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# Haar Wavelet Solution of Poisson's Equation and Their Block Structures

**I. K. Youssef[1], M. H. El Dewaik[2]**

[1]Department of Mathematics, Ain Shams University, Cairo, Egypt

[2]Department of Basic Science, The British University, Cairo, Egypt

**Email address:**
Kaoud22@hotmail.com (I. K. Youssef), Marwa.eldewaik@bue.edu.eg (M. H. El Dewaik)

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**Abstract:** The structure of the algebraic system which results from the use of Haar wavelet when solving Poisson's equation is studied. Haar wavelet technique is used to solve Poisson's equation on a unit square domain. The form of collocation points are used at the mid points of the subintervals i.e at the odd multiple of the sub interval length labeling. It is proved that the coefficient matrix has symmetric block structure. Comparison with the tridagonal block structure obtained by the finite difference with the natural ordering is introduced. The numerical results have illustrated the superiority of the use of Haar wavelet technique. The matrices obtained can be used for any equations containing the Laplace operator.

**Keywords:** Poisson's Equation, Finite Difference, Wavelet, Haar Wavelet

## 1. Introduction

Poisson's equation appears in many applications in physical and engineering problems. Numerical treatment of Poisson's equation by finite difference or finite element method produces structured linear systems. Recently wavelet techniques are used in solving differential equations.

There are different types of wavelet. In literature Haar wavelet is the first ideas related to the concepts of wavelet. Haar wavelet was suggested by the mathematician Alfrd Haar in 1909. However, the concept of the wavelet did not exist at that time. In 1981, the concept was reintroduced by the geophysicist Jean Morlet. Afterward, Morlet and the physicist Alex Grossman invented the term wavelet in 1984. Fortunately, the mathematician Yves Meyer constructed the second orthogonal wavelet called Meyer wavelet in 1985. As more and more scholars joined in this field, the 1st international conference was held in France in 1987. In 1988, Stephane Mallat and Meyer proposed the concept of multiresolution. In the same year, Ingrid Daubechies found a systematical method to construct the compact support orthogonal wavelet. In 1989, Mallat proposed the fast wavelet transform. With the appearance of this fast algorithm, the wavelet transform had numerous applications in the signal processing.

Later many functions are considered in building different wavelets but still Haar wavelet is the principal known wavelet. The Haar wavelet is likewise the least complex conceivable wavelet. Wavelets have picked up a respectable status because of their applications in different disciplines. Effective use of wavelets, have appeared in numerous examples to overcome many difficulty. Prominent effects of their studies are in the fields of signal and image processing, numerical analysis, differential and integral equations, tomography, and so on, [1].

In 1910, Haar showed that certain square wave functions could be translated and scaled to create a basis set that span the space $L^2$. Years later, it was seen that the system of Haar is a particular wavelet system.

In comparison with other techniques, which use the same structure of building bases functions and introduce the solution as a linear combination of those bases [2], the Haar wavelet is simple, can implement standard algorithms with high accuracy for a small number of grid points. The simplicity in building the wavelet bases from any function which use only two operations translation and dilation [3], this can be easily seen in Haar wavelet. The simple form of the mother function in Haar wavelet as we see below makes the processes of dialation and translation an easy work and the introduced wavelet family is orthogonal not only linearly independent. Although, the wavelet function appeared in 1910, their use in the solution of differential equations does not appear until recently [4, 5, 6], last twenty years.

## 2. Haar Functions

The orthogonal set of Haar functions is defined as shown in Figures 1. a -1. h. That is a family of square waves with magnitude of ±1 in some intervals and zeros elsewhere. Just these zeros make the Haar transform faster than other square functions such as Walsh's, [7]. The first curve of Figure 1. a is $h_0(t) = 1$ on the whole interval, $0 \le t \le 1$. This is called the scaling function. The second curve $h_1(t)$ is the fundamental square wave, or the mother wavelet which also spans the whole interval (0, 1). All the other subsequent curves are generated from $h_1(t)$ with two operations: dilation and translation. $h_2(t)$ is obtained from $h_1(t)$ with dilation, which means that $h_1(t)$ is compressed from the whole interval (0, 1) to the half interval (0, 1/2) to generate $h_2(t)$. $h_3(t)$ is the same as $h_2(t)$ but shifted (translated) to the right by 1/2. In general,

$$h_n(t) = h_1(2^j t - k) \tag{1}$$

$$n = 2^j + k, j \ge 0, 0 \le k < j \tag{2}$$

The Haar wavelets are orthogonal in the sense,

$$\int_0^1 h_i(t) h_l(t) dt = 2^{-j} \delta_{il}$$

$$= \begin{cases} 2^{-j} & i = l = 2^j + k \\ 0 & i \ne l \end{cases}$$

Therefore, they form a set of basis functions. And accordingly can be used in approximation of functions as follows.

It is accepted that any square integrable function in the interval [0, 1], $y(t) \epsilon L^2[0,1]$ can be expanded in a Haar series in the form

$$y(t) = \sum_{n=0}^{\infty} c_n h_n(t)$$

Where the coefficients $c_n$ are determined by $c_n = 2^j \int_0^1 y(t) h_n(t) dt$ with, $n = 2^j + k, j \ge 0, 0 \le k < j$

The series expansion of $y(t)$ contains infinite terms. If $y(t)$ is piecewise constant by itself, or may be approximated as piecewise constant during each subinterval, then $y$(t) will be terminated at finite terms,[8] that is

$$y(t) = \sum_{n=0}^{m-1} c_n h_n(t) = \mathbf{C}_m^{\mathrm{T}} \, \mathbf{h}_m(t)$$

Where the coefficients vector $\mathbf{C}_m^{\mathrm{T}}$ and the Haar function vector $\mathbf{h}_m(t)$ are defined as

$$\mathbf{C}_{(m)}^{\mathrm{T}} = [c_0, c_1, \dots, c_{m-1}]$$

And

$$\mathbf{h}_m(t) = [h_0(t), h_1(t), \dots, h_{m-1}(t)]^{\mathrm{T}}$$

where T is denotes the transpose.

To facilitate the comparison with the structured systems appears in the finite difference treatment we use eight collocation points at the points $\frac{j}{16}, j = 1, 3, \cdots, 15$ and the first eight Haar wavelet can be expressed as

$$h_0(t) = [1\ 1\ 1\ 1\ 1\ 1\ 1\ 1],$$

$$h_1(t) = [1\ 1\ 1\ 1\ -1\ -1\ -1\ -1],$$

$$h_2(t) = [1\ 1\ -1\ -1\ 0\ 0\ 0\ 0],$$

$$h_3(t) = [0\ 0\ 0\ 0\ 1\ 1\ -1\ -1]\ h_4(t) = [1\ -1\ 0\ 0\ 0\ 0\ 0\ 0]\ h_5(t) = [0\ 0\ 1\ -1\ 0\ 0\ 0\ 0]$$

$$h_6(t) = [0\ 0\ 0\ 0\ 1\ -1\ 0\ 0]$$

$$h_7(t) = [0\ 0\ 0\ 0\ 0\ 0\ 1\ -1]$$

Which can be written in the matrix form as

$$\boldsymbol{H}_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

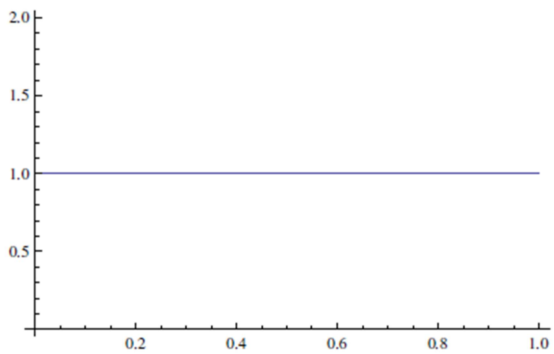



***Figure 1a.*** $h_0(t) = 1,\ 0 \le t \le 1$.

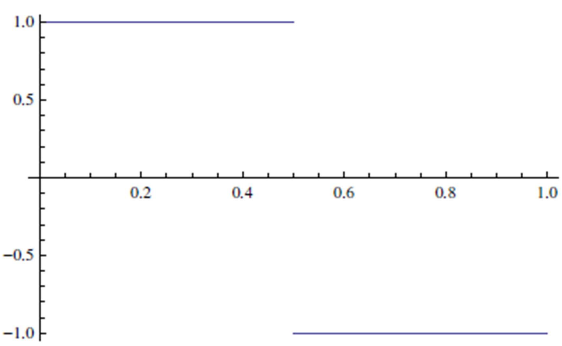



***Figure 1b.*** $h_1(t)$ *is the mother function.*

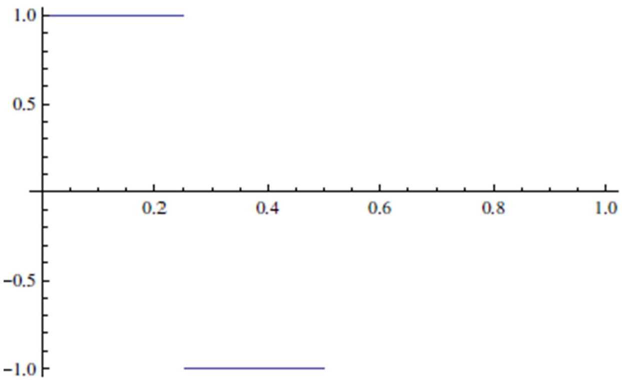



***Figure 1c.*** *$h_2(t)$ illustrates the dilation property.*

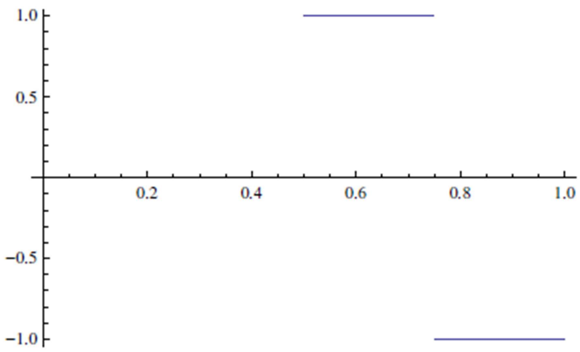



***Figure 1d.*** *$h_3(t)$ illustrates the dilation and translation properties.*

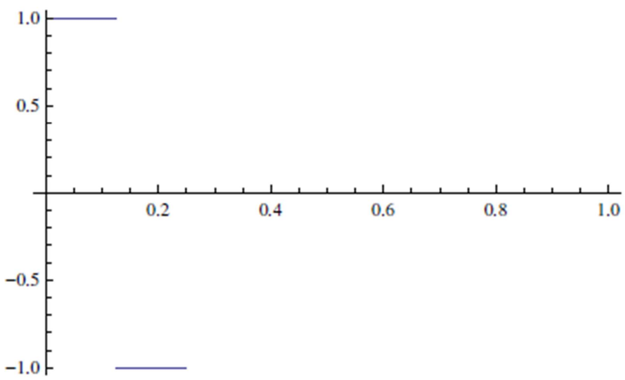



***Figure 1e.*** *$h_4(t)$ illustrates contraction of the dependency domain.*

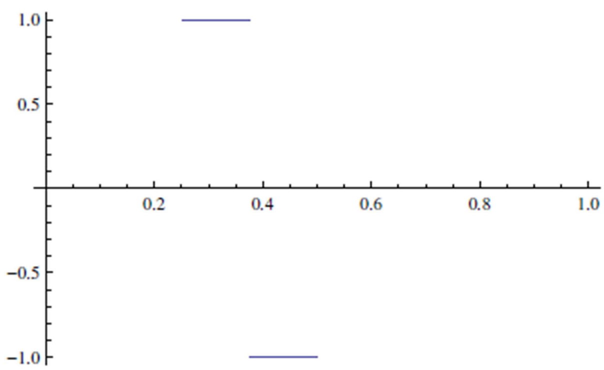



***Figure 1f.*** *$h_5(t)$.*

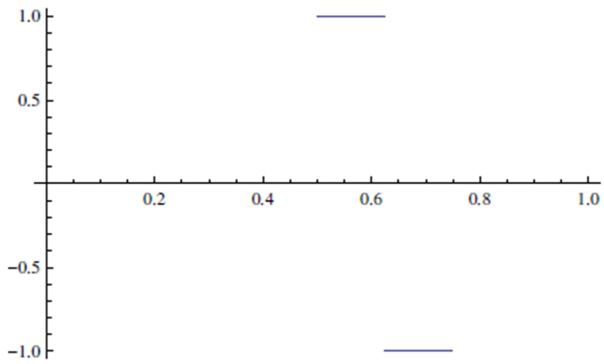



***Figure 1g.*** *$h_6(t)$.*

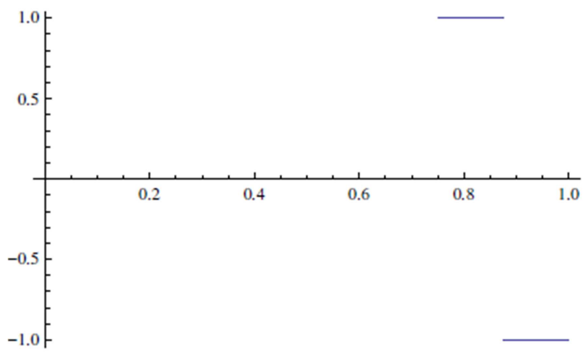



***Figure 1h.*** *$h_7(t)$.*

# 3. Integration of Haar Wavelets

The integrals of the first eight Haar wavelets can be expressed as

$$\int_0^t h_0(t) = t, 0 \le t < 1,$$

And this gives,

$$\cong \frac{1}{16}[1\ 3\ 5\ 7\ 9\ 11\ 13\ 15]$$

At the selected collocation points, similarly one can calculate the other functions to obtain

$$\int_0^t h_1(t) = \begin{cases} t, 0 \le t < \frac{1}{2} \\ 1 - t, \frac{1}{2} \le t < 1 \end{cases} \cong \frac{1}{16}[1\ 3\ 5\ 7\ 7\ 5\ 3\ 1]$$

$$\int_0^t h_2(t) = \begin{cases} t\ 0 \le t < \frac{1}{4} \\ \frac{1}{2} - t, \frac{1}{4} \le t < \frac{1}{2} \end{cases} \cong \frac{1}{16}[1\ 3\ 3\ 1\ 0\ 0\ 0\ 0]$$

$$\int_0^t h_3(t) = \begin{cases} t - \frac{1}{2}\ \frac{1}{2} \le t < \frac{3}{4} \\ 1 - t, \frac{3}{4} \le t < 1 \end{cases} \cong \frac{1}{16}[0\ 0\ 0\ 0\ 1\ 3\ 3\ 1]$$

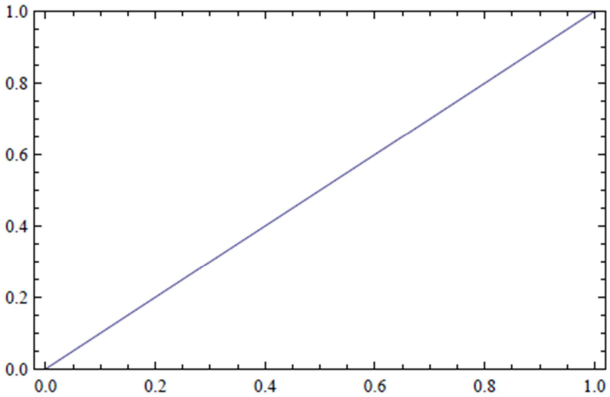
$$\int_0^t h_4(t) = \begin{cases} t\ 0 \le t < \frac{1}{8} \\ \frac{1}{4} - t, \frac{1}{8} \le t < \frac{1}{4} \end{cases} \cong \frac{1}{16}[1\ 1\ 0\ 0\ 0\ 0\ 0\ 0]$$

$$\int_0^t h_5(t) = \begin{cases} t - \frac{1}{4}\ \frac{1}{4} \le t < \frac{3}{8} \\ \frac{1}{2} - t, \frac{3}{8} \le t < \frac{1}{2} \end{cases} \cong \frac{1}{16}[0\ 0\ 1\ 1\ 0\ 0\ 0\ 0]$$

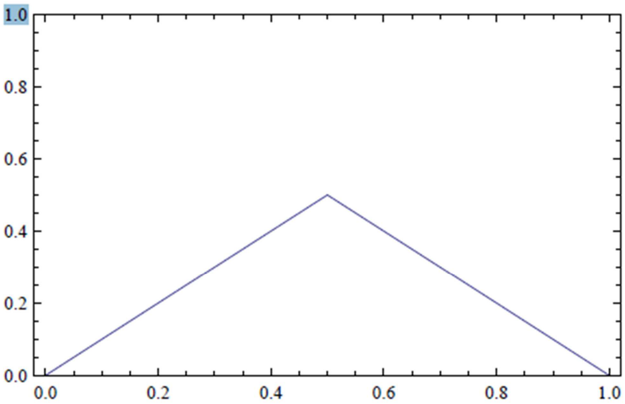
$$\int_0^t h_6(t) = \begin{cases} t - \frac{1}{2}\ \frac{1}{2} \le t < \frac{5}{8} \\ \frac{3}{4} - t, \frac{5}{8} \le t < \frac{3}{4} \end{cases} \cong \frac{1}{16}[0\ 0\ 0\ 0\ 1\ 1\ 0\ 0]$$

$$\int_0^t h_7(t) = \begin{cases} t - \frac{3}{4}\ \frac{3}{4} \le t < \frac{7}{8} \\ 1 - t, \frac{7}{8} \le t < 1 \end{cases} \cong \frac{1}{16}[0\ 0\ 0\ 0\ 0\ 0\ 1\ 1]$$

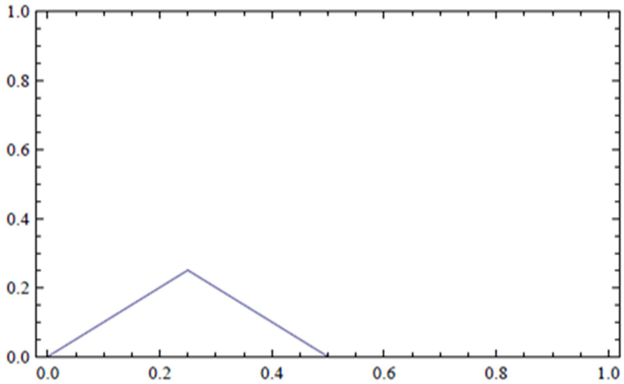
$$\int_0^1 \boldsymbol{H}_8(\boldsymbol{t})\, \boldsymbol{dt} = \frac{1}{16}\begin{bmatrix} 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 \\ 1 & 3 & 5 & 7 & 7 & 5 & 3 & 1 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 3 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$



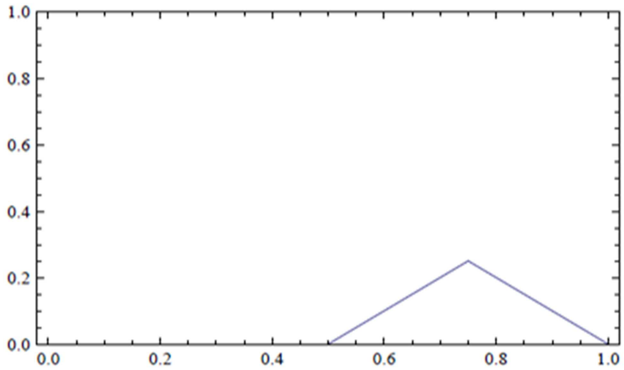

*a.* $p_0 = \int_0^1 h_0(x)dx$.



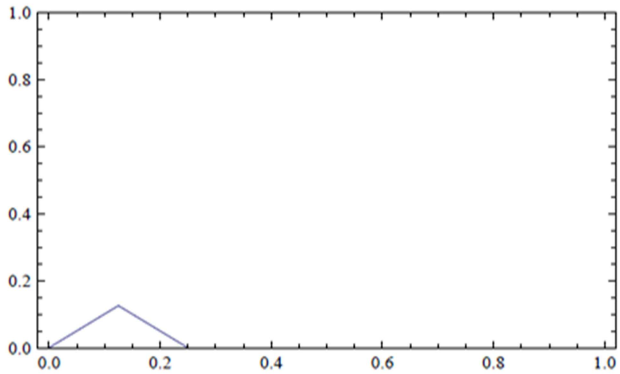

*b.* $p_1 = \int_0^1 h_1(x)dx$.



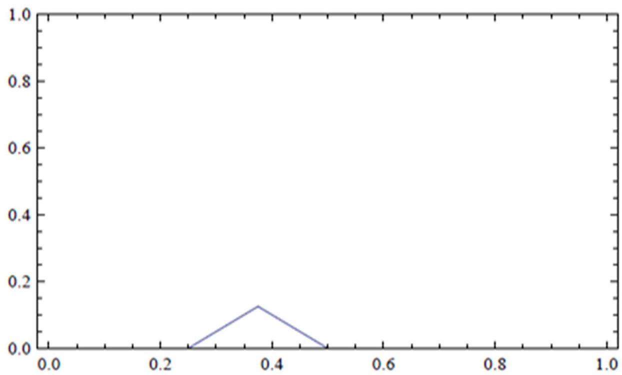

*c.* $p_2 = \int_0^1 h_2(x)dx$.



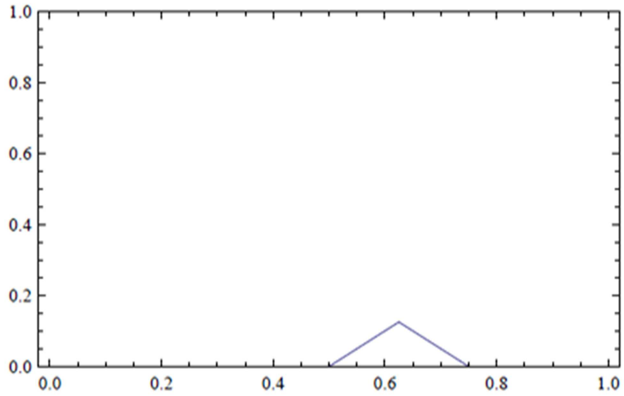

*d.* $p_3 = \int_0^1 h_3(x)dx$.



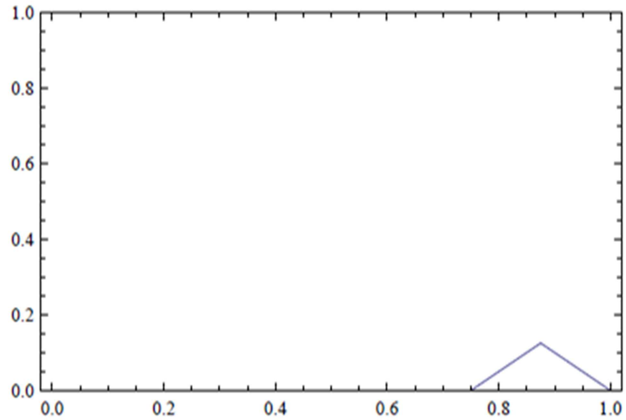

*e.* $p_4 = \int_0^1 h_4(x)dx$.




*f.* $p_5 = \int_0^1 h_5(x)dx$.




g. $p_6 = \int_0^1 h_6(x)dx$.




h. $p_7 = \int_0^1 h_7(x)dx$.

***Figure 2.*** *The graph of the first order integrations of the eight Haar wavelet.*

# 4. The Solution of Elliptic PDE Using Haar Wavelet Method

Poisson's equation has the form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = F(x,y),$$

$$0 \leq x \leq 1, 0 \leq y \leq 1, \tag{3}$$

With boundary conditions

$$\left.\begin{matrix} u(x,0) = f_1(x) \\ u(x,1) = f_2(x) \end{matrix}\right\} 0 \leq x \leq 1 \tag{4}$$

$$\left.\begin{matrix} u(0,y) = g_1(y) \\ u(1,y) = g_2(y) \end{matrix}\right\} 0 \leq y \leq 1. \tag{5}$$

Poisson's equation is used as a model problem in the numerical treatment of elliptic partial differential equations.

According to the two-dimensional multi-resolution analysis, [1], any function u(x,y) which is square integrable on [0,1]×[0,1] can be expressed in terms of two dimensional Haar series as follows

$$u(x,y) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i,j} h_i(x) h_j(y) \tag{6}$$

This series can be taken as an approximation for the solution of Poisson's equation. Moreover, the expansion of u(x,y) can be terminated.

$$u(x,y) = \sum_{i=1}^{2M_1} \sum_{j=1}^{2M_2} a_{i,j} h_i(x) h_j(y) \tag{7}$$

Where the wavelet coefficients $a_{i,j}$ i=1,2,…,2 $M_1$, j=1,2,…,2 $M_2$are to be determined.

The approach of Haar wavelet depends on writing the dominant derivative term in the form

$$u_{xxyy} = \sum_{i=1}^{2M_1} \sum_{j=1}^{2M_2} a_{i,j} h_i(x) h_j(y) \tag{8}$$

And accordingly one can obtain

$$u_{xx}(x,y) = \sum_{i=1}^{2M_1} \sum_{j=1}^{2M_2} a_{i,j} h_i(x) \left[q_j(y) - y q_j(1)\right] + y f_2''(x) + (1-y) f_1''(x) \tag{9}$$

Through, integrating equation (8) two times with respect to y and using equation (4).
Similarly,

$$u_{yy}(x,y) = \sum_{i=1}^{2M_1} \sum_{j=1}^{2M_2} a_{i,j} [q_i(x) - x q_i(1)] h_j(y) + x g_2''(y) + (1-x) g_1''(y) \tag{10}$$

Is obtained by integrating equation (8) two times with respect to x and using equation (5)
Then we Integrate equation (9) two times with respect to x and using equation (5), we obtain

$$u(x,y) = \sum_{i=1}^{2M_1} \sum_{j=1}^{2M_2} a_{i,j} [q_i(x) - x q_i(1)] \left[q_j(y) - y q_j(1)\right] + x g_2(y) + (1-x) g_1(y) + y f_2(x) + (1-y) f_1(x) - x y f_2(1) - x(1-y) f_1(1) - (1-x) y f_2(0) - (1-x)(1-y) f_1(0) \tag{11}$$

The wavelet collocation points are defined by

$$x_l = \frac{l-0.5}{2M_1}, l = 1,2,\ldots,2M_1 \tag{12}$$

$$y_n = \frac{n-0.5}{2M_2}, n = 1,2,\ldots,2M_2 \tag{13}$$

Substituting equations (9) and (10) in equation (3), and replacing $x\ by\ x_l$ and $y\ by\ y_n$ in the obtained equations and equation (11), we arrive at

$$\sum_{i=1}^{2M_1}\sum_{j=1}^{2M_2} a_{i,j}\ A(i,j,l,n) = \emptyset(x_l, y_n) \tag{14}$$

Where

$$A(i,j,l,n) = h_i(x_l)\left[q_j(y_n) - y_n q_j(1)\right] + \left[q_i(x_l) - x_l q_i(1)\right] h_j(y_n) \tag{15}$$

$$\emptyset(x_l, y_n) = (y_n - 1) f_1''(x_l) - y_n f_2''(x_l) + (x_l - 1) g_1''(y_n) - x_l g_2''(y_n) + F(x_l, y_n) \tag{16}$$

$$u(x_l, y_n) = \sum_{i=1}^{2M_1}\sum_{j=1}^{2M_2} a_{i,j}\left[q_i(x_l) - x_l q_i(1)\right]\left[q_j(y_n) - y_n q_j(1)\right] + x_l g_2(y_n) + (1 - x_l)\, g_1(y_n) + y_n f_2(x_l) + (1 - y_n) f_1(x_l) - x_l y_n f_2(1) - x_l(1 - y_n) f_1(1) - (1 - x_l) y_n f_2(0) - (1 - x_l)(1 - y_n) f_1(0) \tag{17}$$

The coefficients $a_{i,j,}$ i=1,2,…,$2M_1$, j=1,2,…,$2M_2$ are found from equation (14). Then we substitute in equation (17) to obtain the Haar solution at the collocation points $x_l, l = 1,2, \dots 2M_1, y_n$, n $= 1,2, \dots, 2M_2$.

# 5. The Coefficient Matrix of the Resulting Linear System

In this section the properties of the resulting linear system using Haar wavelet method are investigated. The coefficient matrix is symmetric matrix as shown in the following

$$\begin{bmatrix} \boldsymbol{D}_1 & \boldsymbol{A}_1 & \boldsymbol{A}_2 & \boldsymbol{A}_3 \\ \boldsymbol{A}_1^T & \boldsymbol{D}_2 & \boldsymbol{B}_1 & \boldsymbol{B}_2 \\ \boldsymbol{A}_2^T & \boldsymbol{B}_1^T & \boldsymbol{D}_3 & \boldsymbol{C}_1 \\ \boldsymbol{A}_3^T & \boldsymbol{B}_2^T & \boldsymbol{C}_1^T & \boldsymbol{D}_4 \end{bmatrix}, \tag{18}$$

Where

$$\boldsymbol{D}_1 = \frac{1}{64}\begin{bmatrix} -7 & -11 & -11 & -7 \\ -11 & -15 & -15 & -11 \\ -11 & -15 & -15 & -11 \\ -7 & -11 & -11 & -7 \end{bmatrix}, \quad \boldsymbol{D}_2 = \frac{1}{64}\begin{bmatrix} -3 & -3 & 3 & 3 \\ -3 & -3 & 3 & 3 \\ 3 & 3 & -3 & -3 \\ 3 & 3 & -3 & -3 \end{bmatrix} \boldsymbol{D}_3 = \frac{1}{128}\begin{bmatrix} 0 & 4 & 3 & 1 \\ 4 & -8 & -3 & -1 \\ 3 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \boldsymbol{D}_4 = \frac{1}{128}\begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & -3 & 3 \\ -1 & -3 & -8 & 4 \\ 1 & 3 & 4 & 0 \end{bmatrix}$$

$$\boldsymbol{A}_1 = \frac{1}{64}\begin{bmatrix} -5 & -5 & 5 & 5 \\ -9 & -9 & 9 & 9 \\ -9 & -9 & 9 & 9 \\ -5 & -5 & 5 & 5 \end{bmatrix}, \quad \boldsymbol{A}_2 = \frac{1}{128}\begin{bmatrix} -7 & 11 & 3 & 1 \\ -15 & 19 & 3 & 1 \\ -15 & 19 & 3 & 1 \\ -7 & 11 & 3 & 1 \end{bmatrix}, \boldsymbol{A}_3 = \frac{1}{128}\begin{bmatrix} -1 & -3 & -11 & 7 \\ -1 & -3 & -19 & 15 \\ -1 & -3 & -19 & 15 \\ -1 & -3 & -11 & 7 \end{bmatrix}$$

$$\boldsymbol{B}_1 = \frac{1}{128}\begin{bmatrix} -3 & 7 & 3 & 1 \\ -3 & 7 & 3 & 1 \\ 3 & -7 & -3 & -1 \\ 3 & -7 & -3 & -1 \end{bmatrix}, \quad \boldsymbol{B}_2 = \frac{1}{128}\begin{bmatrix} -1 & -3 & -7 & 3 \\ -1 & -3 & -7 & 3 \\ 1 & 3 & 7 & -3 \\ 1 & 3 & 7 & -3 \end{bmatrix}$$

$$\boldsymbol{C}_1 = \frac{1}{128}\begin{bmatrix} -1 & -3 & -4 & 0 \\ 1 & 3 & 8 & -4 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

While in case of the finite difference method the resulting coefficient matrix is block tridiagonal matrix with the natural ordering is considered [9], [10].

# 6. Numerical Results and Discussion

The Poisson Equation:

$$\frac{\partial^2 U(x,y)}{\partial\, x^2} + \frac{\partial^2 U(x,y)}{\partial\, y^2} = f(x,y),$$

$0 \le x \le 1, 0 \le y \le 1$ and $f(x,y) = 2(x^2 + y^2)$
Subject to the boundary conditions:

$$U(x,0) = U(0,y) = 0,$$

$$U(x,1) = x^2, U(1,y) = y^2$$

It can be seen that the exact solution is

$$U(x,y) = (xy)^2.$$

The absolute-error is identified by

$$\text{Error} = \frac{1}{(m-1)^2}\sqrt{\sum_{i,j=1}^{m-1}(U_{i,j} - u_{i,j})^2},$$

in which $U_{i,j}$ and $u_{i,j}$ are the exact and numerical solutions respectively.

This problem is solved using Haar wavelet method. The results show higher accuracy compared with the finite difference method, [11]. The absolute error is calculated and is found of order O($10^{-19}$).

Two 3-D mesh graphs are obtained for the Haar wavelet solution and the exact solution. The graphs are similar due to the smallness of the error.

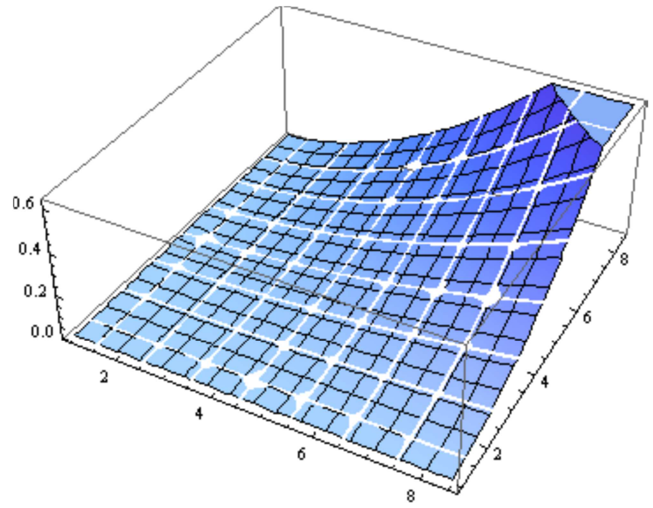



***Figure 3a.*** *3-D meshed surface plot of Haar Wavelet solution.*

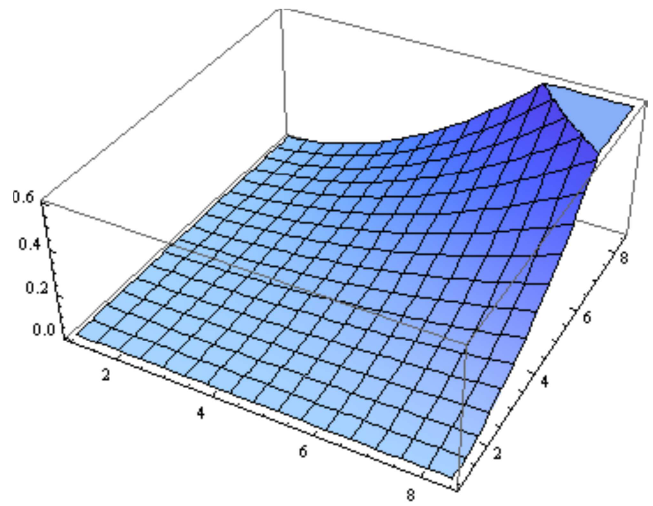



***Figure 3b.*** *3-D meshed surface plot of the exact solution.*

# 7. Conclusion

The coefficient matrix of the linear system, (18) has the symmetric block structure due to the local support of the Haar wavelet bases and the symmetry of the order of derivatives in Poisson's equation. Comparison with the corresponding matrix appears in the finite difference treatment help in building the block structure [7]. The simplicity in the format of the bases has facilitated the evaluations of the integrals as well as calculations. Comparison, with known results have proved the accuracy of the Haar wavelet methods. Other functions can be used building bases and this will be our interest in a subsequent work.

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