

On the Periodicity of a Max-type Fuzzy Difference Equations

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Abstract: Our aim in this paper is to discuss the periodicity and boundedness of a max-type fuzzy difference equation. When studying the periodicity of the solution to the max-fuzzy difference equation, the equation is first converted into a difference system composed of two related difference equations through the cut set theory of the fuzzy number, then the periodicity of each solution sequence in the system is obtained by means of inequality technique, mathematical induction and other theoretical methods, thus the periodicity of the solution is proved. As researching the boundedness of the solution for the fuzzy difference equation, the difference system is also obtained through the cut set theory of the fuzzy number, then analyze the boundedness to each solution sequence according to the periodicity with the solution sequence, through examining the value of the finite subsequence in each solution sequence, the boundedness with these subsequences can be obtained, and then the boundedness for each solution sequence made up of complete subsequences can be known, thus the boundedness of the solution is proved. Finally, the results obtained in this paper are simulated by using the software package MATLAB 2016, the numerical results not only show the dynamic behavior of the solutions to the fuzzy difference systems, but also verify the effectiveness of the theoretical results.

Keywords: Fuzzy Difference Equation, Max-Type, Cut Theory, Periodicity, Boundedness

1. Introduction

The difference equation is a kind of mathematical model describing discrete dynamic system. As an important branch of the dynamic system, the difference equation theory is widely used in population statistics and analysis, economic finance, control science, communication science and computer science (see, e.g., [1-5] and the reference therein). With the progress and development of science and technology, the difference equation will become a more and more popular research topic. As an important part of the difference equation, the max-type difference equation has important applications in many fields, such as models of automatic control theory and so on (see, e.g., [6-10] and thereference therein). Because of this, scholars have studied many theories and applications of the max-type difference equation in recent years. Since periodical research of differential equation theory has been very mature and perfect, some scholars try to investigate the

periodicity of difference equation with the theory of differential equation (see, e.g., [11-17] and the reference therein).

Making a historical flash back for the equation we study in this paper, we need to mention that in 2006, Yang et al. [18] studied the following second order max-type difference equation with index parameters

$$x_n = \max \left\{ \frac{1}{x_{n-1}^\alpha}, \frac{A}{x_{n-2}} \right\}, n \in N_0 \quad (1)$$

where the parameters $A > 0, \alpha \in (0, 1)$, the author proved that the positive solution of the equation (1) converges to 1 when the parameter $A \leq 1$, and the positive solution of the equation (1) tends to four-periodic solution when the parameter $A > 1$.

At the same time we could see that in 2012, S. Stevic [19] studied the periodicity of the solutions for the difference equations (2),

$$\begin{cases} x_n^{(1)} = \max_{1 \leq i \leq m_1} \{f_{1i}(x_{n-k_{i,1}}^{(1)}, x_{n-k_{i,2}}^{(1)}, \dots, x_{n-k_{i,l}}^{(1)}, n), x_{n-s}^{(1)}\}, \\ x_n^{(2)} = \max_{1 \leq i \leq m_1} \{f_{2i}(x_{n-k_{i,1}}^{(1)}, x_{n-k_{i,2}}^{(1)}, \dots, x_{n-k_{i,l}}^{(1)}, n), x_{n-s}^{(2)}\}, \\ \vdots \\ x_n^{(l)} = \max_{1 \leq i \leq m_l} \{f_{li}(x_{n-k_{i,1}}^{(1)}, x_{n-k_{i,2}}^{(1)}, \dots, x_{n-k_{i,l}}^{(1)}, n), x_{n-s}^{(l)}\}, \end{cases} \quad (2)$$

where $n, s, l, m_j, k_{i,t}^{(j)} \in N, j, t \in \{1, \dots, l\}$, and $j \in \{1, \dots, l\}, i \in \{1, \dots, m_j\}$. When the parameters $j, i \in \{1, \dots, m_j\}$ are permanent, then the final period value of the solutions to the equations (2) is s . In addition, S. Stevic also dissected the periodicity of the min-type difference equations corresponding to the equations (2).

In addition, in 2002, Voulou [20] analyzed the periodicity of the positive solution of the max-type difference equation (3),

$$x_n = \max \left\{ \frac{A}{x_{n-k}}, \frac{B}{x_{n-m}} \right\}, n = 0, 1, \dots \quad (3)$$

Where k, m are arbitrary positive integers, the parameters A, B and the initial conditions $x_{-1}, x_{-2}, \dots, x_{-d}$ are arbitrary positive real numbers, in which $d = \max\{k, m\}$. The periodicity of the solution for the equation is proved by the thought of classified discussions.

Recently, many scholars have further expanded the form of the equation (3). In 2009, Ali Gelisken [21] resolved the dynamic properties of the solution to the max-type difference equation (4),

$$x_n = \max \left\{ \frac{A}{x_{n-p}^\alpha}, \frac{B}{x_{n-k}^\beta} \right\}, n = 0, 1, \dots \quad (4)$$

where p, k are positive integers, $0 < \alpha, \beta < 1$ and $A, B > 0$. In this paper, the author obtained that every positive solution converges to $\bar{x} = \max\{A^{1/(1+\alpha)}, B^{1/(1+\beta)}\}$.

Besides, in 1999, Szalkai [22] presented the periodicity of the solution for the max-type difference equation (5),

$$x_{n+1} = \max \left\{ \frac{A}{x_n}, \frac{A}{x_{n-1}}, \dots, \frac{A}{x_{n-k}} \right\}, n = 0, 1, \dots \quad (5)$$

where k is a positive integer, the parameters A and the initial conditions $\{x_i\}_{i=-k}^0$ are real numbers. In this paper, the author proved that every positive solution of the equation is periodic with $k+2$.

With the sustainable development of the max-type difference equation theory, scholars have combined it with the fuzzy sets theory (see, e.g., [23-30] and the reference therein), thus creating the max-type fuzzy difference equation, and at the same time doing numerous studies on the max-type fuzzy difference equation. In 2004, Stefanidou [31] generalized the difference equation (5) from the real number range to the fuzzy number range, that is, the parameter A and the initial values $\{x_i\}_{i=-k}^0$ are positive fuzzy numbers, then the solutions

of the equation are also positive fuzzy numbers. In this paper, the author gave the corresponding fuzzy conditions for the solution of the equation to be periodic, unbounded and unsustainable. Meanwhile, the dynamic properties of the max-type fuzzy difference equation (6), a special form of the equation (3), are also discussed in reference [31]

$$x_{n+1} = \max \left\{ \frac{A_0}{x_n}, \frac{A_1}{x_{n-1}} \right\}, n = 0, 1, \dots, \quad (6)$$

where the parameter A_0, A_1 and the initial values x_{-1}, x_0 are positive fuzzy numbers given arbitrarily. In this paper, the author obtained that the solution of the equation (6) is periodic, unbounded and non-persistent. In 2006, Stefanidou [32] further extended the form of the equation (6), and considered the periodicity of the solution for the max-type type fuzzy difference equation (7),

$$x_{n+1} = \max \left\{ \frac{A_0}{x_{n-k}}, \frac{A_1}{x_{n-m}} \right\}, n = 0, 1, \dots, \quad (7)$$

where the parameter A_0, A_1 and the initial values $x_{-d}, \dots, x_0, d = \max\{k, m\}$ are positive fuzzy numbers given arbitrarily.

Inspired by the discussions above, in this paper, we will study the periodic property of the solutions to the following fuzzy difference equation (8)

$$x_{n+1} = \max \left\{ \frac{A}{x_n}, \frac{A}{x_{n-1}}, \dots, \frac{A}{x_{n-(k-1)}} \right\}, n = 0, 1, 2, \dots, \quad (8)$$

where k is a positive integer, the parameters A and the initial conditions $x_{-k}, x_{-k+1}, \dots, x_0$ are positive fuzzy numbers.

2. Preliminaries and Notations

For the convenience of readers, we need to give the following notations, definitions and preliminary results, see [33-37].

Definition 2.1 For a set B we denote by \bar{B} the closure of B , we say that a function $A: R \rightarrow [0, 1]$ is a fuzzy number if the follow conclusions are true,

(1) A is normal, i.e., there exists $x \in R$ such that $A(x) = 1$;

(2) A is a fuzzy convex set, i.e.,

$$A(tx_1 + (1-t)x_2) \geq \min\{A(x_1), A(x_2)\}, \forall t \in [0, 1], x_1, x_2 \in R;$$

(3) A is upper semicontinuous on R ;

(4) The support of A , i.e., $\text{supp } A = \overline{\cup_{\alpha \in (0, 1]} [A]_\alpha} = \overline{\{x \in R : A(x) > 0\}}$ is compact.

where the α -cuts of A are closed intervals, define as $[A]_\alpha = \{x \in R : A(x) \geq \alpha\}$. if $\text{supp } A \subset (0, +\infty)$, then fuzzy number A is obviously positive.

Definition 2.2 Let A, B be fuzzy numbers which are satisfying $[A]_\alpha = [A_{l,\alpha}, A_{r,\alpha}]$, $[B]_\alpha = [B_{l,\alpha}, B_{r,\alpha}]$,

$\alpha \in (0, 1]$, we denote the following metric

$$D(A, B) = \sup \max \{ |A_{l,\alpha} - B_{l,\alpha}|, |A_{r,\alpha} - B_{r,\alpha}| \}$$

where \sup is taken for all $\alpha \in (0, 1]$. Then (R_f, D) is a complete metric space. For the convenience of application in the future, we define $\hat{0} \in R_f$ as

$$\hat{0}(x) = \begin{cases} 1, & x = 0, \\ 0, & x \neq 0. \end{cases}$$

Thus, $[0]_\alpha = [0, 0], 0 < \alpha \leq 1$.

Considering the following difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-k}), n = 0, 1, 2, \dots \quad (9)$$

where I_x is a real number interval and $f: I_x^{k+1} \rightarrow I_x$ is a continuous function.

$$\begin{aligned} x_{n+1} &= f(x_n, x_{n-1}, \dots, x_{n-k}, y_n, y_{n-1}, \dots, y_{n-k}), n = 0, 1, 2, \dots, \\ y_{n+1} &= g(x_n, x_{n-1}, \dots, x_{n-k}, y_n, y_{n-1}, \dots, y_{n-k}), n = 0, 1, 2, \dots, \end{aligned} \quad (10)$$

where I_x, I_y are real number intervals and $f: I_x^{k+1} \times I_y^{k+1} \rightarrow I_x, g: I_x^{k+1} \times I_y^{k+1} \rightarrow I_y$ are continuous functions.

Definition 2.3 For the difference equation (9), the periodicity is defined as follows

(1) A sequence $\{x_n\}_{n=0}^\infty$ is called P -period, where P is a positive integer, for all $n \in N$, we have $x_{n+p} = x_n$;

(2) A sequence $\{x_n\}_{n=0}^\infty$ is called the min P -period, for all $n \in N$, we have $x_{n+p} = x_n$, and P is the smallest positive integer that holds this formula.

Definition 2.4 For the difference system (10), the periodicity is defined as follows

(1) If $x_n = x_{n+p}, y_n = y_{n+p}$ are satisfied for any $n \in N$, where P is a positive integer, then it is said that the solutions $\{(x_i, y_i)\}_{i=-k}^{+\infty}$ of the difference system (10) are periodic with P .

(2) If $x_n = x_{n+p}, y_n = y_{n+p}$ are satisfied for any $n \in N$, at the same time P is the smallest positive integer satisfying the difference system (10), then it is said that the solutions $\{(x_i, y_i)\}_{i=-k}^{+\infty}$ of the system (10) are the min periodic with P .

(3) If every solution of the difference system (10) is periodic with P , where P is a positive integer, then the solution of the system (10) is called P -period.

Definition 2.5 Let $\{x_n\}$ be a sequence of fuzzy numbers, then

(1) If there is a positive real number M (or N) that holds $\text{supp } x_n \in (M, +\infty]$ (or $\text{supp } x_n \in (0, N]$), $n = 0, 1, 2, \dots$, then the sequence $\{x_n\}$ of fuzzy numbers is persistent (or bounded);

(2) If there are two positive real numbers M, N satisfy $\text{supp } x_n \in (M, N]$, $n = 0, 1, 2, \dots$, then the sequence $\{x_n\}$ of fuzzy numbers is both persistent and bounded.

3. Main Results and Proofs

According to α -cut set theory, the max-type fuzzy difference equation (8) can be transformed into the following ordinary difference system consisting of two ordinary difference equations

$$\begin{aligned} y_{n+1} &= \max \left\{ \frac{B}{z_n}, \frac{B}{z_{n-1}}, \dots, \frac{B}{z_{n-(k-1)}}, y_{n-k} \right\}, n = 0, 1, 2, \dots, \\ z_{n+1} &= \max \left\{ \frac{C}{y_n}, \frac{C}{y_{n-1}}, \dots, \frac{C}{y_{n-(k-1)}}, z_{n-k} \right\}, n = 0, 1, 2, \dots, \end{aligned} \quad (11)$$

where $B, C, y_{-k}, y_{-k+1}, \dots, y_0, z_{-k}, z_{-k+1}, \dots, z_0 \in (0, +\infty)$.

Theorem 3.1 For the difference system (11), each solution sequence $\{y_n\}, \{z_n\}$ has periodicity, and the period is $(k+1)$.

Proof: According to the difference system (11), we can gain the following several inequalities

$$y_n \geq y_{n-(k+1)}, z_n \geq z_{n-(k+1)}, y_{n+1} \geq \frac{B}{z_{n-i}}, (i = 0, 1, \dots, (k-1))$$

By the synthesis method, we can deduce

$$y_n \geq \frac{B}{z_{n-(i+1)}}, y_{n-k} \geq \frac{B}{z_{n-(i+k+1)}} = \frac{B}{z_{n-i-(k+1)}} \geq \frac{B}{z_{n-i}}, (i = 0, 1, \dots, (k-1))$$

The inequalities above reflect the size relations of each algebra contained in the right-hand formula to an equation in the difference system (11), then through the value of each algebra relationship and definition of the equation in the system (11), we can obtain

$$y_{n+1} = y_{n-k}, \text{ Namely } y_n = y_{n-(k+1)}$$

Therefore the solution sequence $\{y_n\}$ has periodicity, and the period is $(k+1)$.

And once again, according to the difference system (11), the following some inequalities can be obtained

$$y_n \geq y_{n-(k+1)}, z_n \geq z_{n-(k+1)}, z_{n+1} \geq \frac{C}{y_{n-i}}, (i = 0, 1, \dots, (k-1))$$

From the synthesis, we can get the following inequalities

$$z_n \geq \frac{C}{y_{n-(i+1)}}, z_{n-k} \geq \frac{C}{y_{n-(i+k+1)}} = \frac{C}{y_{n-i-(k+1)}} \geq \frac{C}{y_{n-i}}, (i = 0, 1, \dots, (k-1))$$

The inequalities above reflect the size relations of the algebraic expressions on the right-hand side of another equation in the difference system (11), then through the value of the size relationship and the definition of the equation in the system (11), we can gain

$$z_{n+1} = z_{n-k}, \text{ Namely } z_n = z_{n-(k+1)}$$

Thereby, the solution sequence $\{z_n\}$ has periodicity, and the period is $(k+1)$. The proof is completed.

Theorem 3.2 For the difference system (11), if the parameters B, C and the initial conditions $y_{-k}, y_{-k+1}, \dots, y_0, z_{-k}, z_{-k+1}, \dots, z_0$ are positive real numbers, then the solution sequence $\{y_n\}, \{z_n\}$ of the system (11) are bounded and persistent.

Proof: Theorem 3.1 indicates that the value of each term in the solution sequence $\{y_n\}$ must be one of the following values

$$(A_1) \ y_{(k+1)n} = y_0,$$

$$(A_2) \ y_{(k+1)n+1} = y_1 = \max\left\{\frac{B}{z_0}, \frac{B}{z_{-1}}, \dots, \frac{B}{z_{-k+1}}, y_{-k}\right\},$$

$$(A_3) \ y_{(k+1)n+2} = y_2 = \max\left\{\frac{B}{z_1}, \frac{B}{z_0}, \dots, \frac{B}{z_{-k+2}}, y_{-k+1}\right\} = \\ \max\left\{\min\left\{\frac{By_0}{C}, \frac{By_{-1}}{C}, \dots, \frac{By_{-k+1}}{C}, \frac{B}{z_0}, \frac{B}{z_{-k+2}}, y_{-k+1}\right\}, \frac{B}{z_0}, \dots, \frac{B}{z_{-k+2}}, y_{-k+1}\right\},$$

$$(A_4) \ y_{(k+1)n+3} = y_3 = \max\left\{\frac{B}{z_2}, \frac{B}{z_1}, \dots, \frac{B}{z_{-k+3}}, y_{-k+2}\right\},$$

\vdots
 \vdots

$$(A_k) \ y_{(k+1)n+(k-1)} = y_{k-1} = \max\left\{\frac{B}{z_{k-2}}, \frac{B}{z_{k-3}}, \dots, \frac{B}{z_{-1}}, y_{-2}\right\},$$

$$(A_{k+1}) \ y_{(k+1)n+k} = y_k = \max\left\{\frac{B}{z_{k-1}}, \frac{B}{z_{k-2}}, \dots, \frac{B}{z_0}, y_{-1}\right\}.$$

Likewise, the value of each term in the solution sequence $\{z_n\}$ must have one of the following values

$$(B_1) \ z_{(k+1)n} = z_0,$$

$$(B_2) \ z_{(k+1)n+1} = z_1 = \max\left\{\frac{C}{y_0}, \frac{C}{y_{-1}}, \dots, \frac{C}{y_{-k+1}}, z_{-k}\right\},$$

$$(B_3) \ z_{(k+1)n+2} = z_2 = \max\left\{\frac{C}{y_1}, \frac{C}{y_0}, \dots, \frac{C}{y_{-k+2}}, z_{-k+1}\right\} = \\ \max\left\{\min\left\{\frac{Cz_0}{B}, \frac{Cz_{-1}}{B}, \dots, \frac{Cz_{-k+1}}{B}, \frac{C}{y_{-k}}, \frac{C}{y_0}, \dots, \frac{C}{y_{-k+2}}, z_{-k+1}\right\}, \frac{C}{y_0}, \dots, \frac{C}{y_{-k+2}}, z_{-k+1}\right\},$$

$$(B_4) \ z_{(k+1)n+3} = z_3 = \max\left\{\frac{C}{y_2}, \frac{C}{y_1}, \dots, \frac{C}{y_{-k+3}}, z_{-k+2}\right\},$$

\vdots
 \vdots

$$(B_k) \ z_{(k+1)n+(k-1)} = z_{k-1} = \max\left\{\frac{C}{y_{k-2}}, \frac{C}{y_{k-3}}, \dots, \frac{C}{y_{-1}}, z_{-2}\right\},$$

$$(B_{k+1}) \ z_{(k+1)n+k} = z_k = \max\left\{\frac{C}{y_{k-1}}, \frac{C}{y_{k-2}}, \dots, \frac{C}{y_0}, z_{-1}\right\}.$$

As can be seen from the equation (B_3) , z_2 is bounded and persistent, so y_3 in the equation (A_4) is bounded and persistent.

Similarly from the equation (A_3) , we can see that y_2 is

bounded and persistent, then we can know that z_3 in the equation (B_4) is bounded and persistent.

By induction, we can conclude that: from the equation (B_k) , we can get that z_{k-1} is bounded and persistent, then we can figure out that y_k in the equation (A_{k+1}) is bounded and persistent.

In a similar way, from the equation (A_k) , y_{k-1} is bounded and persistent, so z_k in the equation (B_{k+1}) is bounded and persistent.

According to the mathematical complete induction, the solution sequences $\{y_n\}, \{z_n\}$ of the system (11) are bounded and continuous. The proof is completed.

4. Numerical Simulation

In order to verify the conclusions obtained in this paper, some numerical simulations are carried out by using the software package MATLAB 2016.

Example 4.1 Considering the case of the difference system (11) at $k = 4$, i.e. the following max-type fuzzy difference equation

$$y_{n+1} = \max\left\{\frac{B}{z_n}, \frac{B}{z_{n-1}}, \frac{B}{z_{n-2}}, \frac{B}{z_{n-3}}, y_{n-4}\right\}, n = 0, 1, 2, \dots, \\ z_{n+1} = \max\left\{\frac{C}{y_n}, \frac{C}{y_{n-1}}, \frac{C}{y_{n-2}}, \frac{C}{y_{n-3}}, z_{n-4}\right\}, n = 0, 1, 2, \dots, \quad (12)$$

The following figure shows the simulation results of the numerical solutions for the difference system (12).

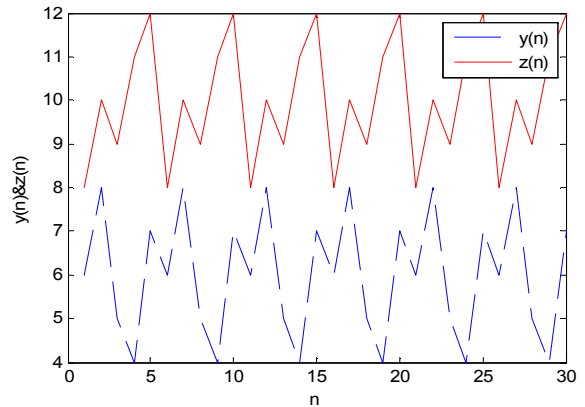


Figure 1. The numerical solutions of the difference system (12).

Figure 1. shows the numerical solutions of $\{y_n\}, \{z_n\}$ for the difference system (12) when $B = 5, C = 9, y_{-4} = 6, y_{-3} = 8, y_{-2} = 5, y_{-1} = 4, y_0 = 7, z_{-4} = 8, z_{-3} = 10, z_{-2} = 9, z_{-1} = 11, z_0 = 12$, it can be seen from Figure 1. that the solutions of the difference system (12) are 5-period, bounded and persistent, which verifies the conclusions of Theorem 3.1 and Theorem 3.2.

5. Conclusion

In this paper, we study the periodicity and boundedness of

the solutions to a max-type difference equation. Firstly, by using iteration method, inequality skills and mathematical induction, we proved that the max-type difference equation has period of $(k+1)$ when the parameters and the initial conditions are all positive fuzzy numbers. Secondly, it is proved that the solutions for the max-type difference equation is bounded and persistent when the parameters and the initial conditions are all positive fuzzy numbers. Finally, the effectiveness of the theoretical results is verified by the numerical experiments with the software package MATLAB 2016.

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