



# On the Log-Concavity of the Root of the Catalan-Larcombe-French Numbers

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## To cite this article:

Yang Wen. On the Log-Concavity of the Root of the Catalan-Larcombe-French Numbers. *American Journal of Mathematical and Computer Modelling*. Vol. 2, No. 4, 2017, pp. 95-98. doi: 10.11648/j.ajmcm.20170204.11

Received: October 29, 2016; Accepted: March 31, 2017; Published: April 17, 2017

**Abstract:** Recently, some combinatorial properties for the Catalan-Larcombe-French numbers have been proved by Sun and Wu, and Zhao. Recently, Z. W. Sun conjectured that the root of the Catalan-Larcombe-French numbers is log-concave. In this paper, we confirm Sun's conjecture by establishing the lower and upper bound for the ratios of the Catalan-Larcombe-French numbers.

**Keywords:** The Catalan-Larcombe-French Number, Log-Concavity, Recurrence Relation

## 1. Introduction

The aim of this paper is to prove the log-concavity of  $\{\sqrt[n]{P_n}\}_{n=1}^{\infty}$ , where  $P_n$  is the  $n$ -th Catalan-Larcombe-French number. This confirms a conjecture given by Zhi Wei Sun [8].

Recall that an infinite sequence  $a_n$  is said to be log-concave if for  $n \geq 1$ ,

$$a_n^2 \geq a_{n-1}a_{n+1}$$

The Catalan-Larcombe-French numbers  $P_n$  were first defined by Catalan in terms of the "Segner numbers". The  $n$ -th Catalan-Larcombe-French number is generated by either of the finite sums

$$P_n = 2^n \sum_{p=0}^{[n/2]} (-4)^p \binom{2(n-p)}{n-p}^2 \binom{n-p}{p}, n \geq 0$$

(where the function  $[x]$ , for arbitrary  $x$  real, is the greatest integer not exceeding  $x$ ) or

$$P_n = \frac{1}{n!} \sum_{p+q=n} \binom{2p}{p} \binom{2q}{q} \frac{(2p)!(2q)!}{p!q!}, n \geq 0$$

The infinite sequence  $\{P_n\}_{n=0}^{\infty}$  is known as the Catalan-Larcombe-French sequence (Sequence No. A053175 in Sloane's database [7]). These numbers occur in the theory of elliptic integrals, and there are relations to the arithmetic-geometric-mean. Furthermore, the sequence

satisfies the following recurrence relation:

$$P_n = \frac{8(3n^2-3n+1)}{n^2} P_{n-1} - \frac{128(n-1)^2}{n^2} P_{n-2} \quad (1)$$

for  $n \geq 2$ , with the initial values given by  $P_0 = 1$  and  $P_1 = 8$ . For more details, see [1-6].

The combinatorial properties for the Catalan-Larcombe-French sequence have been considered. The log-behavior of the Catalan-Larcombe-French sequence was studied by Zhao [12]. Moreover, she proved that the sequence  $\{P_n\}_{n=0}^{\infty}$  is log-balanced. Xia and Yao [11] proved that the sequences  $\left\{\frac{P_{n+1}}{P_n}\right\}_{n=0}^{\infty}$  and  $\{\sqrt[n]{P_n}\}_{n=0}^{\infty}$  are strictly increasing. The 2-log-convexity of the sequence  $\{P_n\}_{n=0}^{\infty}$  was proved by Sun and Wu [8]. Furthermore, Sun and Jin [9] proved the log-concavity for the sequence  $\left\{\frac{P_n}{P_{n-1}}\right\}_{n=1}^{\infty}$ , which confirmed a conjecture due to Sun and Wu [8].

Recently, Sun [10] posed the following conjecture:

Conjecture 1.1. The sequence  $\{\sqrt[n]{P_n}\}_{n=1}^{\infty}$  is log-concave.

Moreover, the sequence  $\{\sqrt[n+1]{P_{n+1}}/\sqrt[n]{P_n}\}_{n=1}^{\infty}$  is strictly decreasing.

In this paper, we will establish lower and upper bounds for  $\frac{P_n}{P_{n-1}}$  by utilizing the recurrence relation of  $P_n$  and then present a proof of Conjecture 1.1.

## 2. Lower and Upper Bounds for $\frac{P_n}{P_{n-1}}$

In order to prove Conjecture 1.1, we need to prove several inequalities for The Catalan-Larcombe-French numbers  $P_n$  and establish lower and upper bounds for  $\frac{P_n}{P_{n-1}}$ . We first establish a lower bound for  $\frac{P_n}{P_{n-1}}$ .

Lemma 2.1. For  $n \geq 7$ , we have

$$f(n) < \frac{P_n}{P_{n-1}} \quad (2)$$

where

$$f(n) = \frac{16(8n^3 - 20n^2 + 14n - 11)}{(2n-1)^3} \quad (3)$$

*Proof.* We are ready to prove Lemma 2.1 by induction on  $n$ . It is easy to check that Inequality (2) is true when  $n = 7$ . Suppose that Lemma 2.1 holds when  $n = m \geq 7$ , namely,

$$\frac{16(8m^3 - 20m^2 + 14m - 11)}{(2m-1)^3} < \frac{P_m}{P_{m-1}} \quad (4)$$

In order to prove this lemma, we need to prove that (2) holds when  $n = m+1$ , that is,

$$\frac{16(8(m+1)^3 - 20(m+1)^2 + 14(m+1) - 11)}{(2(m+1)-1)^3} < \frac{P_{m+1}}{P_m} \quad (5)$$

Thanks to (1) and (4), we deduce that

$$\begin{aligned} \frac{P_{m+1}}{P_m} &= \frac{8(3(m+1)^2 - 3(m+1) + 1)}{(m+1)^2} - \frac{128m^2}{(m+1)^2} \frac{P_{m-1}}{P_m} \\ &> \frac{8(3(m+1)^2 - 3(m+1) + 1)}{(m+1)^2} \\ &\quad - \frac{128m^2}{(m+1)^2} \frac{(2m-1)^3}{16(8m^3 - 20m^2 + 14m - 11)} \\ &= \frac{8(16m^5 - 24m^4 - 16m^3 - 10m^2 - 19m - 11)}{(m+1)^2(8m^3 - 20m^2 + 14m - 11)} \end{aligned} \quad (6)$$

Thanks to (6), we have

$$\begin{aligned} \frac{P_{m+1}}{P_m} &- \frac{16(8(m+1)^3 - 20(m+1)^2 + 14(m+1) - 11)}{(2(m+1)-1)^3} \\ &> \frac{8(16m^5 - 24m^4 - 16m^3 - 10m^2 - 19m - 11)}{(m+1)^2(8m^3 - 20m^2 + 14m - 11)} \\ &\quad - \frac{16(8(m+1)^3 - 20(m+1)^2 + 14(m+1) - 11)}{(2(m+1)-1)^3} \\ &= \frac{8(32m^6 - 80m^5 - 384m^4 - 488m^3 - 254m^2 - 273m - 209)}{(m+1)^2(8m^3 - 20m^2 + 14m - 11)(2m+1)^3} > 0. \end{aligned} \quad (7)$$

Inequality (5) follows from (7). Therefore, Lemma 2.1 is proved by induction. The proof is complete.

We are now in a position to establish an upper bound for  $\frac{P_n}{P_{n-1}}$ .

Lemma 2.2. For  $n \geq 0$ , we have

$$\frac{P_n}{P_{n-1}} < f(n+1), \quad (8)$$

where  $f(n)$  is defined by (3).

*Proof.* We also prove Lemma 2.2 by induction on  $n$ . It is easy to verify that (8) is true when  $n = 7$ . Assume that Lemma 2.2 holds when  $n = m \geq 7$ , namely,

$$\frac{P_m}{P_{m-1}} < \frac{16(8(m+1)^3 - 20(m+1)^2 + 14(m+1) - 11)}{(2(m+1)-1)^3}. \quad (9)$$

In order to prove (8), it suffices to prove that (8) is true when  $n = m+1$ , that is,

$$\frac{P_{m+1}}{P_m} < \frac{16(8(m+2)^3 - 20(m+2)^2 + 14(m+2) - 11)}{(2(m+2)-1)^3} \quad (10)$$

It follows from (1) and (9) that

$$\begin{aligned} \frac{P_{m+1}}{P_m} &= \frac{8(3(m+1)^2 - 3(m+1) + 1)}{(m+1)^2} - \frac{128m^2}{(m+1)^2} \frac{P_{m-1}}{P_m} \\ &< \frac{8(3(m+1)^2 - 3(m+1) + 1)}{(m+1)^2} \\ &\quad - \frac{128m^2}{(m+1)^2} \frac{(2(m+1)-1)^3}{16(8(m+1)^3 - 20(m+1)^2 + 14(m+1) - 11)} \\ &= \frac{8(16m^5 + 24m^4 + 8m^3 - 30m^2 - 29m - 9)}{(m+1)^2(8m^3 + 4m^2 - 2m - 9)}. \end{aligned} \quad (11)$$

Therefore, by (11), we deduce that

$$\begin{aligned} \frac{P_{m+1}}{P_m} &- \frac{16(8(m+2)^3 - 20(m+2)^2 + 14(m+2) - 11)}{(2(m+2)-1)^3} \\ &< \frac{8(16m^5 + 24m^4 + 8m^3 - 30m^2 - 29m - 9)}{(m+1)^2(8m^3 + 4m^2 - 2m - 9)} \\ &\quad - \frac{16(8(m+2)^3 - 20(m+2)^2 + 14(m+2) - 11)}{(2(m+2)-1)^3} \\ &= - \frac{8(32m^6 + 80m^5 + 288m^4 + 744m^3 + 978m^2 + 689m + 225)}{(m+1)^2(8m^3 + 4m^2 - 2m - 9)(2m+3)^3} < 0. \end{aligned} \quad (12)$$

Inequality (5) follows from (12) and Lemma 2.2 is proved by induction. This completes the proof.

## 3. Proof of Conjecture 1.1

In this section, we provide a proof of Conjecture 1.1 by utilizing the lower and upper bounds for  $\frac{P_n}{P_{n-1}}$  established in Section 2. We first prove the following lemma:

Lemma 3.1. For  $n \geq 7$ ,

$$\left(1 - \frac{2}{n^2+n+2}\right)^{n^2+n+2} \geq \left(1 - \frac{1}{29}\right)^{58} \quad (13)$$

*Proof.* It is easy to see that for  $x > y > 0$ ,

$$x^{n+1} - y^{n+1} = (x-y)(x^n + x^{n-1}y + \cdots + xy^{n-1} + y^n) > (n+1)(x-y)y^n. \quad (14)$$

If we set  $x = 1 - \frac{2}{n+1}$  and  $y = 1 - \frac{2}{n}$  in (14), then we get

$$\left(1 - \frac{2}{n+1}\right)^{n+1} - \left(1 - \frac{2}{n}\right)^{n+1} > (n+1) \left(\frac{2}{n} - \frac{2}{n+1}\right) \left(1 - \frac{2}{n}\right)^n = \frac{2}{n} \left(1 - \frac{2}{n}\right)^n. \quad (15)$$

Therefore, it follows from (15) that

$$\begin{aligned} \left(1 - \frac{2}{n+1}\right)^{n+1} &> \left(1 - \frac{2}{n}\right)^{n+1} + \frac{2}{n} \left(1 - \frac{2}{n}\right)^n \\ &= \left(1 - \frac{2}{n} + \frac{2}{n}\right) \left(1 - \frac{2}{n}\right)^n = \left(1 - \frac{2}{n}\right)^n \end{aligned} \quad (16)$$

Therefore, the sequence  $\left\{\left(1 - \frac{2}{n}\right)^n\right\}_{n=2}^{\infty}$  is increasing. In particular, its subsequence  $\left\{\left(1 - \frac{2}{n^2+n+2}\right)^{n^2+n+2}\right\}_{n=7}^{\infty}$  is also increasing. Hence, for  $n \geq 7$ ,

$$\begin{aligned} \left(1 - \frac{2}{n^2+n+2}\right)^{n^2+n+2} &\geq \left(1 - \frac{2}{7^2+7+2}\right)^{7^2+7+2} \\ &= \left(1 - \frac{1}{29}\right)^{58}. \end{aligned} \quad (17)$$

which is nothing but (17). The proof of this lemma is complete.

We are now in a position to turn to prove Conjecture 1.1.

*Proof of Conjecture 1.1.* It is easy to check that Conjecture 1.1 is true when  $1 \leq n \leq 6$ . Hence, we only need to consider the case  $n \geq 7$ . By (2) and (8), we deduce that for  $n \geq 7$ ,

$$\left(\frac{P_{n+1}}{P_n}\right)^{(n+1)(n+2)} > f^{(n+1)(n+2)}(n+1) \quad (18)$$

$$\begin{aligned} \left(\frac{P_{n+1}}{P_n}\right)^{(n+1)(n+2)} - \left(\frac{P_{n+2}}{P_{n+1}}\right)^{n(n+1)} P_{n+1}^2 &> f^{(n+1)(n+2)}(n+1) - f^{n^2+n+2}(n+3) P_7^2 f^{2n-14}(n+1) \\ &= f^{2n-14}(n+1) f^{n^2+n+2}(n+3) \left( \left(\frac{f(n+1)}{f(n+3)}\right)^{n^2+n+2} f^{14}(n+1) - P_7^2 \right) \\ &> f^{2n-14}(n+1) f^{n^2+n+2}(n+3) \left( \left(1 - \frac{2}{n^2+n+2}\right)^{n^2+n+2} f^{14}(n+1) P_7^2 \right) \end{aligned} \quad (24)$$

By (13), (24) and the fact that  $f(7) < f(n+1)$  for  $n \geq 7$ ,

$$\begin{aligned} \left(\frac{P_{n+1}}{P_n}\right)^{(n+1)(n+2)} - \left(\frac{P_{n+2}}{P_{n+1}}\right)^{n(n+1)} P_{n+1}^2 &> f^{2n-14}(n+1) f^{n^2+n+2}(n+3) \left( \left(1 - \frac{1}{29}\right)^{58} f^{14}(n+1) - P_7^2 \right) \\ &> f^{2n-14}(n+1) f^{n^2+n+2}(n+3) \left( \left(1 - \frac{1}{29}\right)^{58} f^{14}(7) - P_7^2 \right) \end{aligned} \quad (25)$$

With Maple, it is easy to verify that

$$\left(1 - \frac{1}{29}\right)^{58} f^{14}(7) - P_7^2 > 0 \quad (26)$$

Combining (3.13) and (3.14) yields

$$\frac{P_{n+1}}{P_n} < f(n+2) < \frac{P_{n+2}}{P_{n+1}} < f(n+3) \quad (19)$$

and

$$\begin{aligned} \frac{P_8}{P_7} < f(9) < \frac{P_9}{P_8} < f(10) < \dots \\ < \frac{P_n}{P_{n-1}} < f(n+1) < \frac{P_{n+1}}{P_n}, \end{aligned} \quad (20)$$

where  $f(n)$  is defined by (3). By (20), we see that for  $n \geq 7$ ,

$$P_{n+1}^2 = P_7^2 \left(\frac{P_8}{P_7} \frac{P_9}{P_8} \dots \frac{P_n}{P_{n-1}}\right)^2 \frac{P_{n+1}^2}{P_n^2} \leq P_7^2 f^{2n-14}(n+1) \frac{P_{n+1}^2}{P_n^2}. \quad (21)$$

In view of (19) and (21), we see that

$$\begin{aligned} \left(\frac{P_{n+2}}{P_{n+1}}\right)^{n(n+1)} P_{n+1}^2 &< f^{n(n+1)}(n+3) P_{n+1}^2 \\ &< f^{n^2+n}(n+3) P_7^2 f^{2n-14}(n+1) \frac{P_{n+1}^2}{P_n^2} \\ &< f^{n^2+n+2}(n+3) P_7^2 f^{2n-14}(n+1). \end{aligned} \quad (22)$$

It is easy to check that for  $n \geq 7$ ,

$$\begin{aligned} \frac{f(n+1)}{f(n+3)} - \left(1 - \frac{2}{n^2+n+2}\right) \\ = \frac{2(64n^5+112n^4-576n^3-1992n^2-2196n-1125)}{(2n+1)^3(8n^3+52n^2+110n+67)(n^2+n+2)} > 0. \end{aligned} \quad (23)$$

Based on (3.6), (3.10) and (3.11), we deduce that for  $n \geq 7$ ,

$$\left(\frac{P_{n+1}}{P_n}\right)^{(n+1)(n+2)} > \left(\frac{P_{n+2}}{P_{n+1}}\right)^{n(n+1)} P_{n+1}^2 \quad (27)$$

Inequality (27) can be rewritten as

$$P_{n+1}^{2n^2+4n} > P_n^{(n+1)(n+2)} P_{n+2}^{n(n+1)}$$

Therefore,

$$(P_{n+1}^{2n^2+4n})^{\frac{1}{n(n+1)(n+2)}} > (P_n^{(n+1)(n+2)} P_{n+2}^{n(n+1)})^{\frac{1}{n(n+1)(n+2)}}$$

which yields

$$P_{n+1}^{\frac{2}{n+1}} > P_n^{\frac{1}{n}} P_{n+2}^{\frac{1}{n+2}}$$

The above inequality can be rewritten as

$$\frac{n+1}{\sqrt[n]{P_n}} > \frac{n+2}{\sqrt[n+1]{P_{n+1}}} \quad (28)$$

It follows from (28) that the sequence  $\{\sqrt[n]{P_n}\}_{n=1}^{\infty}$  is log-concave and the sequence  $\{\sqrt[n+1]{P_{n+1}}/\sqrt[n]{P_n}\}_{n=1}^{\infty}$  is strictly decreasing. This completes the proof of Conjecture 1.1.

## 4. Conclusion

The Catalan-Larcombe-French numbers play important roles in combinatorics and number theory. Many combinatorial properties and congruence properties for the Catalan-Larcombe-French numbers have been proved. In this paper, by establishing several inequalities for the Catalan-Larcombe-French numbers, we obtain the lower and upper bound for the quotient  $\frac{P_n}{P_{n-1}}$  and prove that the  $n$ th Root of the Catalan-Larcombe-French Numbers are log-concave, which confirm a conjecture given by Sun.

## Acknowledgements

I wish to thank the referee for comments and suggestions which have resulted in an improved version of the paper.

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