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Numerical Approach for Solving Stiff Differential Equations Through the Extended Trapezoidal Rule Formulae

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Abstract: The most popular methods for the solution of stiff initial value problems for ordinary differential equations are the backward differentiation formulae (BDF). In this paper, we focus on the derivation of the fourth, sixth and eighth order extended trapezoidal rule of first kind (ETRs) formulae through Hermite polynomial as basis function which we named FETR, SETR and EETR respectively. We then interpolate and collocate at some points of interest to generate the desire method. The stability analysis on our methods suggests that they are not only convergent but possess regions suitable for the solution of stiff ordinary differential equations (ODEs). The methods were very efficient when implemented in block form, they tend to perform better over existing methods.

Keywords: Stiffness, Hermite Polynomial, ETRs, A-Stability, Ordinary Differential Equations

1. Introduction

A very important special class of differential equations taken up in the initial value problems termed as stiff differential equations result from the phenomenon with widely differing time scales [3, 7]. There is no universally acceptable definition of stiffness. Stiffness is a subtle, difficult and important concept in the numerical solution of ordinary differential equations. It depends on the differential equation, the initial condition and the interval under consideration. The initial value problems with stiff ordinary differential equation systems occur in many field of engineering science, particularly in the studies of electrical circuits, vibrations, chemical reactions and so on. Stiff differential equations are ubiquitous in astrochemical kinetics, many non-industrial areas like weather prediction and biology.

A set of differential equations is 'stiff' when an excessively small step is needed to obtain correct integration. In other words we can say a set of differential equations is 'stiff' when it contains at least two 'time constants' (where time is supposed to be the joint independent variable) that differ by several orders of magnitude. A more rigorous definition of stiffness was also given by Shampine and Gear: "By a stiff problem we mean one for which no solution component is unstable (no Eigen-value of the Jacobian matrix has a real part which is at all large and positive) and at least some component is very stable (at least one Eigen-value has a real part which is large and negative). Further, we will not call a problem stiff unless its solution is slowly varying with respect to most negative part of the Eigen-values. Consequently a problem may be stiff for some intervals and not for others.

We seek to propose a new numerical integrator for the solution of first-order ordinary differential equation of the form:

$$y'(x) = f(x, y), \; y(a) = y_0, I = \left[x_0, x_N\right] \tag{1}$$

where, f is continuous within the interval of integration, we assume that f satisfies Lipchitz condition which guarantees the existence and uniqueness of solution of (1).

A lot of work has been conducted by many scholarson finding the solution of (1). They have adopted several methods using different polynomials as basis functions, some of these scholars include; [6, 7, 3, 4, 1, 2, 9, 10].

2. Derivation of the Method

The approach in this paper basically entails substituting into (1) a trail solution of the form

$$y:\left[x_0,x_N\right]\rightarrow R^m \tag{2}$$

where $f:\left[x_0,x_N\right]\times R^m\rightarrow R^m$ an d $I_j=a+jh$, are the interpolation and collocation points, $j=0,\ldots,N-1$, is the Hermite polynomial generated by the formula:

$$h=\frac{b-a}{N-1} \tag{3}$$

For the sake of reporting, we present some few terms of the Hermite polynomial as

$$k-step\ ,\sum_{j=0}^{k}\alpha_j y_{n+j}=h\left[\sum_{j=0}^{k}\beta_j(x)f_{n+j}\right],y_{n+j}\ ,f_{n+j}\ ,$$
$$y(x_{n+j})\ ,f(x_{n+j},y(x_{n+j}))$$

From (2)

$$e^{-\lambda t} \tag{4}$$

Substituting (4) into (1) we obtained

$$\lambda \tag{5}$$

2.1. Derivation of the Block ETRs Method

2.1.1. Fourth Order Block ETRs (FETR)

Interpolating (2) at $Y_1^{[n]},Y_2^{[n]},\ldots,Y_s^{[n]}$ and collocating (5) at $f\left(Y_1^{[n]}\right),f\left(Y_2^{[n]}\right),\ldots,f\left(Y_s^{[n]}\right)$ gives a system of equation which can be put in the form

$$Y_1^{[n-1]},Y_2^{[n-1]},\ldots,Y_r^{[n-1]} \tag{6}$$

Solving (6) for the $Y_1^{[n]},Y_2^{[n]},\ldots,Y_r^{[n]}$ yields

$$Y^{[n]},f\left(Y^{[n]}\right),$$

$$y^{[n-1]}$$

$$y^{[n]}$$

$$Y^{[n]}=\begin{bmatrix}Y_1^{[n]}\\Y_2^{[n]}\\ \cdot\\ \cdot\\ \cdot\\ Y_s^{[n]}\end{bmatrix}$$

$$f\left(Y^{[n]}\right)=\begin{bmatrix}f\left(Y_1^{[n]}\right)\\f\left(Y_2^{[n]}\right)\\ \cdot\\ \cdot\\ \cdot\\ f\left(Y_s^{[n]}\right)\end{bmatrix} \tag{7}$$

substituting (7) into (2) for $y^{[n-1]}=\begin{bmatrix}y_1^{[n-1]}\\y_2^{[n-1]}\\ \cdot\\ \cdot\\ \cdot\\ y_r^{[n-1]}\end{bmatrix}$ and $y^{[n]}=\begin{bmatrix}y_1^{[n]}\\y_2^{[n]}\\ \cdot\\ \cdot\\ \cdot\\ y_r^{[n]}\end{bmatrix}$ give an equation which is evaluated at some points of interest to generate the following set of main and additional equations.

Additional Equations

$$Y_1^{[n]},Y_2^{[n]},\ldots,Y_s^{[n]}$$

$$Y^{[n]}=h\left(A\oplus I_m\right)f\left(Y^{[n]}\right)+\left(U\oplus I_m\right)y^{[n-1]} \tag{8}$$

2.1.2. Sixth Order Block ETRs (SETR)

Interpolating (2) at $y^{[n]}=h\left(B\oplus I_m\right)f\left(Y^{[n]}\right)+\left(V\oplus I_m\right)y^{[n-1]}$ and collocating (5) at $\oplus$ we obtained

$$Y^{[n]}=hAf\left(Y^{[n]}\right)+Uy^{[n-1]} \tag{9}$$

whose solution gives the coefficients of $y^{[n]}=hBf\left(Y^{[n]}\right)+Vy^{[n-1]}$ as

$$M=\begin{bmatrix}A & U\\ B & V\end{bmatrix}$$

$$y(x)=\sum_{j=0}^{k-1}\alpha_j(x)y_{n+j}+\alpha_q(x)y_{n+q}+h\left[\sum_{j=k-2}^{2}\beta_j(x)f_{n+j}+\beta_q(x)f_{n+q}\right]$$

$$\alpha_j(x)$$

$$\beta_j(x)$$

$$j=0(1)\frac{3}{2},2$$

$$\beta_q(x)$$

$$\alpha_q(x) \tag{10}$$

substituting (10) into (2) for $q=\frac{a}{b}$, $\frac{2r+1}{2},r=1,2$ and evaluating at points of interest to generate: Additional Equations

$$\phi_0(x) = 1 - \frac{1225}{303}\frac{\xi}{h} + \frac{8035}{1212}\frac{\xi^2}{h^2} - \frac{3419}{606}\frac{\xi^3}{h^3} + \frac{3187}{1212}\frac{\xi^4}{h^4} - \frac{193}{303}\frac{\xi^5}{h^5} + \frac{19}{303}\frac{\xi^6}{h^6}$$

$$\phi_1(x) = -\frac{22016}{101}\frac{\xi^3}{h^3} + \frac{17268}{101}\frac{\xi^2}{h^2} - \frac{3876}{101}\frac{\xi^5}{h^5} + \frac{13329}{101}\frac{\xi^4}{h^4} - \frac{5040}{101}\frac{\xi}{h} + \frac{436}{101}\frac{\xi^6}{h^6}$$

$$\phi_2(x) = -\frac{4017}{101}\frac{\xi^5}{h^5} + \frac{60687}{404}\frac{\xi^2}{h^2} - \frac{40631}{202}\frac{\xi^3}{h^3} + \frac{475}{101}\frac{\xi^6}{h^6} - \frac{4365}{101}\frac{\xi}{h} + \frac{52203}{404}\frac{\xi^4}{h^4}$$

$$\phi_{\frac{3}{2}}(x) = \frac{29440}{303}\frac{\xi}{h} - \frac{99328}{303}\frac{\xi^2}{h^2} + \frac{23872}{303}\frac{\xi^5}{h^5} - \frac{2752}{303}\frac{\xi^6}{h^6} + \frac{128704}{303}\frac{\xi^3}{h^3} - \frac{79936}{303}\frac{\xi^4}{h^4} \tag{11}$$

2.1.3. Eighth Order Block ETRs (EETR)

Following the same procedure as in fourth and sixth order ETRs, we obtained Additional Equations

$$\psi_1(x) = \frac{11947}{303}\frac{\xi^4}{h^3} - \frac{6953}{101}\frac{\xi^3}{h^2} + \frac{17704}{303}\frac{\xi^2}{h} - \frac{1940}{101}\xi + \frac{364}{303}\frac{\xi^6}{h^5} - \frac{1112}{101}\frac{\xi^5}{h^4}$$

$$\psi_2(x) = \frac{7359}{202}\frac{\xi^4}{h^3} - \frac{8167}{202}\frac{\xi^2}{h} + \frac{5584}{101}\frac{\xi^3}{h^2} + \frac{1155}{101}\xi - \frac{142}{101}\frac{\xi^6}{h^5} + \frac{1166}{101}\frac{\xi^5}{h^4}$$

$$\psi_{\frac{3}{2}}(x) = \frac{704}{303}\frac{\xi^2}{h} - \frac{64}{101}\xi + \frac{704}{303}\frac{\xi^4}{h^3} + \frac{32}{303}\frac{\xi^6}{h^5} - \frac{336}{101}\frac{\xi^3}{h^2} - \frac{80}{101}\frac{\xi^5}{h^4}$$

$$\left[\begin{array}{cccccc|cccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-\frac{101}{5040} & -\frac{388}{1008} & 0 & \frac{231}{1008} & -\frac{4}{315} & 0 & -\frac{97}{112} & \frac{5888}{3024} & 0 & -\frac{35}{432} \\
0 & -\frac{159}{112} & -\frac{303}{56} & -\frac{387}{244} & \frac{3}{56} & 0 & \frac{4023}{448} & 0 & -\frac{891}{112} & -\frac{11}{448} \\
0 & -\frac{113}{633} & 0 & \frac{530}{1477} & -\frac{256}{4431} & \frac{101}{13293} & 0 & \frac{68864}{39879} & -\frac{152}{211} & -\frac{257}{39879} \\
0 & -\frac{75}{404} & 0 & \frac{675}{808} & \frac{15}{101} & 0 & -\frac{675}{1616} & \frac{225}{101} & -\frac{325}{404} & -\frac{9}{1616} \\
0 & \frac{366}{101} & 0 & -\frac{738}{101} & \frac{192}{101} & 0 & \frac{2187}{101} & -\frac{3584}{101} & \frac{1485}{101} & \frac{13}{101} \\
\hline
0 & \frac{366}{101} & 0 & -\frac{738}{101} & \frac{192}{101} & 0 & \frac{2187}{101} & -\frac{3584}{101} & \frac{1485}{101} & \frac{13}{101} \\
0 & -\frac{113}{633} & 0 & \frac{530}{1477} & -\frac{256}{4431} & \frac{101}{13293} & 0 & \frac{68864}{39879} & -\frac{152}{211} & -\frac{257}{39879} \\
0 & -\frac{159}{112} & -\frac{303}{56} & -\frac{387}{244} & \frac{3}{56} & 0 & \frac{4023}{448} & 0 & -\frac{891}{112} & -\frac{11}{448} \\
-\frac{101}{5040} & -\frac{388}{1008} & 0 & \frac{231}{1008} & -\frac{4}{315} & 0 & -\frac{97}{112} & \frac{5888}{3024} & 0 & -\frac{35}{432}
\end{array}\right]$$

$$I=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}$$

$$A=\begin{bmatrix}0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\end{bmatrix} \quad (12)$$

2.2. Main Equation for FETR, SETR and EETR Respectively

$$B_1=\begin{bmatrix}\frac{673}{360}&-\frac{104}{45}&\frac{211}{120}&-\frac{32}{45}&\frac{43}{360}\\[2ex]\frac{1323}{640}&-\frac{77}{40}&\frac{1053}{640}&-\frac{27}{40}&\frac{73}{640}\\[2ex]\frac{92}{45}&-\frac{224}{135}&\frac{29}{15}&-\frac{32}{45}&\frac{16}{135}\\[2ex]\frac{2375}{1152}&-\frac{125}{72}&\frac{875}{384}&-\frac{35}{72}&\frac{125}{1152}\\[2ex]\frac{81}{40}&-\frac{8}{5}&\frac{81}{40}&0&\frac{11}{40}\end{bmatrix}$$

$$B_0=\begin{bmatrix}0&0&0&0&\frac{11}{40}\\[2ex]0&0&0&0&\frac{35}{128}\\[2ex]0&0&0&0&\frac{37}{135}\\[2ex]0&0&0&0&\frac{35}{128}\\[2ex]0&0&0&0&\frac{11}{40}\end{bmatrix}$$

$$y'(x)=f(x,y), \quad (13)$$

The class of methods we have developed shall each be implemented in block form to generate approximate solutions $y(a)=y_0$ simultaneously without the need for starters.

3. Analysis of Basic Properties of the Newly Derived Block Methods

3.1. Order of the Block ETRs Method

3.1.1. Consider Equation (8) and Its Main Equation in (13)

Applying (5), we obtained

$$I=\left[x_0,x_N\right] \quad (14)$$

Expanding (14) in Taylor Series gives

$$y:\left[x_0,x_N\right]\rightarrow R^m \quad (15)$$

Following (15), we obtained the order and error constants of (8), including its main equation in (13) as $f:\left[x_0,x_N\right]\times R^m\rightarrow R^m$ and $I_j=a+jh,$ respectively.

3.1.2. Consider Equation (2.6)

Applying (5) on (11), including its main equation in (13), yields

$$j=0,\dots,N-1, \quad (16)$$

Expanding (16) in Taylor Series gives

$$h=\frac{b-a}{N-1}$$

$$k-step$$

$$\sum_{j=0}^{k}\alpha_j y_{n+j}=h\left[\sum_{j=0}^{k}\beta_j(x)f_{n+j}\right]$$

$$y_{n+j}$$

$$f_{n+j}$$

$$y(x_{n+j}) \quad (17)$$

Following (17), we obtained a uniform order $f(x_{n+j},y(x_{n+j}))$ for (11) and its main equation in (13), whose error constants were calculated as $e^{-\lambda t}$.

3.1.3. Consider Equation (2.7)

Application of (5) on (12), including its main equation in (13) yields

$$\lambda$$

$$Y_1^{[n]},Y_2^{[n]},\dots,Y_s^{[n]} \quad (18)$$

Expanding (18) in Taylor Series gives

$$f\left(Y_1^{[n]}\right),f\left(Y_2^{[n]}\right),.,f\left(Y_s^{[n]}\right)$$

$$Y_1^{[n-1]},Y_2^{[n-1]},\dots,Y_r^{[n-1]}$$

$$Y_1^{[n]},Y_2^{[n]},\dots,Y_r^{[n]}$$

$$Y^{[n]},f\left(Y^{[n]}\right),$$

$$y^{[n-1]}$$

$$y^{[n]}$$

$$Y^{[n]} = \begin{bmatrix} Y_1^{[n]} \\ Y_2^{[n]} \\ \cdot \\ \cdot \\ \cdot \\ Y_s^{[n]} \end{bmatrix}$$

$$f\left(Y^{[n]}\right) = \begin{bmatrix} f\left(Y_1^{[n]}\right) \\ f\left(Y_2^{[n]}\right) \\ \cdot \\ \cdot \\ \cdot \\ f\left(Y_s^{[n]}\right) \end{bmatrix} \tag{19}$$

From (19), we get the order of equation (12) and main equation (13) as $y^{[n-1]} = \begin{bmatrix} y_1^{[n-1]} \\ y_2^{[n-1]} \\ \cdot \\ \cdot \\ \cdot \\ y_r^{[n-1]} \end{bmatrix}$ and error constants as

$$y^{[n]} = \begin{bmatrix} y_1^{[n]} \\ y_2^{[n]} \\ \cdot \\ \cdot \\ \cdot \\ y_r^{[n]} \end{bmatrix}.$$

3.2. Zero Stability of the Main Method(s)

3.2.1. Zero Stability of Main Method for SETR

Consider the characteristic equation associated with the main discrete scheme of (11) given by

$$Y_1^{[n]}, Y_2^{[n]}, \ldots, Y_s^{[n]} \; Y^{[n]} = h\left(A \oplus I_m\right) f\left(Y^{[n]}\right) + \left(U \oplus I_m\right) y^{[n-1]} \tag{20}$$

where $y^{[n]} = h\left(B \oplus I_m\right) f\left(Y^{[n]}\right) + \left(V \oplus I_m\right) y^{[n-1]}$ is the eigen-value(s) of the Jacobian of (1). $\oplus$ and $Y^{[n]} = hAf\left(Y^{[n]}\right) + Uy^{[n-1]}$ are the first and second characteristic polynomials of the scheme in (11) respectively and which is given by

$$y^{[n]} = hBf\left(Y^{[n]}\right) + Vy^{[n-1]} \tag{21}$$

$$M = \begin{bmatrix} A & U \\ B & V \end{bmatrix} \tag{22}$$

Solving (21), we obtained

$$y(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + \alpha_q(x) y_{n+q} + h\left[\sum_{j=k-2}^{2} \beta_j(x) f_{n+j} + \beta_q(x) f_{n+q}\right] \tag{23}$$

Zero-Stability property requires that the roots of (23) must satisfy $\alpha_j(x)$ and every root with $\beta_j(x)$ must have multiplicity 1.

3.2.2. Zero Stability of Main Method for EETR

Similarly, the first and second characteristic polynomials of the main scheme in (12) is given by

$$j = 0(1)\frac{3}{2}, 2 \tag{24}$$

$$\beta_q(x) \tag{25}$$

From (24), it can easily be shown that themain scheme of the eighth order ETRs is zero stable.

3.3. Zero Stability of the Block Method(s)

Definition 1: The block method SETR is said to be zero-stable provided the roots $\alpha_q(x)$ of the first characteristics polynomial $q = \frac{a}{b}$ specified by

$$\frac{2r+1}{2}, r = 1, 2 \tag{26}$$

satisfies $\phi_0(x) = 1 - \frac{1225}{303}\frac{\xi}{h} + \frac{8035}{1212}\frac{\xi^2}{h^2} - \frac{3419}{606}\frac{\xi^3}{h^3} + \frac{3187}{1212}\frac{\xi^4}{h^4} - \frac{193}{303}\frac{\xi^5}{h^5} + \frac{19}{303}\frac{\xi^6}{h^6}$ and for those roots with $\phi_1(x) = -\frac{22016}{101}\frac{\xi^3}{h^3} + \frac{17268}{101}\frac{\xi^2}{h^2} - \frac{3876}{101}\frac{\xi^5}{h^5} + \frac{13329}{101}\frac{\xi^4}{h^4} - \frac{5040}{101}\frac{\xi}{h} + \frac{436}{101}\frac{\xi^6}{h^6}$ the multiplicity does not exceed two.

Consistency and Convergence: Since each of the block methods (8), (11), (12) and their individual main equations in (13) has order $\phi_2(x) = -\frac{4017}{101}\frac{\xi^5}{h^5} + \frac{60687}{404}\frac{\xi^2}{h^2} - \frac{40631}{202}\frac{\xi^3}{h^3} + \frac{475}{101}\frac{\xi^6}{h^6} - \frac{4365}{101}\frac{\xi}{h} + \frac{52203}{404}\frac{\xi^4}{h^4}$ they are all consistent and by Henrici (1962). Convergence= Consistency and zero stability.

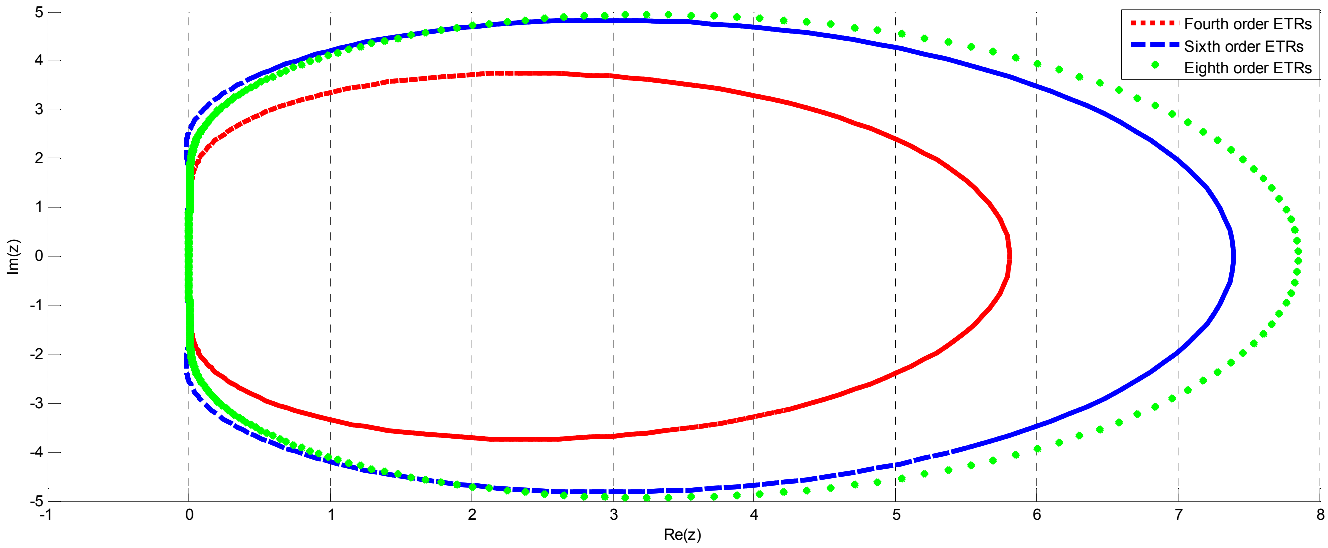

Figure 1. *Stability Region for the Fourth, Sixth and Eighth Order Block ETRs Method.*

4. Numerical Implementation of *FETR, SETR*

Problem 1: consider a highly stiff ODE of the form

$$\phi_{\frac{3}{2}}(x) = \frac{29440}{303}\frac{\xi}{h} - \frac{99328}{303}\frac{\xi^2}{h^2} + \frac{23872}{303}\frac{\xi^5}{h^5} - \frac{2752}{303}\frac{\xi^6}{h^6} + \frac{128704}{303}\frac{\xi^3}{h^3} - \frac{79936}{303}\frac{\xi^4}{h^4},$$

$$\psi_1(x) = \frac{11947}{303}\frac{\xi^4}{h^3} - \frac{6953}{101}\frac{\xi^3}{h^2} + \frac{17704}{303}\frac{\xi^2}{h} - \frac{1940}{101}\xi + \frac{364}{303}\frac{\xi^6}{h^5} - \frac{1112}{101}\frac{\xi^5}{h^4}$$

$$\psi_2(x) = \frac{7359}{202}\frac{\xi^4}{h^3} - \frac{8167}{202}\frac{\xi^2}{h} + \frac{5584}{101}\frac{\xi^3}{h^2} + \frac{1155}{101}\xi - \frac{142}{101}\frac{\xi^6}{h^5} + \frac{1166}{101}\frac{\xi^5}{h^4},$$

$$\psi_{\frac{3}{2}}(x) = \frac{704}{303}\frac{\xi^2}{h} - \frac{64}{101}\xi + \frac{704}{303}\frac{\xi^4}{h^3} + \frac{32}{303}\frac{\xi^6}{h^5} - \frac{336}{101}\frac{\xi^3}{h^2} - \frac{80}{101}\frac{\xi^5}{h^4},$$

$$\left[\begin{array}{cccccc|cccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-\frac{101}{5040} & -\frac{388}{1008} & 0 & \frac{231}{1008} & -\frac{4}{315} & 0 & -\frac{97}{112} & \frac{5888}{3024} & 0 & -\frac{35}{432} \\
0 & -\frac{159}{112} & -\frac{303}{56} & -\frac{387}{244} & \frac{3}{56} & 0 & \frac{4023}{448} & 0 & -\frac{891}{112} & -\frac{11}{448} \\
0 & -\frac{113}{633} & 0 & \frac{530}{1477} & -\frac{256}{4431} & \frac{101}{13293} & 0 & \frac{68864}{39879} & -\frac{152}{211} & -\frac{257}{39879} \\
0 & -\frac{75}{404} & 0 & \frac{675}{808} & \frac{15}{101} & 0 & -\frac{675}{1616} & \frac{225}{101} & -\frac{325}{404} & -\frac{9}{1616} \\
0 & \frac{366}{101} & 0 & -\frac{738}{101} & \frac{192}{101} & 0 & \frac{2187}{101} & -\frac{3584}{101} & \frac{1485}{101} & \frac{13}{101} \\
\hline
0 & \frac{366}{101} & 0 & -\frac{738}{101} & \frac{192}{101} & 0 & \frac{2187}{101} & -\frac{3584}{101} & \frac{1485}{101} & \frac{13}{101} \\
0 & -\frac{113}{633} & 0 & \frac{530}{1477} & -\frac{256}{4431} & \frac{101}{13293} & 0 & \frac{68864}{39879} & -\frac{152}{211} & -\frac{257}{39879} \\
0 & -\frac{159}{112} & -\frac{303}{56} & -\frac{387}{244} & \frac{3}{56} & 0 & \frac{4023}{448} & 0 & -\frac{891}{112} & -\frac{11}{448} \\
-\frac{101}{5040} & -\frac{388}{1008} & 0 & \frac{231}{1008} & -\frac{4}{315} & 0 & -\frac{97}{112} & \frac{5888}{3024} & 0 & -\frac{35}{432}
\end{array}\right]$$

Exact Solution $I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Table 1. *Maximum Errors for Example 1.*

X	Skwame et al., 2012 4th Order Method
0.1	3.12 $A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ $B_1 = \begin{bmatrix} \frac{673}{360} & -\frac{104}{45} & \frac{211}{120} & -\frac{32}{45} & \frac{43}{360} \\ \frac{1323}{640} & -\frac{77}{40} & \frac{1053}{640} & -\frac{27}{40} & \frac{73}{640} \\ \frac{92}{45} & -\frac{224}{135} & \frac{29}{15} & -\frac{32}{45} & \frac{16}{135} \\ \frac{2375}{1152} & -\frac{125}{72} & \frac{875}{384} & -\frac{35}{72} & \frac{125}{1152} \\ \frac{81}{40} & -\frac{8}{5} & \frac{81}{40} & 0 & \frac{11}{40} \end{bmatrix}$
0.2	2.49 $I_j = a + jh,\ j = 0, ..., N-1,$
0.3	7.81 $f(x_{n+j}, y(x_{n+j}))\ e^{-\lambda t}$
0.4	6.24 $y^{[n-1]}\ y^{[n]}$
0.5	1.95 $y^{[n]} = h(B \oplus I_m) f(Y^{[n]}) + (V \oplus I_m) y^{[n-1]}\ \oplus$
0.6	1.56 $j = 0(1)\frac{3}{2}, 2\ \beta_q(x)$
0.7	4.87 $\phi_{\frac{3}{2}}(x) = \frac{29440}{303}\frac{\xi}{h} - \frac{99328}{303}\frac{\xi^2}{h^2} + \frac{23872}{303}\frac{\xi^5}{h^5} - \frac{2752}{303}\frac{\xi^6}{h^6} + \frac{128704}{303}\frac{\xi^3}{h^3} - \frac{79936}{303}\frac{\xi^4}{h^4}$ $\psi_1(x) = \frac{11947}{303}\frac{\xi^4}{h^3} - \frac{6953}{101}\frac{\xi^3}{h^2} + \frac{17704}{303}\frac{\xi^2}{h} - \frac{1940}{101}\xi + \frac{364}{303}\frac{\xi^6}{h^5} - \frac{1112}{101}\frac{\xi^5}{h^4}$
0.8	3.90 $B_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{11}{40} \\ 0 & 0 & 0 & 0 & \frac{35}{128} \\ 0 & 0 & 0 & 0 & \frac{37}{135} \\ 0 & 0 & 0 & 0 & \frac{35}{128} \\ 0 & 0 & 0 & 0 & \frac{11}{40} \end{bmatrix}$ $Y^{[n]} = hAf(Y^{[n]}) + Uy^{[n-1]}$
0.9	1.22 $f: [x_0, x_N] \times R^m \to R^m\ I_j = a + jh,$
1.0	9.76 $y(x_{n+j})\ f(x_{n+j}, y(x_{n+j}))$
1.1	2.05 $Y^{[n]}, f(Y^{[n]}),\ y^{[n-1]}$
1.2	2.44 $Y^{[n]} = h(A \oplus I_m) f(Y^{[n]}) + (U \oplus I_m) y^{[n-1]}\ y^{[n]} = h(B \oplus I_m) f(Y^{[n]}) + (V \oplus I_m) y^{[n-1]}$
1.3	7.62 $\alpha_j(x)\ \beta_j(x)$
1.4	6.10 $\phi_1(x) = -\frac{22016}{101}\frac{\xi^3}{h^3} + \frac{17268}{101}\frac{\xi^2}{h^2} - \frac{3876}{101}\frac{\xi^5}{h^5} + \frac{13329}{101}\frac{\xi^4}{h^4} - \frac{5040}{101}\frac{\xi}{h} + \frac{436}{101}\frac{\xi^6}{h^6}$

X	***Skwame et al., 2012*** 4^{th} ***Order Method***
	$\phi_2(x) = -\frac{4017}{101}\frac{\xi^5}{h^5} + \frac{60687}{404}\frac{\xi^2}{h^2} - \frac{40631}{202}\frac{\xi^3}{h^3} + \frac{475}{101}\frac{\xi^6}{h^6} - \frac{4365}{101}\frac{\xi}{h} + \frac{52203}{404}\frac{\xi^4}{h^4}$
1.5	1.90 $A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ $B_1 = \begin{bmatrix} \frac{673}{360} & -\frac{104}{45} & \frac{211}{120} & -\frac{32}{45} & \frac{43}{360} \\ \frac{1323}{640} & -\frac{77}{40} & \frac{1053}{640} & -\frac{27}{40} & \frac{73}{640} \\ \frac{92}{45} & -\frac{224}{135} & \frac{29}{15} & -\frac{32}{45} & \frac{16}{135} \\ \frac{2375}{1152} & -\frac{125}{72} & \frac{875}{384} & -\frac{35}{72} & \frac{125}{1152} \\ \frac{81}{40} & -\frac{8}{5} & \frac{81}{40} & 0 & \frac{11}{40} \end{bmatrix}$
1.6	1.52 $f:[x_0, x_N]\times R^m \to R^m \; I_j = a + jh,$
1.7	4.76 $y(x_{n+j}) \;\; f(x_{n+j}, y(x_{n+j}))$
1.8	3.81 $Y^{[n]}, f\left(Y^{[n]}\right), \; y^{[n-1]}$
1.9	1.19 $Y^{[n]} = h\left(A \oplus I_m\right) f\left(Y^{[n]}\right) + \left(U \oplus I_m\right) y^{[n-1]} \;\; y^{[n]} = h\left(B \oplus I_m\right) f\left(Y^{[n]}\right) + \left(V \oplus I_m\right) y^{[n-1]}$
2.0	9.52 $y(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + \alpha_q(x) y_{n+q} + h\left[\sum_{j=k-2}^{2} \beta_j(x) f_{n+j} + \beta_q(x) f_{n+q}\right] \; \alpha_j(x)$

X	**Skwame et al., 2012 6th Order Method**
0.1	5.00 $B_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{11}{40} \\ 0 & 0 & 0 & 0 & \frac{35}{128} \\ 0 & 0 & 0 & 0 & \frac{37}{135} \\ 0 & 0 & 0 & 0 & \frac{35}{128} \\ 0 & 0 & 0 & 0 & \frac{11}{40} \end{bmatrix}$ $y'(x) = f(x, y),$
0.2	1.67 $h = \frac{b-a}{N-1} \;\; k - step$
0.3	5.00 $\lambda \;\; Y_1^{[n]}, Y_2^{[n]}, \ldots, Y_s^{[n]}$
0.4	3.00 $Y^{[n]} = \begin{bmatrix} Y_1^{[n]} \\ Y_2^{[n]} \\ . \\ . \\ . \\ Y_s^{[n]} \end{bmatrix} \; f\left(Y^{[n]}\right) = \begin{bmatrix} f\left(Y_1^{[n]}\right) \\ f\left(Y_2^{[n]}\right) \\ . \\ . \\ . \\ f\left(Y_s^{[n]}\right) \end{bmatrix}$
0.5	1.50 $Y^{[n]} = hAf\left(Y^{[n]}\right) + Uy^{[n-1]} \;\; y^{[n]} = hBf\left(Y^{[n]}\right) + Vy^{[n-1]}$
0.6	5.00 $\alpha_q(x) \;\; q = \frac{a}{b}$

X	Skwame et al., 2012 6th Order Method
0.7	1.50 $\psi_2(x) = \frac{7359}{202}\frac{\xi^4}{h^3} - \frac{8167}{202}\frac{\xi^2}{h} + \frac{5584}{101}\frac{\xi^3}{h^2} + \frac{1155}{101}\xi - \frac{142}{101}\frac{\xi^6}{h^5} + \frac{1166}{101}\frac{\xi^5}{h^4}$ $\psi_{\frac{3}{2}}(x) = \frac{704}{303}\frac{\xi^2}{h} - \frac{64}{101}\xi + \frac{704}{303}\frac{\xi^4}{h^3} + \frac{32}{303}\frac{\xi^6}{h^5} - \frac{336}{101}\frac{\xi^3}{h^2} - \frac{80}{101}\frac{\xi^5}{h^4}$
0.8	9.00 $Y^{[n]} = hAf\left(Y^{[n]}\right) + Uy^{[n-1]}$ $M(z) = V + zB(I - zA)^{-1}U$
0.9	4.50 $j = 0, ..., N-1,\ h = \frac{b-a}{N-1}$
1.0	1.50 $e^{-\lambda t}$ λ
1.1	4.50 $y^{[n]}$ $Y^{[n]} = \begin{bmatrix} Y_1^{[n]} \\ Y_2^{[n]} \\ . \\ . \\ . \\ Y_s^{[n]} \end{bmatrix}$
1.2	2.70 $\oplus$ $Y^{[n]} = hAf\left(Y^{[n]}\right) + Uy^{[n-1]}$
1.3	1.35 $j = 0(1)\frac{3}{2}, 2$ $\beta_q(x)$
1.4	4.50 $\phi_{\frac{3}{2}}(x) = \frac{29440}{303}\frac{\xi}{h} - \frac{99328}{303}\frac{\xi^2}{h^2} + \frac{23872}{303}\frac{\xi^5}{h^5} - \frac{2752}{303}\frac{\xi^6}{h^6} + \frac{128704}{303}\frac{\xi^3}{h^3} - \frac{79936}{303}\frac{\xi^4}{h^4}$ $\psi_1(x) = \frac{11947}{303}\frac{\xi^4}{h^3} - \frac{6953}{101}\frac{\xi^3}{h^2} + \frac{17704}{303}\frac{\xi^2}{h} - \frac{1940}{101}\xi + \frac{364}{303}\frac{\xi^6}{h^5} - \frac{1112}{101}\frac{\xi^5}{h^4}$
1.5	1.35 $B_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{11}{40} \\ 0 & 0 & 0 & 0 & \frac{35}{128} \\ 0 & 0 & 0 & 0 & \frac{37}{135} \\ 0 & 0 & 0 & 0 & \frac{35}{128} \\ 0 & 0 & 0 & 0 & \frac{11}{40} \end{bmatrix}$ $\Phi(w,z) = \det(wI - M(z))$
1.6	8.10 $j = 0, ..., N-1,\ h = \frac{b-a}{N-1}$
1.7	4.05 $e^{-\lambda t}$ λ
1.8	1.35 $y^{[n]}$ $Y^{[n]} = \begin{bmatrix} Y_1^{[n]} \\ Y_2^{[n]} \\ . \\ . \\ . \\ Y_s^{[n]} \end{bmatrix}$
1.9	4.05 $\oplus$ $Y^{[n]} = hAf\left(Y^{[n]}\right) + Uy^{[n-1]}$
2.0	2.43 $\beta_j(x)$ $j = 0(1)\frac{3}{2}, 2$

X	FETR
0.1	1.5576 $y(a) = y_0$ $I = [x_0, x_N]$
0.2	1.7321 $\sum_{j=0}^{k}\alpha_j y_{n+j} = h\left[\sum_{j=0}^{k}\beta_j(x) f_{n+j}\right]$ y_{n+j}

X	FETR
0.3	1.1797 $f\left(Y_1^{[n]}\right), f\left(Y_2^{[n]}\right), \ldots, f\left(Y_s^{[n]}\right)\ Y_1^{[n-1]}, Y_2^{[n-1]}, \ldots, Y_r^{[n-1]}$
0.4	3.8528 $y^{[n-1]} = \begin{bmatrix} y_1^{[n-1]} \\ y_2^{[n-1]} \\ . \\ . \\ . \\ y_r^{[n-1]} \end{bmatrix} \quad y^{[n]} = \begin{bmatrix} y_1^{[n]} \\ y_2^{[n]} \\ . \\ . \\ . \\ y_r^{[n]} \end{bmatrix}$
0.5	4.1553 $M = \begin{bmatrix} A & U \\ B & V \end{bmatrix}\ y(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + \alpha_q(x) y_{n+q} + h\left[\sum_{j=k-2}^{2} \beta_j(x) f_{n+j} + \beta_q(x) f_{n+q}\right]$
0.6	2.1700 $\frac{2r+1}{2}, r = 1, 2\ \ \phi_0(x) = 1 - \frac{1225}{303}\frac{\xi}{h} + \frac{8035}{1212}\frac{\xi^2}{h^2} - \frac{3419}{606}\frac{\xi^3}{h^3} + \frac{3187}{1212}\frac{\xi^4}{h^4} - \frac{193}{303}\frac{\xi^5}{h^5} + \frac{19}{303}\frac{\xi^6}{h^6}$
0.7	1.8980 $\left[\begin{array}{cccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{101}{5040} & -\frac{388}{1008} & 0 & \frac{231}{1008} & -\frac{4}{315} & 0 & -\frac{97}{112} & \frac{5888}{3024} & 0 & -\frac{35}{432} \\ 0 & -\frac{159}{112} & -\frac{303}{56} & -\frac{387}{244} & \frac{3}{56} & 0 & \frac{4023}{448} & 0 & -\frac{891}{112} & -\frac{11}{448} \\ 0 & -\frac{113}{633} & 0 & \frac{530}{1477} & -\frac{256}{4431} & \frac{101}{13293} & 0 & \frac{68864}{39879} & -\frac{152}{211} & -\frac{257}{39879} \\ 0 & -\frac{75}{404} & 0 & \frac{675}{808} & \frac{15}{101} & 0 & -\frac{675}{1616} & \frac{225}{101} & -\frac{325}{404} & -\frac{9}{1616} \\ 0 & \frac{366}{101} & 0 & -\frac{738}{101} & \frac{192}{101} & 0 & \frac{2187}{101} & -\frac{3584}{101} & \frac{1485}{101} & \frac{13}{101} \\ \hline 0 & \frac{366}{101} & 0 & -\frac{738}{101} & \frac{192}{101} & 0 & \frac{2187}{101} & -\frac{3584}{101} & \frac{1485}{101} & \frac{13}{101} \\ 0 & -\frac{113}{633} & 0 & \frac{530}{1477} & -\frac{256}{4431} & \frac{101}{13293} & 0 & \frac{68864}{39879} & -\frac{152}{211} & -\frac{257}{39879} \\ 0 & -\frac{159}{112} & -\frac{303}{56} & -\frac{387}{244} & \frac{3}{56} & 0 & \frac{4023}{448} & 0 & -\frac{891}{112} & -\frac{11}{448} \\ -\frac{101}{5040} & -\frac{388}{1008} & 0 & \frac{231}{1008} & -\frac{4}{315} & 0 & -\frac{97}{112} & \frac{5888}{3024} & 0 & -\frac{35}{432} \end{array}\right]\ I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$
0.8	5.1155 $y'(x) = f(x, y),\ \ y(a) = y_0$
0.9	4.4037 $k-step\ \sum_{j=0}^{k} \alpha_j y_{n+j} = h\left[\sum_{j=0}^{k} \beta_j(x) f_{n+j}\right]$
1.0	2.1834 $Y_1^{[n]}, Y_2^{[n]}, \ldots, Y_s^{[n]}\ \ f\left(Y_1^{[n]}\right), f\left(Y_2^{[n]}\right), \ldots, f\left(Y_s^{[n]}\right)$
1.1	4.6956 $f\left(Y^{[n]}\right) = \begin{bmatrix} f\left(Y_1^{[n]}\right) \\ f\left(Y_2^{[n]}\right) \\ . \\ . \\ . \\ f\left(Y_s^{[n]}\right) \end{bmatrix} \quad y^{[n-1]} = \begin{bmatrix} y_1^{[n-1]} \\ y_2^{[n-1]} \\ . \\ . \\ . \\ y_r^{[n-1]} \end{bmatrix}$

X	FETR
1.2	1.5467 $y^{[n]} = hBf\left(Y^{[n]}\right) + Vy^{[n-1]}$ $M = \begin{bmatrix} A & U \\ B & V \end{bmatrix}$
1.3	6.8444 $\alpha_q(x)$ $q = \frac{a}{b}$
1.4	1.6501 $\psi_2(x) = \frac{7359}{202}\frac{\xi^4}{h^3} - \frac{8167}{202}\frac{\xi^2}{h} + \frac{5584}{101}\frac{\xi^3}{h^2} + \frac{1155}{101}\xi - \frac{142}{101}\frac{\xi^6}{h^5} + \frac{1166}{101}\frac{\xi^5}{h^4}$ $\psi_{\frac{3}{2}}(x) = \frac{704}{303}\frac{\xi^2}{h} - \frac{64}{101}\xi + \frac{704}{303}\frac{\xi^4}{h^3} + \frac{32}{303}\frac{\xi^6}{h^5} - \frac{336}{101}\frac{\xi^3}{h^2} - \frac{80}{101}\frac{\xi^5}{h^4}$
1.5	1.7794 $y'(x) = f(x, y),\ y(a) = y_0$
1.6	2.0002 $k-step$ $\sum_{j=0}^{k}\alpha_j y_{n+j} = h\left[\sum_{j=0}^{k}\beta_j(x) f_{n+j}\right]$
1.7	1.4414 $Y_1^{[n]}, Y_2^{[n]}, \ldots, Y_s^{[n]}$ $f\left(Y_1^{[n]}\right), f\left(Y_2^{[n]}\right), \ldots, f\left(Y_s^{[n]}\right)$
1.8	8.5234 $f\left(Y^{[n]}\right) = \begin{bmatrix} f\left(Y_1^{[n]}\right) \\ f\left(Y_2^{[n]}\right) \\ \cdot \\ \cdot \\ \cdot \\ f\left(Y_s^{[n]}\right) \end{bmatrix}$ $y^{[n-1]} = \begin{bmatrix} y_1^{[n-1]} \\ y_2^{[n-1]} \\ \cdot \\ \cdot \\ \cdot \\ y_r^{[n-1]} \end{bmatrix}$
1.9	2.6913 $y^{[n]} = hBf\left(Y^{[n]}\right) + Vy^{[n-1]}$ $M = \begin{bmatrix} A & U \\ B & V \end{bmatrix}$
2.0	5.8666 $\beta_q(x)$ $\alpha_q(x)$

X	SETR
0.1	8.9860 $y : [x_0, x_N] \to R^m$ $f : [x_0, x_N] \times R^m \to R^m$
0.2	3.5800 f_{n+j} $y(x_{n+j})$
0.3	2.2793 $Y_1^{[n]}, Y_2^{[n]}, \ldots, Y_r^{[n]}$ $Y^{[n]}, f\left(Y^{[n]}\right),$
0.4	2.3103 $Y_1^{[n]}, Y_2^{[n]}, \ldots, Y_s^{[n]}$ $Y^{[n]} = h\left(A \oplus I_m\right) f\left(Y^{[n]}\right) + \left(U \oplus I_m\right) y^{[n-1]}$
0.5	9.6111 $\alpha_j(x)$ $\beta_j(x)$
0.6	3.5877 $\phi_1(x) = -\frac{22016}{101}\frac{\xi^3}{h^3} + \frac{17268}{101}\frac{\xi^2}{h^2} - \frac{3876}{101}\frac{\xi^5}{h^5} + \frac{13329}{101}\frac{\xi^4}{h^4} - \frac{5040}{101}\frac{\xi}{h} + \frac{436}{101}\frac{\xi^6}{h^6}$ $\phi_2(x) = -\frac{4017}{101}\frac{\xi^5}{h^5} + \frac{60687}{404}\frac{\xi^2}{h^2} - \frac{40631}{202}\frac{\xi^3}{h^3} + \frac{475}{101}\frac{\xi^6}{h^6} - \frac{4365}{101}\frac{\xi}{h} + \frac{52203}{404}\frac{\xi^4}{h^4}$
0.7	1.3030 $A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ $B_1 = \begin{bmatrix} \frac{673}{360} & -\frac{104}{45} & \frac{211}{120} & -\frac{32}{45} & \frac{43}{360} \\ \frac{1323}{640} & -\frac{77}{40} & \frac{1053}{640} & -\frac{27}{40} & \frac{73}{640} \\ \frac{92}{45} & -\frac{224}{135} & \frac{29}{15} & -\frac{32}{45} & \frac{16}{135} \\ \frac{2375}{1152} & -\frac{125}{72} & \frac{875}{384} & -\frac{35}{72} & \frac{125}{1152} \\ \frac{81}{40} & -\frac{8}{5} & \frac{81}{40} & 0 & \frac{11}{40} \end{bmatrix}$
0.8	4.9168 $I = [x_0, x_N]$ $y : [x_0, x_N] \to R^m$
0.9	1.6269 y_{n+j} f_{n+j}
1.0	1.2028 $Y_1^{[n-1]}, Y_2^{[n-1]}, \ldots, Y_r^{[n-1]}$ $Y_1^{[n]}, Y_2^{[n]}, \ldots, Y_r^{[n]}$

<table>
<tr><th>X</th><th>SETR</th></tr>
<tr><td>1.1</td><td>4.4549 $y^{[n]} = \begin{bmatrix} y_1^{[n]} \\ y_2^{[n]} \\ . \\ . \\ . \\ y_r^{[n]} \end{bmatrix}$ $Y_1^{[n]}, Y_2^{[n]}, \ldots, Y_s^{[n]}$</td></tr>
<tr><td>1.2</td><td>1.6290 $M(z) = V + zB(I - zA)^{-1} U$ $y(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + \alpha_q(x) y_{n+q} + h\left[\sum_{j=k-2}^{2} \beta_j(x) f_{n+j} + \beta_q(x) f_{n+q}\right]$</td></tr>
<tr><td>1.3</td><td>6.0645 $\frac{2r+1}{2}, r = 1,2$ $\phi_0(x) = 1 - \frac{1225}{303}\frac{\xi}{h} + \frac{8035}{1212}\frac{\xi^2}{h^2} - \frac{3419}{606}\frac{\xi^3}{h^3} + \frac{3187}{1212}\frac{\xi^4}{h^4} - \frac{193}{303}\frac{\xi^5}{h^5} + \frac{19}{303}\frac{\xi^6}{h^6}$</td></tr>
<tr><td>1.4</td><td>2.1259 $\left[\begin{array}{cccccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{101}{5040} & -\frac{388}{1008} & 0 & \frac{231}{1008} & -\frac{4}{315} & 0 & -\frac{97}{112} & \frac{5888}{3024} & 0 & -\frac{35}{432} \\ 0 & -\frac{159}{112} & -\frac{303}{56} & -\frac{387}{244} & \frac{3}{56} & 0 & \frac{4023}{448} & 0 & -\frac{891}{112} & -\frac{11}{448} \\ 0 & -\frac{113}{633} & 0 & \frac{530}{1477} & -\frac{256}{4431} & \frac{101}{13293} & 0 & \frac{68864}{39879} & -\frac{152}{211} & -\frac{257}{39879} \\ 0 & -\frac{75}{404} & 0 & \frac{675}{808} & \frac{15}{101} & 0 & -\frac{675}{1616} & \frac{225}{101} & -\frac{325}{404} & -\frac{9}{1616} \\ 0 & \frac{366}{101} & 0 & -\frac{738}{101} & \frac{192}{101} & 0 & \frac{2187}{101} & -\frac{3584}{101} & \frac{1485}{101} & \frac{13}{101} \\ \hline 0 & \frac{366}{101} & 0 & -\frac{738}{101} & \frac{192}{101} & 0 & \frac{2187}{101} & -\frac{3584}{101} & \frac{1485}{101} & \frac{13}{101} \\ 0 & -\frac{113}{633} & 0 & \frac{530}{1477} & -\frac{256}{4431} & \frac{101}{13293} & 0 & \frac{68864}{39879} & -\frac{152}{211} & -\frac{257}{39879} \\ 0 & -\frac{159}{112} & -\frac{303}{56} & -\frac{387}{244} & \frac{3}{56} & 0 & \frac{4023}{448} & 0 & -\frac{891}{112} & -\frac{11}{448} \\ -\frac{101}{5040} & -\frac{388}{1008} & 0 & \frac{231}{1008} & -\frac{4}{315} & 0 & -\frac{97}{112} & \frac{5888}{3024} & 0 & -\frac{35}{432} \end{array}\right]$ $I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$</td></tr>
<tr><td>1.5</td><td>1.1312 $I = [x_0, x_N]$ $y : [x_0, x_N] \rightarrow R^m$</td></tr>
<tr><td>1.6</td><td>4.1787 y_{n+j} f_{n+j}</td></tr>
<tr><td>1.7</td><td>1.5316 $Y_1^{[n-1]}, Y_2^{[n-1]}, \ldots, Y_r^{[n-1]}$ $Y_1^{[n]}, Y_2^{[n]}, \ldots, Y_r^{[n]}$</td></tr>
<tr><td>1.8</td><td>5.6758 $y^{[n]} = \begin{bmatrix} y_1^{[n]} \\ y_2^{[n]} \\ . \\ . \\ . \\ y_r^{[n]} \end{bmatrix}$ $Y_1^{[n]}, Y_2^{[n]}, \ldots, Y_s^{[n]}$</td></tr>
<tr><td>1.9</td><td>2.0273 $M(z) = V + zB(I - zA)^{-1} U$ $\Phi(w, z) = \det(wI - M(z)).$</td></tr>
<tr><td>2.0</td><td>9.4748 $q = \frac{a}{b}$ $\frac{2r+1}{2}, r = 1,2$</td></tr>
</table>

Problem 2

$$\phi_0(x)=1-\frac{1225}{303}\frac{\xi}{h}+\frac{8035}{1212}\frac{\xi^2}{h^2}-\frac{3419}{606}\frac{\xi^3}{h^3}+\frac{3187}{1212}\frac{\xi^4}{h^4}-\frac{193}{303}\frac{\xi^5}{h^5}+\frac{19}{303}\frac{\xi^6}{h^6},$$

$$\phi_1(x)=-\frac{22016}{101}\frac{\xi^3}{h^3}+\frac{17268}{101}\frac{\xi^2}{h^2}-\frac{3876}{101}\frac{\xi^5}{h^5}+\frac{13329}{101}\frac{\xi^4}{h^4}-\frac{5040}{101}\frac{\xi}{h}+\frac{436}{101}\frac{\xi^6}{h^6},$$

$$\phi_2(x)=-\frac{4017}{101}\frac{\xi^5}{h^5}+\frac{60687}{404}\frac{\xi^2}{h^2}-\frac{40631}{202}\frac{\xi^3}{h^3}+\frac{475}{101}\frac{\xi^6}{h^6}-\frac{4365}{101}\frac{\xi}{h}+\frac{52203}{404}\frac{\xi^4}{h^4},$$

$$\phi_{\frac{3}{2}}(x)=\frac{29440}{303}\frac{\xi}{h}-\frac{99328}{303}\frac{\xi^2}{h^2}+\frac{23872}{303}\frac{\xi^5}{h^5}-\frac{2752}{303}\frac{\xi^6}{h^6}+\frac{128704}{303}\frac{\xi^3}{h^3}-\frac{79936}{303}\frac{\xi^4}{h^4}$$

Exact Solution $\psi_1(x)=\frac{11947}{303}\frac{\xi^4}{h^3}-\frac{6953}{101}\frac{\xi^3}{h^2}+\frac{17704}{303}\frac{\xi^2}{h}-\frac{1940}{101}\xi+\frac{364}{303}\frac{\xi^6}{h^5}-\frac{1112}{101}\frac{\xi^5}{h^4}$

Table 2. *Maximum Errors for Example 1.*

X	*Exact Solution*	*Khadijah 2015 4th Order Method*	*FETR*	*Absolute Error*
0.2	0.019801326693245	0.818738	0.0198014832636218	1.5657 $\psi_2(x)=\frac{7359}{202}\frac{\xi^4}{h^3}-\frac{8167}{202}\frac{\xi^2}{h}+\frac{5584}{101}\frac{\xi^3}{h^2}+\frac{1155}{101}\xi-\frac{142}{101}\frac{\xi^6}{h^5}+\frac{1166}{101}\frac{\xi^5}{h^4}$ $\psi_{\frac{3}{2}}(x)=\frac{704}{303}\frac{\xi^2}{h}-\frac{64}{101}\xi+\frac{704}{303}\frac{\xi^4}{h^3}+\frac{32}{303}\frac{\xi^6}{h^5}-\frac{336}{101}\frac{\xi^3}{h^2}-\frac{80}{101}\frac{\xi^5}{h^4}$ 2.0807 $\left[\begin{array}{cccccc\|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{101}{5040} & -\frac{388}{1008} & 0 & \frac{231}{1008} & -\frac{4}{315} & 0 & -\frac{97}{112} & \frac{5888}{3024} & 0 & -\frac{35}{432} \\ 0 & -\frac{159}{112} & -\frac{303}{56} & -\frac{387}{244} & \frac{3}{56} & 0 & \frac{4023}{448} & 0 & -\frac{891}{112} & -\frac{11}{448} \\ 0 & -\frac{113}{633} & 0 & \frac{530}{1477} & -\frac{256}{4431} & \frac{101}{13293} & 0 & \frac{68864}{39879} & -\frac{152}{211} & -\frac{257}{39879} \\ 0 & -\frac{75}{404} & 0 & \frac{675}{808} & \frac{15}{101} & 0 & -\frac{675}{1616} & \frac{225}{101} & -\frac{325}{404} & -\frac{9}{1616} \\ 0 & \frac{366}{101} & 0 & -\frac{738}{101} & \frac{192}{101} & 0 & \frac{2187}{101} & -\frac{3584}{101} & \frac{1485}{101} & \frac{13}{101} \\ \hline \end{array}\right.$
0.4	0.076883653613364	0.670321	0.0768857343339293	$\left.\begin{array}{cccccc\|cccc} 0 & \frac{366}{101} & 0 & -\frac{738}{101} & \frac{192}{101} & 0 & \frac{2187}{101} & -\frac{3584}{101} & \frac{1485}{101} & \frac{13}{101} \\ 0 & -\frac{113}{633} & 0 & \frac{530}{1477} & -\frac{256}{4431} & \frac{101}{13293} & 0 & \frac{68864}{39879} & -\frac{152}{211} & -\frac{257}{39879} \\ 0 & -\frac{159}{112} & -\frac{303}{56} & -\frac{387}{244} & \frac{3}{56} & 0 & \frac{4023}{448} & 0 & -\frac{891}{112} & -\frac{11}{448} \\ -\frac{101}{5040} & -\frac{388}{1008} & 0 & \frac{231}{1008} & -\frac{4}{315} & 0 & -\frac{97}{112} & \frac{5888}{3024} & 0 & -\frac{35}{432} \end{array}\right]$ $I=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

X	*Exact Solution*	*Khadijah 2015 4th Order Method*	*FETR*	*Absolute Error*
0.6	0.164729788588728	0.545813	0.164732324486997	1.2061 $A=\begin{bmatrix}0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\end{bmatrix}$ $B_1=\begin{bmatrix}\frac{673}{360}&-\frac{104}{45}&\frac{211}{120}&-\frac{32}{45}&\frac{43}{360}\\\frac{1323}{640}&-\frac{77}{40}&\frac{1053}{640}&-\frac{27}{40}&\frac{73}{640}\\\frac{92}{45}&-\frac{224}{135}&\frac{29}{15}&-\frac{32}{45}&\frac{16}{135}\\\frac{2375}{1152}&-\frac{125}{72}&\frac{875}{384}&-\frac{35}{72}&\frac{125}{1152}\\\frac{81}{40}&-\frac{8}{5}&\frac{81}{40}&0&\frac{11}{40}\end{bmatrix}$
0.8	0.273850962926309	0.449325	0.273853652953028	2.6900 $B_0=\begin{bmatrix}0&0&0&0&\frac{11}{40}\\0&0&0&0&\frac{35}{128}\\0&0&0&0&\frac{37}{135}\\0&0&0&0&\frac{35}{128}\\0&0&0&0&\frac{11}{40}\end{bmatrix}$ $\Phi(w,z)=\det(wI-M(z)).$
1.0	0.393469340287367	0.367873	0.393473662912352	4.3226 $\Phi(w,z)=\det(wI-M(z))$ $\Phi(w,z)=\det(wI-M(z))$

5. Conclusion

We derived block Extended Trapezoidal Rule of First (ETRs) that are A-stable of up to order 8. The methods were shown tocompete favorablythan other existing methodsin terms of accuracy (see tables 1 and 2). Our newly derived methods in block form are shown to have extensive regions of stability and in particular are A-stable up to order 8 and so very suitable for stiff system of ordinary differential equations. The continuous formulation for each step number k is evaluated at the end point of the interval to recover the individual main method for *FETR, SETR and EETR* in (13) and their counterparts schemes in (8), (11) and (12) respectively, to this end, the idea of additional conditions is discarded. Furthermore, we do not need any pair in our implementation. Our block methods preserve the A-stability property of the trapezoidal rule (refer to figure 1), they are also less expensive in terms of the number of functions evaluation per step.

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