + h(y_\alpha) - h(x)\right] \geq \inf_{w \in T(y)} Re\langle w, y - x \rangle + h(y) - h(x)$ whenever $\limsup_\alpha \left[\inf_{w \in T(y_\alpha)} Re\langle w, y_\alpha - y \rangle + h(y_\alpha) - h(y)\right] \geq \inf_{w \in T(y)} Re\langle w, y - x \rangle + h(y) - h(x)$.

Then there exists a point $\hat{y} \in X$ such that

(i) $\hat{y} \in S(\hat{y})$ and

(ii) there exists a point $\hat{w} \in T(\hat{y})$ with $Re\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

Remark 3.5. (1) Theorems 3.1, 3.2, 3.3 and 3.4 of this paper are further extensions of the results obtained in [4] on generalized quasi-variational inequalities of pseudo-monotone type III and strongly pseudo-monotone type III operators.

(2) In 1985, Shih and Tan ([4]) obtained results on generalized quasi-variational inequalities in locally convex topological vector spaces and their results were obtained on compact sets where the set-valued mappings were either lower semi-continuous or upper semi-continuous. Our present paper is another extension of the original work in [4] using pseudo-monotone type III and strongly pseudo-monotone type III operators on non-compact sets.

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