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Mathematical Electrodynamics: Groups, Cohomology Classes, Unitary Representations, Orbits and Integral Transforms in Electro-Physics

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Abstract: Through consider applications of the $(\mathfrak{g}, K)-$ modules as $\pounds-$ modules to the Lie groups $SL(2,R), SU(2), SO(4), U(4,R), SU(2,2), SO(4,R), SU(p,q)$, and $Sp(n,K)$, the evaluating of integrals on equivariant and invariant holomorphic vector bundles under the action of these groups, are created and developed electromagnetic models of the space-time with their field observable obtained as images of integral transforms that are solutions of the field equations modulo electromagnetic fields. Finally is constructed through the equivalences obtained by these integral transforms the moduli space involving the non-commutative rings in electro-physics.

Keywords: Electromagnetic Space-Time Models, Electromagnetic Intertwining Operators, Mathematical Electrodynamics, Maxwell Fields, Ultra-Hyperbolic Wave Equation

1. Introduction

The beginning of the mathematical electrodynamics[1] is given in the problem of the equivalence in the geometrical Langland problem and their special attention, to the field theory, in the geometrical Langlands ramification[2] [1, 2] to the extension of the fields and particular Maxwell fields as gauges of other fields.

These equivalences that are our interests in the electromagnetism context, are given between in densities spaces with weights until of arbitrary order of complex holomorphic bundles in electrodynamics and their corresponding cohomologies of intertwining integral operators [3] of the Maxwell fields to obtain a cohomological theory of the electromagnetic interactions of the Universe and their generalization and extension to interaction more general as cosmic rays, gravity and quantum interactions of any spin type that have to see with charge constitution (fermions) [4, 5] or the matter (bosons).

We are interested in the cohomology of the intertwining operators on cohomological spaces of the type

$$H^{\bullet}(V, O(B)) \cong_{\mathcal{P}} H^{\bullet}(\nu^{-1}(B), \mu^{-1}O(B)), \qquad (1)$$

which define the general solution class to the symmetric part (auto-dual) of the Maxwell equations and with the possibility of generalize to the anti-symmetric case to after use it in the solution of the massless fields equations, being this limit of a Lobachevski Universe in the infinite.

Said cohomology is naturally deduced to start of the re-interpretation of the contour integrals of the spaces $\mathcal{R}_{\sigma}(Ad(G^0) \otimes C^{\infty}(\Sigma))$, [6, 7][3] as isomorphisms between

1 Mathematical electrodynamics: It´s the part of the mathematical physics that studies the electromagnetic phenomena through their moduli stacks obtained by geometrical invariants of these integral transforms on complex space-time models.

2 Related of this research program the mathematical electrodynamics applies integral geometry methods the obtaining of geometrical ramifications of the differential operators that can be considered in the electromagnetic wave equation and their massless field equations to the construction of their solutions. These solutions are realized through cohomological classes.

3 The space consists of the images obtained through geometrical integral transforms as the Radon or Penrose transforms. These integrals to certain objects (cycles) can derive in contour integrals [3], for example to sources, monopoles and multi-poles we have the Conway integrals or Cauchy integrals. Also the Radon transform can be related with the Fourier transform to describe co-cycles in non-Euclidean spaces such as for example horocycles in the hyperbolic disc which can describe electromagnetic waves propagation in a 2-dimensional model of the

sheaves of cohomology groups on certain regions of a projective space P, and kernels or co-kernels of certain conformally invariant differential operators on corresponding regions of the Minkowski space. This is precisely the idea of the integral geometry that establish the using of a solutions cohomology of the Maxwell fields to their generalization and extension to the context of the space-time. σ, in this case is an smooth embedding of smooth sub-manifolds Σ, in a Riemannian manifold with homogeneous space G/G^0, with G^0, a connect component of G. $\mathcal{R}_\sigma$, results the Radon transform of the space $Ad(G^0) \otimes C^\infty(\Sigma)$, which represents a section of the vector bundle of geodesics in the complex Riemannian manifold M, and whose image in P, result to be belonging lines to the corresponding homogeneous vector bundle.

We can consider a reductive homogeneous space. Then are considered their open $G-$ orbits corresponding to the complex vector $G-$ bundle and is preceded to calculate an intertwining integral operator on said orbits belonging to a cohomology with coefficients in sections of the complex vector $G-$ bundle. Said integral is a realization of the corresponding irreducible unitary representations of the Lie group whose actions on fibers of the vector bundle are trivial and are $G-$ equivariants, being an isomorphism between cohomology classes of the open orbits of the corresponding homogeneous space. In twistor theory the representations are the cohomology classes $H^s(Q^\infty, \mathcal{L})$, with $s \leq r$, being $s = \dim_R Q$, which consider a homogeneous bundle of lines $\mathcal{L}$, of the corresponding flag manifold whose maximal compact submanifold Q, underlying in a fix open orbit G_0, in a submanifold X_0, of the flag manifold.

Finally, in using a connecting homomorphism of Mayer-Vietoris, we have a $SU(2,2)-$ invariant pairing on $H^1(Q, E)$, with E, a vector bundle defined by the Hermitian form $<,>$.

In a L^2-cohomology said inner product comes given by the integrals

$$< \omega, \eta >= \int_Q \omega \wedge \eta, \qquad (2)$$

$\forall \omega, \eta \in \Lambda^2(M)$.

This idea can be extrapolated to the computing of the curvature and torsion considering said Hermitian product on Maxwell fields F, using the auto-duality and anti-auto-duality of F.

In electrodynamics will be of interest the isomorphisms such that

$$\Gamma(Q,E) \cong_T \{\phi \text{ of conformal weight } 1, \text{ on } M, \text{ satisfying } \nabla^{(A'}_{(A}\nabla^{B')}_{B)}\phi + \Phi^{A'B'}_{AB}\phi = 0\}, \qquad (3)$$

with the possibility to extend them to the quantification of the curvature and torsion of the space-time to the macro and microscopic levels as the image of an integral operator in generalized topological spaces.

__Fig. 1.__ The Radon integral is calculated on every horocycle (hyperbolic wave front). The points where these hocycles are tangent to circle of the hyperbolic disc can be interpreted as sources where can be applied the Conway integral.

2. 1, and 2-Dimensional Cohomologies in Electromagnetism

Let M, be a Minkowski space with a signature metric $(+ - - -)$. There are two ways to discover the free Maxwell fields on M:

a) As a $2-$ form F, with the conditions $dF = 0$, and $d * F = 0$, where $*$, is the star Hodge operator.

b) As a equivalence class of $1-$ forms Φ, under the condition $d * d\Phi = 0$, where two $1-$ forms are said to be equivalent if their difference is df, to some function f.

Locally, that is to say, in the vicinity of the cubic space R^3, are the same and the relation between these come given for

$$F = d\Phi, \qquad (4)$$

The $1-$ form $\Phi \in \Lambda^1(R^4)$, is called a electric potential to the Maxwell field $F \in \Lambda^2(R^4)$.

Now globally the mapping $F \mapsto \Phi$, is of any way injective or surjective. Thus there is a topological obstruction to find a potential Φ, to a Maxwell field given F, and a seemed topological obstruction to find to f, such that $df = \Phi$, when $d\Phi = 0$. This global *equivalence* is very important, since is required determine experimentally which or if any of the descriptions is physically relevant.

space-time (see the figure 1). Also the "energy" of observables such as curvature or torsion can be described by certain integrals that derive of the Radon transform related with other integral transforms [8-10].

The equivalence class of potentials Φ, can be precisely re-interpreted as a connection on a trivial lines bundle and permitting non-trivial lines bundles with connection endowed of the arbitrary fields F, (as the curvature of their connection). The Aharanov-Bohm endows of a justification to this re-formulation and their resultant physical theories and generalizations to the respect (connections on vector bundles) are known as gauge theories. This in twistor geometry has their equivalent as the Ward correspondence to the Yang-Mills self-dual fields. This last in Lie groups theory means that in the electrodynamics we replace the Abelian group $U(1)$, by a non-Abelian group, for example $SU(n)$.

Likewise, the Ward correspondence establishes bijective mappings between principal fibered on the sphere S^4 (as the lines bundles in $P^3(R)$) with Lie group $SU(2)$.

Then a gauge transformation is an automorphism of the corresponding trivial vector $G-$ bundle (that to the potential class is a trivial lines bundle).

Now, of this global situation is necessary to have in consideration that in dimension 4, are had the following particular characteristics:

a) The Yang-Mills functional is conformally invariant, that is to say, only depends in the conformal class of the metric in M. This implies in particular that given that the Euclidean mapping in R^4, is conformally equivalent to the sphere S^4, least one point, then the bundles that are asymptotically trivial on R^4, are extended to S^4.

b) The Hodge operator $*:\Lambda^2(M)\rightarrow\Lambda^2(M)$, is an involution and thus the $2-$ forms are discomposed in auto-dual and anti-auto-dual forms. Then considering the equivalence of the two descriptions in $\Lambda^2(R^4)$, at least very nearest in $\Lambda^2(M)$, is of interest to obtain reflected this equivalence in the space of solutions of both descriptions of the Maxwell fields. To it, is necessary to evaluate the corresponding cohomological space that says such equivalence through of their isomorphism with the cohomology of lines $G-$ bundle determined on a adequate heuristic modelling with bases in an integral geometry of the space-time M.

To this goal, we consider the Maxwell fields on a compactified Minkowski space M, (that is to say, suppose that M, is compact), then $M\cong S^1\times S^3$ [4]. But of the exact short sequence:

$$0\rightarrow H^1(M,R)\xrightarrow{d}H^2(M,R)\rightarrow H^3(S^1\times S^1,R)\rightarrow 0, \quad (5)$$

we deduce that $H^2(M,R)=0$, where we conclude that the mapping

$$d:\{\text{potentials}\}\rightarrow\{\text{fields}\}, \quad (6)$$

is surjective.

If for one moment we weakening our notion of equivalence such that equivalent potentials are now those such that go to give to the same field then we obtain the isomorphism with the cohomological space $H^1(M,R)/\approx_\Phi$, given by the co-bordering operator

$$\begin{array}{ccc} d:\{\text{potential equivalenc e class}\} & \overset{\cong}{\rightarrow} & \{\text{fields}\} \\ \uparrow\downarrow & & \uparrow\downarrow \\ H^1(M,R)/\approx_\Phi & \cong & H^2(M,R) \end{array} , \quad (7)$$

Thus in this instance, the group of confomal transformations preserving global orientation acts on M, setting potentials to potentials and field to fields.

Since d, is an invariant conformally operator on $2-$ forms in a $4-$ dimensional space, also d, is an intertwining operator on these two representations of this group. By twistor theory, the group of these transformations well known for their $4-$ fold gauge is identified by $SU(2,2)$.

Inside the context of the representation theory will be of our interest the Maxwell representations of $SU(2,2)$, that are unitarizables, this with the goal of construct an invariant inner product (a symmetric positive defined Hermitian form).

One canonical way to establish this in such twistor construction through Maxwell representations of $SU(2,2)$, is to observe the invariant Hermitian couplings between two different realizations of the same representation. To it, is necessary to use an interwining integral operator on the connect component of the space $S^1\times S^3$, where a vector field is integrable. The integrals in this respect belong to an integral operator class characterized by the cohomology with coefficients in sections of a given holomorphic bundle for a fibered $E\rightarrow S^2$. Likewise, $SU(2,2)$, as compact component of $SL(4,\mathbb{C})$, acts conformally on the generated product for the mentioned integrals.

To the context of the Maxwell fields these integrals degenerate in a symmetric integral operator, which is a Fourier transform on the components S^{2-}, and S^{2+}, of the Maxwell fields space in the limits of the De Sitter space, that is to say, the algebra $\Lambda^2(R^4)$, when the Maxwell fields are asymptotically simples in S^4.

We consider a Lie algebra $\mathfrak{g}$, of G, and let $(,)$, be the bilinear natural form or the Killing form of $\mathfrak{g}$. Suppose that G, is $SU(n)$, or $U(n)$, and $X,Y\in\mathfrak{g}$, anti-Hermitian matrices of order n, then

$$(X,Y)=-\mathrm{tr}(XY), \quad (8)$$

We consider here that $(,)$, is a positive defined Hermitian form on $\mathfrak{g}$. Then the Yang-Mills functional (generalized solution) to the auto-dual equations of Yang-Mills comes given by the Lagrangian

[4] $S^1\times S^3$, is a discomposing of S^4, adequate to integrate certain $3-$ form on S^4.

$$\Lambda_{YM} = -\frac{1}{4}(F_{\alpha\beta}, F^{\alpha\beta}), \tag{9}$$

which comes from the Maxwell Lagrangian

$$\Lambda_M = -\frac{1}{4} < F_{\alpha\beta}, F^{\alpha\beta} >, \tag{10}$$

with $*F_{\alpha\beta} = \Psi_{\alpha\beta\gamma\delta} F^{\gamma\delta}$, where $\Psi_{\alpha\beta\gamma\delta}$, is a volume form in M, and $*$, the Hodge operator, then the Maxwell equations are precisely the Euler-Lagrange equations corresponding to the variation problem to said Maxwell fields, which comes given for

$$d * F = 0, \tag{11}$$

If $*$, is the Hodge operator in M, we have $** = -1$, when this acts on elements of the space $\Lambda^2(R^4)$. Therefore we can to consider the proper spaces of $*$, corresponding to the eigenvalues i, and $-i$.

Likewise, the equations to the $2-$ forms of the space $\Lambda^2(R^4)$, such that

$$*\sigma = \pm i\sigma, \tag{12}$$

are the auto-dual and anti-auto-dual equations of Yang Mills. In particular, if that $2-$ form is a Maxwell field F, we have that said field admits the discomposing

$$F = F^+ + F^-, \tag{13}$$

which generates all the space $D^{+2} \oplus D^{-2}$, of auto-dual Maxwell fields where every representation space D^{+2}, and D^{-2}, is a irreducible unitary representation of the group $SU(2,2)$. Here the representation space to (13) is explicitly:

$$\begin{aligned}&\{\text{autodual Maxwell fields of positive frequencyon } M\} \oplus \\ &\{\text{autodual Maxwell fields of negative frequencyon } M\}, \end{aligned} \tag{14}$$

Here the positive frequency (or respectively negative frequency) means that in the Fourier discomposing only the terms in $e^{-ik_\alpha x^\alpha}$, which occur to k_α, pointing to the future (or respectively to the past). Likewise in function of the $2-$ forms F, the auto-dual and anti-auto-dual forms can be written as:

$$F^+ = \frac{1}{2}(F - i * F), \tag{15}$$

that is the auto-dual part of the field F, and

$$F^- = \frac{1}{2}(F + i * F), \tag{16}$$

that is the anti-auto-dual part of the field F.

3. Orbital Integrals and Integral Transforms to Maxwell Fields and Their Orbits

Let M^+, and P^+, be open orbits of $SU(2,2)$, on $M = \mathrm{Gr}_2(\mathbb{C}^4)$, and $P = \mathbb{C}P_3 = P(\mathbb{C}^4)$, respectively. The projective dual twistor space is $P^* = P((\mathbb{C}^4)^*)$. The sheaves of holomorphic functions and of holomorphic $3-$ forms (that is to say, belonging to the space $\Lambda^3(S^3)$) are denoted by $\mathcal{O}$, and Ω^3 respectively.

The Penrose transform to this case results natural, and in particular this isomorphism with cohomology results to be $SU(2,2)-$ equivariant.

It´s clear that the twistor description of this representation is one of the more clear that the version of Maxwell fields to effects of induction of $G-$ modules (that is to say, representations or classification spaces). The action of $SU(2,2)$, is automatic. Now, is natural to do the question, how the scalar product of the Hermitian form arises in this twistor correspondence.

Indeed, motivated by the classic construction, is well known that this can be possible through an intertwining operator. The cohomology $H^1(|P^+|, \Omega^3)$, (where $|P^+|$, represents the closure of the open orbit P^+) which gives an ascent directly to fields on M^+, considering that the description "potential modulo gauge" of the same fields find their interpretation on the dual twistor theory as $H^1(|P^{*-}|, \mathcal{O})$. Note that $|M^+|$, is simply connect (really contractive) and has Stein base neighborhood such that the original description of field and the description "potential module gauge" meet. Thus the intertwining integral operator, which to establish the crucial ingredient in the classic construction is interpreted in the twistor description as a twistor intertwining integral operator, being this the twistor transform

$$\mathcal{T} : H^1(|P^+|, \Omega^3) \to H^1(|P^{*-}|, \mathcal{O}), \tag{17}$$

Considering this operator, is easy to describe the scalar product in a way in which is manifested the $SU(2,2)-$ invariance. If $\Gamma \in H^1(|P^{*-}|, \mathcal{O})$, then $\Gamma \in H^1(|P^-|, \mathcal{O})$, and to obtain the invariance under conformal transformations of said product, is necessary the multi-linear coupling of $G-$ modules $H^1(|P^+|, \Omega^3)$, and $H^1(|P^-|, \mathcal{O})$, given by

$$H^1(|P^+|, \Omega^3) \otimes H^1(|P^-|, \mathcal{O}) \to \mathbb{C}, \tag{18}$$

Such pairing can be obtained taking representative forms σ, on $|P^+|$, of the type (3,1), and τ, on $|P^-|$, of the type (0,1), and integrating the $5-$ forms $\sigma \wedge \tau$, on P^0. This equivalent in the homomorphism language to

$$H^1(|P^+|, \Omega^3) \otimes H^1(|P^-|, \mathcal{O}) \xrightarrow{\wedge} H^2(P^0, \Omega^3) \xrightarrow{\delta} H^1(P, \Omega^3) \to \mathbb{C}, \tag{19}$$

and by duality in twistor theory, can be deduced the wanted scalar product. Thus $F, G \in H^1(|P^+|, \Omega^3)$, we define

$$< F, G >= F.\overline{TG}, \tag{20}$$

As exercise, consulting [11] can be demonstrate the meeting of this twistor with the classic construction on the massless fields. Of fact, studying the solutions of the ultra-hyperbolic wave equation proposed through the John integral given by

$$\phi(w,x,y,z) = \int_{-\infty}^{+\infty} f(w+x\zeta, y+\zeta z, \zeta)d\zeta, \tag{21}$$

having that

$$\begin{array}{c} \dfrac{\partial^2\phi}{\partial w\partial z} - \dfrac{\partial^2\phi}{\partial x\partial y} = 0 \\ \uparrow\downarrow \qquad \uparrow\downarrow \\ \int_R \zeta \dfrac{\partial^2\phi}{\partial p\partial q} d\zeta = 0 \end{array}, \tag{22}$$

Penrose describes this formula to the case of the Maxwell fields determining the solution of the wave equation in R^4. Of fact, he finds a generalizing, which guarantees solutions of the Maxwell fields also known these as massless fields equations. Then a solution ϕ, of the massless field equations represented these last for the electromagnetic field equation

$$\Box^2\phi = 0, \tag{23}$$

in M, can be represented in $P^3(\mathbb{C})$, through the line integral

$$\phi(w,x,y,z) = \int_{S^1} f((w+x)+(y+iz)\zeta, (y-iz)+(w-x)\zeta, \zeta)d\zeta, \tag{24}$$

A geometrical interpretation of this problem comes given through the Radon transform (using lines bundles):

$$\mathcal{R} : C^\infty(\mathbb{C}^3) \to C^2(R^4),^5 \tag{25}$$

with rule of correspondence

$$f(p,q,\zeta) \mapsto \int_{L\subset R^4} f(p,q,\zeta)d\zeta, \tag{26}$$

However, to establish the same idea of calculate integrals on orbital classes of homogeneous spaces, we consider the flag manifold the lines space

$$\mathbb{F} = \{L | L \subset R^4\}, \tag{27}$$

the Penrose integral (24) takes the form under the scheme of the auto-dual solutions of the Yang-Mills equations (that is to say to the fields given by (15)) on S^4, as

5 $C^2(R^4) = \{\phi | \Box^2\phi = 0\}$.

$$\begin{array}{ccc} & \mathbb{F} & \\ {}_\mu\swarrow & & \searrow_\nu \\ P^3(\mathbb{C}) & & M \\ {}_\pi\searrow & & \swarrow_\phi \\ & S^4 & \end{array}, \tag{28}$$

which does possible translate fibered bundles of S^4, in fibered bundles of $P^3(\mathbb{C})$, having the Atiyah-Ward Theorem.

But, what happen with the fields given in (16) whose representation are given by $\Gamma \in H^1(|P^-|, \Omega^3)$?

The twistor construction has very much advantages on the classic construction planted from a point of view of Bateman and the integrals as given by (21), likewise the exposition given by Penrose to solutions of the electromagnetic wave equation in the space R^4, in the sense to give the possibility of to be generalized to the ambit of spaces of major dimension, or of structure that are defined under topologies more weakness. Very important representations of reductive Lie groups occur naturally as a cohomology on a homogeneous space, such is the case of some induced representations on generalized orbital spaces.

In this point, once can to have the possibility of know when these representations are unitarizable. To this goal, results useful to work with cohomology eluding of this way, arguments of the space-time that can hinder to the study of the unitarization of the representations, on all in the electrodynamical representations of Maxwell fields and their quantum possible extensions where the inherency of the space-time is intrinsic.

In this treatment we want to establish the inherency of the Maxwell fields on the space-time producing certain world sheets or electromagnetic images of the space-time including their singularities, which also can be re-interpreted by twistor geometry as zeros of certain polynomials constructed from a certain homogeneous lines bundles. This formulation carry us to the solutions obtained through integrals on lines or curves (cycles) where these solutions are co-cycles in the spaces of the holomorphic functions.

However there is not that forget that is wanted to obtain $G-$ modules that can be classified as $\pounds-$ modules with the goal of obtain all the unitary representations of the integral operators that act in the general solution of the Maxwell equations.

Thus we want to understand and generalize so much as to be possible the twistor transform $\mathcal{T} : H^1(|P^+|, \Omega^3) \to H^1(|P^{*-}|, \mathcal{O})$, as an integral operator in electrodynamics of intertwining type to the obtaining of solutions in the space-time modulo the Maxwell fields inside of the context of the unitary representations. Furthermore also we want to use the formulating of the twistor transform as an ensemble of classic intertwining integral operators and gives their corresponding representations is necessary to check that the Hermitian form that comes from the twistor transform is symmetric. Finally, once can to verify this on $K-$types. This has been made for example to the leader representations of $SU(p,q)$.

Consider a complex vector space $\mathbb{T}$, of dimension $N+1$, with Hermitian form Φ, with signature (p,q), $p+q=N+1$. Assume that by simplicity that $2 \le p \le q$ (if $p=1$, (or pair if $p=0$), the general procedure is of fact correct but the conclusions are under certain little modification [12].

Let $G = SU(p,q)$, be the subgroup of $SL(N+1,\mathbb{C})$, which preserves Φ. The projective space $P^N(\mathbb{C})$, is divided in three orbits, P^+, P^-, and P^0, under the action of G. Here P^+, respectively P^-) is the lines space $x \subset \mathbb{T}$, such that $\Phi|_x$, is positive (respectively negative) defined. The space P^0, some times denoted by PN, consists of those lines on which Φ, is null. Thus P^0, is a real hypersurface in $P^N(\mathbb{C})$. Note that P^+, and P^-, are open subsets indicated as complex manifolds. The fact of that these spaces are the unique open $G-$orbits do the pairing constructions more simple. Let M, be the Grassmannian of $p-$ dimensional subspaces of $\mathbb{T}$. These open $\min(p, q)+1$ $G-$orbits in M, correspond to the possible ways in which Φ, can be reduced as a non-degenerated form on a $p-$dimensional space. It´s of interests in the orbit M^+, on whose points Φ, is reduced to be positive defined. We note that the orbit M^-, is a Stein manifold.

As was demonstrated in [11] the composing of two Penrose transforms establish the isomorphism given for the twistor transform to $SU(p,q)$, that is to say:

$$\mathcal{T} : H^{p-1}(P^+,\mathcal{O}(-n-p)) \to H^{q-1}(P^{*-},\mathcal{O}(n-p)), \tag{29}$$

The sheaf $\mathcal{O}(-k)$, corresponds to the kth-potency of the tautological lines bundle on $P^N(\mathbb{C})$. Thus when $p=n=2$, $\mathcal{O}(-4)=\Omega^3$, and this meets with the mapping (17). The demonstration of that this is an isomorphism is similar to the demonstration of that (17) and [6] are isomorphisms. For example, when $n \ge 1$, once demonstrate that the left side is isomorphic to fields on M^+, satisfying the analogous differential equations to the massless field equations of helicity $n/2$, on the Minkowski space. Seemly, the right side is naturally isomorphic to potentials modulo gauge to the same fields on M^{*-}. But M^{*-}, is the same as in M^+, in the isomorphism. Newly once can to extend the isomorphism to the closures of P^+, and P^-, given the theorem:

Theorem 3. 1. There are a mapping $SU(p,q)-$equivariant

$$\mathcal{T} : H^{p-1}(\left|P^+\right|,\mathcal{O}(-n-p)) \to H^{q-1}(\left|P^{*-}\right|,\mathcal{O}(n-p)),$$

which is an isomorphism.

6 Theorem. To an open homomorphism $U \subset S^4$, exists a natural isomorphism

$$\mathcal{P} : H^1(\pi^{-1}(U),\mathcal{O}(-2)) \to \{\phi, \text{ Harmonic on } R\},$$

called the Penrose transform.

Table I. *To Conformal Actions of the Group* $SU(2,2)$, *in Orbits of the Space-Time.*

Orbits	**Electromagnetic Twistor Representation Space**	**Electromagnetic Space-Time Images**[a]
$\left\|M^+\right\|$	$H^1(\left\|M^+\right\|,\mathcal{O})$	Auto-dual Maxwell Fields of Positive Frequency on M.
$\left\|M^-\right\|$	$H^1(\left\|M^{*-}\right\|,\mathcal{O})$	Auto-dual Maxwell Fields of Negative Frequency on M.
$\left\|P^+\right\|$	$H^1(\left\|P^{*-}\right\|,\mathcal{O})$	Potentials Modulo Gauge on M.
$\left\|P^-\right\|$ $\left\|P^0\right\|$	$H^1(\left\|P^-\right\|,\mathcal{O})$ $H^1(P,\Omega^3)$	Potentials Modulo Gauge on M. Potentials modulo Elemental States on M.
$\overline{\mathcal{O}}_S$	$H^1(\Pi - \coprod,\mathcal{O}(V))$.	Cohomological Contours
U^0	$H^1_{\mathcal{L}}(U'',\mathcal{O}(-2))$	Electromagnetic Poles: Mono-poles and Multi-poles
	$H^1(B^+,\mathcal{O}(V))$	Massless Fields on Future Light Cones
B^-	$H^1(B^-,\mathcal{O}(V))$	Massless Fields on Past Light Cones
P^+	$H^{q-1}(P^{*-},\mathcal{O}(n-2))$	Potentials Modulo Massless Fields with helicity $n/2$
PT	$H^1(PT^*,\mathcal{O})$	Wave Functions for Massless Fields with helicity $n/2$
P/L	$H^1(\overline{P}^+,\mathcal{O}(2s-2))$	L^2- Positive Frequencies Massless Fields of Helicity s on M
B	$H^{1,1}_{[\infty]}(M,\mathcal{O}(n))$	Verma Modules to Classification of the extended Fields in Light Cones on M,
B	$H^{n,n}_{[\infty]}(M,\mathcal{O}(n))$	Verma Modules to Classification of Extended Fields in Light Cones on $M = SL(\mathbb{C},2)$, $n \ge 2$

If we consider the holomorphic de Rham sequence of P, given by

$$0 \to \mathbb{C} \to \Omega^{\bullet}, \tag{30}$$

This is very nearest to the twistor theory of Maxwell theory. The first and last sheaves that are $\mathcal{O}$, and $\mathcal{O}(-4)$, and cohomological space of type H^1, of these gives respectively potential modulo gauge for left-handed Maxwell fields and right-handed Maxwell fields (see the Table 1).

But the constant sheaf $\mathbb{C}$, which is resolved by (30) has significance in the electromagnetic charge that live in $\mathbb{C}$, (as was opposed to some more elaborate representation space of the conformal group, as for example $SU(2,2)$). In this sense we can consider the Penrose transform framework to obtain two pieces to $H^1(U'',\Omega^1)$, [7]

7 Remember as has been mentioned we want isomorphism of the form (1), that is to say:

$$H^1(U'',\mathcal{O}(V)) = H^1(U',\mu^{-1}\mathcal{O}(V)),$$

where considering the natural double fibration, we have $U \subset M$, which is some (suitable convex) region of the space-time, also $\nu^{-1}(U) = U' \subset \mathbb{F}$, and $\mu(U') = U'' \subset P$.

$$\{\text{gauge-restricted gauge for right-handed potentials on } U\}, \qquad (31)$$

and the respectively

$$\{\text{left-handed fields on } U\}, \qquad (32)$$

Also the Penrose transform of $H^1(U'', \Omega^2)$, is as (31).

In these identifications, "gauge-restricted" refers to the imposition of the conformally invariant conditions (see the figure 3):

$$\Box^2 f = 0, \qquad (33)$$

and

$$\Box \nabla^{\alpha} \Phi_{\alpha} = 0, \qquad (34)$$

as was discussed in the before paragraphs of this section and the section 2. Then the Penrose transform of the complex (30) contains all the spaces of fields that one is interested in for Maxwell theory (see the Table [7]).

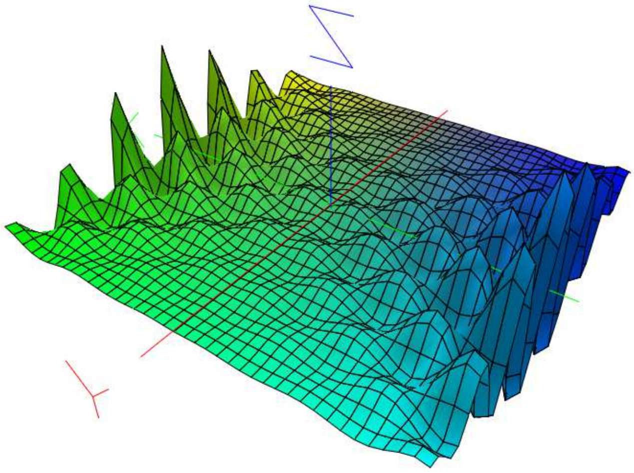

Fig. 2. *Electromagnetic waves in conformal actions of the group* $SU(2,2)$, *on a* $2-$ *dimensional flat model of the space-time. The ultra-hyperbolic wave equation is satisfied. In both sides of axis* X, *appear the auto-dual Maxwell fields of positive frequency and negative frequency on* M, *respectively that go being added in each time to each orbit. This corresponds to partial waves expansions in 2-dimensions [13].*

4. Electrodynamical Representations of the Cosmos and Their Elements

To can to give cosmic elements of electromagnetic nature we need correct electromagnetic representations of the space-time, where we have also that consider the superconducting and MHD phenomena in their sidereal composition, but using the twistor geometry to certain re-interpretations of the electromagnetic fields in the Universe.

We consider $\mathfrak{C}$, [8]a lines bundle such that $\mathfrak{C} \cong P^3(\mathbb{C})$, and also we consider the algebra in electrophysics [7]:

$$E \otimes H = \{F \in \Omega^2(M) | \pounds F = F\}, \qquad (35)$$ [9]

whose Maxwell fields satisfy the transformation law to coordinates systems:

$$F'_{ab} = \nabla'_a A'_b - \nabla'_b A'_a = \eta_{ac}\eta_{cd}(\nabla_c A_d - \nabla_d A_c) = \eta_{ac}\eta_{cd} F_{cd}, \qquad (36)$$

It´s necessary consider the moduli problem from a point of view of the equivalences that we want though the integral transforms. With this goal, we consider the algebra (35) as graded Lie algebra that control the moduli problem between derived categories planted in the moduli space $\mathscr{M}^0$, [10][14, 15], and the algebra of non-commutative rings of $\pounds-$ modules. That is to say we want an electromagnetic representation of Cosmos given by $SO(4) \cong SU(2) \otimes T$, where $T \cong U(1)$, the torus. By the isomorphism with $SO(4)$, (representation through $U(1)-$connections of the Universe) the endomorphisms of the group G, are unitary representations of $SO(4)$. An example of the application of these representations is the $2-$hyperbolic space (figure 1), whose curvature can be represented through the value of certain $G-$ invariant integrals on lines of a lines bundle on the principal $G-$ bundle of (35), and whose integrals come from the relation between co-cycles of the cohomology on G/L, with L, an open orbit and fields on G/K, (for example the fields on horo-cycles as solutions of the 2-dimensional hyperbolic wave equation when $G = SU(1,1)$, and $SO(2)$). Other case is to consider to G/L, as flag manifold and with $G = SL(2, \mathbb{C})$, and $L = T$. G/K, is a $3-$dimensional hyperbolic space and the space G/L, is a minitwistor space. The integrals that relate the cohomology on G/L, to this case of G/K, are the Whittaker integrals.

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8 $\mathfrak{C} = \{C(p) | C(p) \subset T_p(M), \forall p \in M\}$. In $\mathbb{C}^4$, (complex Minkowski space) $\mathfrak{C}$, is isomorphic to all linear Lorentzian manifold whose Lorentzian submanifolds have an affine connection $\sigma_t(X_p) = \nabla_{\gamma'(t)} X = \exp_p(tX)$.

9 This is a Lie algebra to superconducting phenomena.

10 A concrete application of the moduli space $\mathscr{M}^G_{G/H}$, on $SU(2)-$ bundles and the moduli space of the relation between hyperbolic waves (horocycles) [14] and the Haar measure of the group action in $SU(2,2)$, on M, is the moduli space of the functors obtained with the help of the integral transforms [15], is $\mathscr{M}^0 = \{D_{\mathrm{P}}(-k) | \Gamma H^{\bullet}{}_{SU(2)}(Hom(\pi_1(X), U(2)) = H^j(\Phi_{\mathrm{F}}(D_{\mathrm{P}}(-k))), \forall D_X, D_Y \in \mathrm{M}(D_{\mathrm{P}} - \text{modules})\}$, where the cohomological space $H^j(\Phi_{\mathrm{F}}(D_{\mathrm{P}}(-k)))$, is the space of the $SU(2)-$ equivariant functors related with the functors given in by the corresponding relation of $D-$ modules transform, where this last come from the use of Penrose transform on $D-$ modules [16].

We consider the flat case. Then we have:

Theorem (F. Bulnes) 4. 1. If $\mathfrak{C} \cong P^3(\mathbb{C})$, in $U \subset \mathbb{C}^4$, then exists in $SO(4,\mathbb{C})$, a $SU(2,2)$ – invariant mapping given by the twistor transform:

$$\mathcal{T} : H^1(P^3(\mathbb{C}), \mathcal{O}(\mathfrak{C})) \overset{\cong}{\rightarrow} H^1(U, \Omega^2(M)), \qquad (37)$$

which gives a moduli problem that identifies to (35) with the tangent bundle $\mathfrak{C}_p$, $\forall p \in \mathbb{C}^4$.

Proof. We must exhibit a co-bordering operator $\mathfrak{d}$, that sees the complex mapping

$$\mathfrak{d} : \{\text{lines or light geodesics}\} \xrightarrow{\text{interwining operator}} \{\text{fields}\}, \qquad (38)$$

The structure of the demonstration will be in the following facts and order to demonstrate: first, we prove that these light geodesics are electromagnetic $1-$ forms. Second, the transformation that carry these light geodesics of the space-time in fields is a conformal transformation where this obtained fields are conformally invariants. Third, these tensors are elements of (35).

We consider the causal structure of the Cosmos and we use the convexity of all subspace $C(p) \in \mathfrak{C}$, and $C(p) \subset \mathfrak{C}_p, \forall p \in M$. Also, we consider the causal underlying group in the differentiable manifold M, whose Lie algebra is the vector space of fields in $\mathfrak{X}(M)$. Furthermore, we consider an infinitesimal generator of a uni-parametric group in the future cone [17], that is to say, the space $\{e^{tX}\}$, whose transformations carry to each point $p \in M$, to a point $q \in e^{tX_p}(M)$, which is temporally preceded for p, when $t > 0$. Then there is a geodesic arc γ_{ab}, from p, to q, whose tangent trajectory in every point is in the future direction.

If the fields are the Maxwell fields F, the uniparametric group generated for F, in the light subspace $C(p)$, to the frequency k_a, is $\{e^{ik_a x^a}\}$. Of fact, to the real case, a stable decomposing of the $2-$forms space F, is the given by the auto-dual decomposing (13). In $\mathbb{C}^4$, is necessary to use a conformal transformation on fibers of the bundle $\mathfrak{C}$, whose frequency is given for the light rays whose uni-parametric group of the future cone is $\{e^{ik_a x^a}\}$, $\forall x^a \in \mathbb{C}^4$.

Let $M \cong \mathbb{C}^4$, a smoothly oriented conformal manifold for the causal structure of $\mathfrak{C}$, and whose metric is defined, save a scale. Then exists a rays sub-bundle of the bundle $\otimes^2 T^*M = \otimes E$, which are positive defined everywhere in the sense of the any element non-null will give a positive defined quadratic form. Given that M, is a Lorentzian manifold, the Laplacian operator in the Maxwell equations can be replaced by the wave differential operator $\square$, where this operator is invariant conformal in the space $S^+ \otimes S^- = TM$. But the light bundles of the vector bundle $\mathfrak{C}$, in M, need of a twistor space $\mathcal{I}$, such that the wave operators derived of the affine connections of the form

$$\nabla_a : \{\text{geodesics of light}\} \rightarrow \{1-\text{forms}\}, \qquad (39)$$

deduced from the composing:

$$\underset{\text{in } \mathbb{C}^4}{O[1]} \rightarrow \mathcal{I} \xrightarrow{\nabla_a} \mathcal{I}_a \rightarrow \underset{\text{in } M}{O_{ab}[1]} \cong \mathfrak{C}, \qquad (40)$$

are conformally invariants, and where $\mathfrak{C}$, is the curved version of the flat version on the space $\mathbb{C}^4$, linked with the vector bundle of lines or rays $O[1]$. These wave operators are. Each part of the composing (40) is conformally invariant [18].

But, given that the compositions $\nabla_a \nabla_b \phi$, are conformally invariants and by the curved translation principle the endomorphism $[\nabla_a \nabla_b - \nabla_b \nabla_a]\phi$, is the curvature of $\mathcal{I}$, (essentially the Weyl curvature of M) whose gauges are $2-$ forms of the space $\Omega^2(M)$, then, in particular, $\nabla_a(\nabla_b \phi) = \nabla_a A_b$, is conformally invariant $\forall \phi$, a wave function or $0-$ form in $U(1)$. But $\nabla_a \nabla_b \phi \in E \otimes H([,])$, as Lie algebra. Of fact, $\nabla_a \nabla_b \phi \in (E \otimes H)/\{[\nabla_a, A_b]\}$, being $A_c = \nabla_c \phi$. Then

$$\nabla_a \nabla_b : \{\text{wave functions}\} \rightarrow \{2\text{ - forms}\}, \qquad (41)$$

For other way, given that the neighborhoods $U \subset C(p) \in \mathfrak{C}$, are connects then exists an isomorphism of cohomological space

$$H^r(P^3(\mathbb{C}), \mathcal{O}(B)) \cong H^r(\nu^{-1}(U), \mu^{-1}\mathcal{O}(B)), \qquad (42)$$

to $\mathcal{O}(B) = \mathfrak{C}$, on $U = P^3(\mathbb{C})$, where $\nu^{-1}\mathfrak{C}$, is the sheaf of germs of sections $\mu^*\mathfrak{C}$, constants along the fibers μ. Thus is natural the exact sequence

$$0 \rightarrow \mu^{-1}\mathfrak{C} \rightarrow \mathfrak{C} \xrightarrow{\nabla_a} \wedge^1(R^4) \xrightarrow{\nabla_a \nabla_b} \wedge^2(R^4) \rightarrow 0, \qquad (43)$$

Then to said sequence, the isomorphism (42) comes given by the Twistor transform given in (37), which proves the theorem.

A re-interpretation of the singularities of the electromagnetic type in the space-time are given as sources of electromagnetic radiative fields[11] where through the twistor geometry these singularities can be zeros (or roots) of certain polynomials of the homogeneous lines bundles. Also these can be poles of certain surfaces, where can be projected fields whose origin are charges Q (see the figure 3). For example considering the sphere S^2, which we can identify through twistor theory as twistor space PT, with their two orbits PT^+, and PT^-, are projectivized the poles as PN, and PS, to each semi-sphere S^+, and S^-, identified these with the two orbits of PT. Likewise the line S^1, (Ecuador circle)

11 These could be multi-radiative electromagnetic fields to hyper-transmission-reception of signals.

divide to the holomorphic functions (H^0 – elements) in S^2, in their parts of positive or negative frequency in S^+, and S^-, respectively. If we consider to the signals that come or go of the Riemann sphere S^2, through the solutions of the complex equations $DAe^{i\omega t} = 0$, these can be interpreted as signals emitted or received by a encoder in S^2, of an energy signal coming from of the space-time (see the figure 3 and 4).

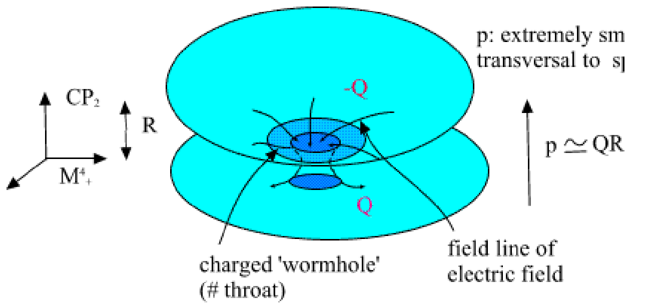


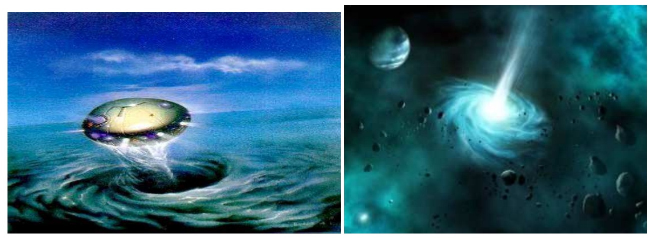
Fig. 3. Electric charge sources in the space-time to produce the developing of the space-time fabric to create wormholes. The 2-dimensional projections of the wormhole is a 2-dimensional conformal action on asymptotic plane of the space-time.

Fig. 4. A) Displacement of the ship using an electromagnetic property of the space to the electro-anti-gravity. This ship projective an electro-anti-gravity charge to produce their movement.

The integrals to electromagnetic sources in twistor theory have equivalents as contour integrals, which carry on to the isomorphism between the certain cohomological spaces to different classes of Maxwell fields. This is not a coincidence, since that as mentioned, the differential forms space to the Maxwell tensors as $2-$ forms and the potentials (likewise the quantum densities of energy that produce J_s) as $1-$ forms admits a geometrical modelling through lines bundles that can to establish the structural equivalence in the complex ambit given for (which is a twistor transform):

$$H^1(\Pi(X),\Omega) \cong H^1(P\mathcal{T},O(-2-2)), \qquad (44)$$

Here arises an unlimited set of transformations that can to explain from a topological point of view much of the geometrical phenomena that happen in the Universe and that can be copied inside the electromagnetic context to the designs and obtaining of prototypes of an electromagnetic vehicle or different electromagnetic devices to different proposes. The space $\Pi(X)$, defines a set of singularities that from the point of view of our integrals give values of the field as monopoles or multi-poles in twistor re-interpretation of holomorphic bundles to electromagnetic frame of the Universe (see the figure 5).



A *B*

Fig. 5. A). Proposed Syncrotronic Machine [19]. B). Black Hole. In both cases the sources are singularities of the spaces that wrap the object. Also can be projected this singularity in the space.

If we consider the multi-poles as the sources of the electromagnetic nature of the space-time (of fact their moduli stack is obtained by equivalences in field theory using some *gerbes* of derived categories as has been mentioned in the section 5), we can to use the loops around of these poles as contours of the cohomological functionals (see the Table 1, arrow 6) to evaluate through the residue theorem their energy and these values can be registered to design spinor waves where these values are amplitudes of these waves, of fact, we can consider the partial wave expansions of the space-time suggested by the conformal actions in 4-dimensional and 2-dimensional spaces.

The Conway integrals can be considered in axisymmetric boundaries and also non-axisymmetric cases where matter is confined within axisymmetric boundaries, for example in a galaxy. If we consider the electromagnetic nature of the iso-rotations for magnetic intersidereal fields in galaxies, we need other formalism based in twister geometry, where elliptic integrals are analogues in the space $H^1_{\mathcal{L}}(U'',\mathcal{O}(-2))$.

The cohomological analogous are "poles" which can be interpreted as "sources" of electromagnetic radiative energy (see the figure 6).

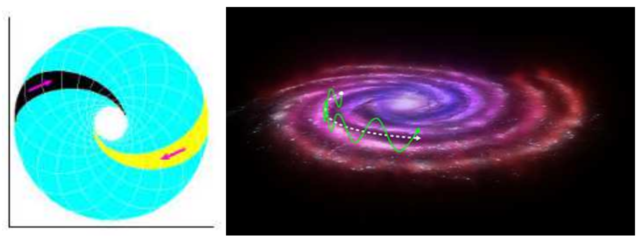


Fig. 6. The fields obtained by the contour integrals around of these sources represent the fields induced by matter in an extended region as a distribution of gravitating disc. The spinor formalism could be used through the responsible electromagnetic energy of the accretion and iso-rotation of a galaxy to design technological devices of these spinors.

5. Conclusions

The cohomological classes obtained to electromagnetism through of integral geometry methods determine with great precision the solution classes of the Maxwell equations and permits generalize and induce this solutions to electromagnetic special fields that could have several technological applications, if we understand their mathematical object under re-interpretations of the twistor theory and values of their integrals, that is to say, the "space-time" doing it to correspond to the "twistor space", likewise the geometrical invariance of images obtained under the integral transforms applied to different orbits where the actions of the fields are topologically equivalents to the obtained to representations is realized in nature way.

Likewise, this set of methods that consider the groups, their actions, and the geometrical images obtained under integral transforms in electro-physics (being these points of these groups), establish the perspective to generate a geometrical ramification theory that envelopes whole the electromagnetism from the electrostatic until superconducting and semi-conducting phenomena in the space-time under same scheme of cohomologically generates solution classes to the Maxwell field equations with the property of can to be extended and inducted to other fields in field theory. These methods and all information of their equivalences under their respective integral type conform the mathematical electrodynamics as integral operator's electrodynamics alternative and as other theory in geometrical correspondences in field theory.

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