The Matching Energy of Random Multipartite Graphs

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Abstract: Let G_n be a simple graph with n vertices. Gutman and Wagner founded the theory of random graphs, they introduced the matching energy of the graph G_n, which was defined as the sum of the absolute values of the eigenvalues of the matching polynomial of the graph G_n. For the Erdös-Rényi type random graph $G_{n,p}$ of order n with a fixed probability p, where p is a real number greater than zero and less than 1, that is, the graph G on n vertices by connecting two vertices with probability p(e), and each edge is independent of other one. Chen, Li and Lian solved a conjecture proposed by Gutman and Wagner, that is, the expectation of the matching energy of $G_{n,p}$ converges to a certain number associated with n and p almost surely. But they only did the result for random bipartite graphs. In this paper, we give some lower bounds for the matching energy of random bipartite graphs. And then we will use Chen et al's method to generalize this conclusion to any random multipartite graphs.

Keywords: Random Graphs, Matching Energy, Multipartite Graphs

1. Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. And its vertex number and edge number are denoted by $|V(G)|$ and $|E(G)|$, respectively. A matching of G is a set of independent edges in G, and a k-matching of G is a matching of G that has exactly k edges, denoted by $M(G, k)$. Let $m(G, k)$ be the number of k-matchings, i.e., the number of selections of k independent edges, or the number of k-element independent edge sets of a graph G. Specifically, $m(G, 1) = m$ and $m(G, k) = 0$ for $k > \frac{n}{2}$. It is both consistent and convenient to set $m(G, 0) = 1$. The matching polynomial of a graph G of order n is defined as

$$\alpha(G) = \alpha(G, \lambda) = \sum_{k \geq 0} (-1)^k m(G, k) \lambda^{n-2k}. \quad (1)$$

The matching polynomial has many important implications in statistical physics and chemistry; see [6, 9, 14]. Many results on the properties of the matching roots have been obtained; see [4, 10-13].

On the other hand, the matching energy of a graph G was introduced by Gutman and Wagner [5], which is defined as the sum of the absolute values of the roots of the matching polynomial of G (see Eq. (3)). Moreover, Gutman and Wagner found that the matching energy is a quantity of relevance with chemical applications [5] and they arrived at the relation

$$TRE(G) = E(G) - ME(G)$$

where *TRE(G)* is the topological resonance energy of the graph G, and *E(G)* is the energy of the graph G, i.e., the sum of the absolute values of the roots of the matching polynomial of *G*. For more information about the applications of matching energy, we refer to [6, 9]. For terminology and notation not given here, we refer to [1].

Definition 1 *[5]*. Let G be a simple graph, and let $\mu_1, \mu_2, \cdots, \mu_n$ be the zeros of its matching polynomial. Then the matching energy of G is defined as

$$ME(G) = \sum_{i=1}^{n} |\mu_i|. \quad (2)$$

Theorem 2 *[5]*. Let G be a simple graph. The matching energy of G can be expressed as the following formula

$$ME(G) = \frac{2}{\pi}\int_0^\infty \frac{1}{x^2}\ln\left[\sum_{k\geq 0} m(G,k)x^{2k}\right]dx. \quad (3)$$

Then Eq. (3) could be considered as the definition of matching energy and Eq. (2) would become a theorem.

Defifinition 3 *[6, 7]*. Let G' and G'' be two graphs (not necessarily with the same order). If $m(G', k) \geq m(G'', k)$ holds for all $k = 1, 2, 3, \cdots$, then we write G′ ≥ G″. If, in addition, for at least one value of k, $m(G', k) > m(G'', k)$, then we write $G' > G''$. If both $G' \geq G''$ and $G'' \geq G'$ hold, then we say that G' and G'' are matching-equivalent, and write $G' \sim G''$. If neither $G' \geq G''$ or $G'' \geq G'$ hold, then we say that G' and G'' are matching-incomparable.

Bearing in mind that Eq. (2) holds, we have

Lemma 4 *[8]*. If G' and G'' are two graphs, then

$$G' \geq G'' \Rightarrow ME(G') \geq ME(G''),$$

and

$$G' > G'' \Rightarrow ME(G') > ME(G'').$$

Next the basic properties of the matching energy and the graphs with extremal matching energy will be introduced.

Theorem 3 *[8]*. If the graph G is a forest, then its matching energy coincides with its energy.

Theorem 4 *[8]*. Let G be a graph and e is one of its edges. Let G-e be the subgraph obtained by deleting from G the edge e, but keeping all the vertices of G. Then

$$ME(G-e) < ME(G).$$

Theorem 5 *[8]*. Among all graphs on n vertices, the empty graph without edges and the complete graph K_n have, respectively, minimum and maximum matching energy.

By theorem 4, the connected graph with minimal *ME* must be a tree. By theorem 3, trees have equal *E*- and *ME*-values. The fact that, among *n*-vertex trees, the star has minimal energy was established long time ago [6]. Therefore, we have

Theorem 6 *[8]*. The connected graph on n vertices having minimum matching energy is the star S_n.

The matching energy of the empty graph is clearly 0, and the matching energy of the star, which equals its ordinary energy, is $ME(S_n) = 2\sqrt{(n-1)}$. There does not seem to be a similarly simple expression for the maximum matching energy $ME(K_n)$, but the following could be proven:

Theorem 7 *[5]*. The matching energy of the complete graph

K_n is asymptotically equal to $\frac{8}{3\pi}.n^{\frac{3}{2}}$. More precisely,

$$ME(K_n) = \frac{8}{3\pi}n^{\frac{3}{2}} + O(n).$$

Theorem 8 *[5]*. Let $G_{n,p}$ be a random graph of order n with fixed probability $p \in (0,1)$. Then

$$ME(G_{n,p}) \geq \frac{\sqrt{p}}{\pi}n^{\frac{3}{2}} + O(\sqrt{n}\ln n),$$

holds asymptotically almost surely.

Theorem 9 *[8]*. Let $G' \cup G''$ be the graph consisting of disconnected components G' and G''. Then

$$ME(G' \cup G'') = ME(G') + ME(G'').$$

Proposition 1. *[8]*. Let F' and F'' be two forests. Then

$$F' \geq F'' \Rightarrow E(F') \geq E(F''),$$

and

$$F' > F'' \Rightarrow E(F') > E(F'').$$

We can directly extend Proposition 1 into:

Theorem 10 *[8]*. If G' and G'' be two forests. Then

$$G' \geq G'' \Rightarrow ME(G') \geq ME(G''),$$

and

$$G' > G'' \Rightarrow ME(G') > ME(G'').$$

Theorem 11 *[5]*. The bipartite graph on n vertices having maximum matching energy is $K_{\left\lfloor\frac{n}{2}\right\rfloor,\left\lceil\frac{n}{2}\right\rceil}$.

In the 1980s the present author established a number of relations of the type $G > H$ [15-19].

Lemma 5 (Sliding). Let G be a connected graph with at least two vertices, and let u be one of its vertices. Denote by $P(n,k,G,u)$ the graph obtained by identifying u with the vertex v_k of a simple path $v_1, v_2, ..., v_n$. Write $n = 4p + i, i \in \{1, 2, 3, 4\}$, and $l = \lfloor(i-1)/2\rfloor$. Then the inequalities

$$ME(P(n,2,G,u) < ME(p(n,4,G,u)) < \cdots <$$
$$ME(p(n,2p+2l,G,u)) < ME(P(n,2p+1,G,u))$$

$$< \cdots < ME(P(n,3,G,u)) < ME(P(n,1,G,u))$$

hold.

Lemma 6 (Ironing). Suppose that G is a connected graph and T is an induced subgraph of G such that T is a tree and T is connected to the rest of G only by a cut vertex v. If T is replaced by a star of the same order, centered at v, then the matching energy decreases (unless T is already such a star). If T is replaced by a path, with one end at v, then the matching energy increases (unless T is already such a path).

Based on Lemmas 5 and 6, it was possible to characterize unicyclic, bicyclic, and tricyclic connected graphs with

smallest and greatest matching energy. The consideration of trees will be skipped since, by theorem 3 it reduces to the well known case of extremal energy.

Denote by U_n the set of all connected unicyclic graphs on n vertices. Let C_n be the n-vertex cycle, and let S_n^+ be the graph obtained by adding a new edge to the star S_n. of course, $C_n, S_n^+ \in U_n$.

Theorem 12 *[5]*. If $G \in U_n \setminus \{S_n^+, C_n\}$, then

$$ME(S_n^+) < ME(G) < ME(C_n).$$

Denote by B_n the set of all connected bicyclic graphs on n vertices. Let S_n^* be the graph obtained by adding two new edges to the same (pendent) vertex of the star S_n.

Theorem 13 *[5, 16]*. If $G \in B_n \setminus \{S_n^*, B_n\}$, then for $n \neq 9$,

$$ME(S_n^*) < ME(G) < ME(B_n).$$

If $n = 9$ and $G \in B_9 \setminus \{S_9^*, B_9', B_9''\}$, then

$$ME(S_9^*) < ME(G) < ME(B_9') = ME(B_9'').$$

Denote by TC_n the set of all connected tricyclic graphs on n vertices. Let $S_n^\bullet$ be the graph obtained by adding three new edges to the same (pendent) vertex of the star S_n.

Theorem 14 *[17]*. If $G \in TC_n \setminus \{S_n^\bullet, TC_n\}$, then for $n \neq 9$,

$$ME(S_n^\bullet) < ME(G) < ME(TC_n).$$

If $n = 9$ and $G \in TC_9 \setminus \{S_9^\bullet, TC_9', TC_9''\}$, then

$$ME(S_9^\bullet) < ME(G) < ME(TC_9') = ME(TC_9'').$$

In 1950s, Erdös-Rényi [3] founded the theory of random graphs. The Erdös-Rényi random graph model $G_n(p)$ consists of all graphs on n vertices in which the edges are chosen independently with probability p, where p is a constant with $0 < p < 1$.

Denote by $K_{n;\beta_1,\cdots,\beta_k}$ the complete k-partite graph on n vertices with vertex set V, whose parts are $V_1, V_2, \cdots, V_k$ $(2 \leq k = k(n) \leq n)$ satisfying $|V_i| = n\beta_i = n\beta_i(n), i = 1, 2, \cdots, k$. The random k-partite graph model $G_{n;\beta_1,\cdots,\beta_k}(p)$ consists of all random k-partite graphs in which the edges are chosen independently with probability p from the set of edges of $K_{n;\beta_1,\cdots,\beta_k}$. Note that $\sum_{i=1}^{k} \beta_i = 1$ and $\beta_i \geq 0$ for $i = 1, 2, \cdots, k$. Denote by $A_{n,k} = A(G_{n;\beta_1,\beta_2,\cdots,\beta_k}) = (x_{ij})_{n \times n}$ the adjacency matrix of the random k-partite graph

$G_{n;\beta_1,\cdots,\beta_k} \in G_{n;\beta_1,\cdots,\beta_k}(p)$, where x_{ij} is a random indicator variable for $v_i v_j$ being an edge with probability p, for $i \in V_l$ and $j \in V \setminus V_l$, $i \neq j$, $1 \leq l \leq k$. Then $A_{n,k}$ satisfies the following properties:

(i) x_{ij}'s, $1 \leq i < j \leq n$, are independent random variables with $x_{ij} = x_{ji}$;

(ii) $\Pr(x_{ij} = 1) = 1 - \Pr(x_{ij} = 0) = p$, if $i \in V_l$ and $j \in V \setminus V_l$, while $\Pr(x_{ij} = 0) = 1 = 1$ if $i \in V_l$ and $j \in V_l$, $1 \leq l \leq k$.

Note that when $k = n$, $G_{n;\beta_1,\cdots,\beta_k} = G_n(p)$, that is, the random multipartite graph model can be viewed as a generalization to the Erdös-Rényi model.

Chen, Li and Lian in [2] solved a conjecture proposed by Gutman and Wagner, which is now a result stated as follows.

Theorem 15 *[2]*. For $p \in (0,1)$, the matching energy $ME(G_{n,p})$ of the random graph $G_{n,p}$ enjoys asymptotically almost surely the following equation:

$$ME(G_{n,p}) = n^{\frac{3}{2}} p^{\frac{1}{2}} \left(\frac{8}{3\pi} + o(1) \right). \tag{4}$$

In this paper, we give some lower bounds for the matching energy of random bipartite graphs. And then we will use Chen et al's method to generalize this conclusion to any random multipartite graphs, and obtain some lower bounds for the matching energy of random multipartite graphs.

2. The Matching Energy of Random Bipartite Graphs

In this section, we will give a lower bound for the matching energy of random bipartite graphs.

Theorem 16. Let G_1 and G_2 be two graphs with the same number of vertices (see Figure 1), and let G_1 be connected and bipartite with $V(G_1) = A \cup B$ and G_2 be disconnected with two connected components G_2^1 and G_2^2, i.e., $G_2 = G_2^1 \cup G_2^2$. Let $|A| = |V(G_2^1)|$ and $|B| = |V(G_2^2)|$. If G is a graph such that $V(G) = V(G_1) = V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$, then $ME(G) \leq ME(G_1 \cup G_2)$.

Proof. For any nonnegative integer k, we claim that

$$m(G,k) \leq m(G_1 \cup G_2, k).$$

We distinguish the following cases.

Case 1. All the k independent edges in $M(G, k)$ appear in G_1, that is, $M(G,k) = M(G_1,k)$.

Case 2. All the k independent edges in $M(G, k)$ appear in G_2, that is, $M(G,k) = M(G_2,k)$.

Case 3. Some of the k independent edges in $M(G, k)$ appear in G_1, and the others appear in G_2.

In other words, any set of k-matchings of G is also a one of $G_1 \cup G_2$. Thus, we have $m(G, k) \leq m(G_1 \cup G_2, k)$. From Lemma 4, it follows that $ME(G) \leq ME(G_1 \cup G_2)$.

Theorem 17. Let $G_{n;\beta_1,\beta_2} \in G_{n;\beta_1,\beta_2}(p)$ be a random bipartite graph. Then

$$n^{\frac{3}{2}} p^{\frac{1}{2}} (\frac{8}{3\pi} + o(1))(1 - \beta_1^{\frac{3}{2}} - \beta_2^{\frac{3}{2}}) \leq ME(G_{n;\beta_1,\beta_2}). \quad (5)$$

Proof. Note that $n = n\beta_1 + n\beta_2$. Let $G_n \in G_n(P)$ be a random graph, and set $G_{n;\beta_1,\beta_2} \subseteq G_n$, that is, $G_{n;\beta_1,\beta_2}$ is a spanning bipartite subgraph of G_n. Then by Theorem 16, G_n, $G_{n;\beta_1,\beta_2}$ and $G_{n\beta_1} \cup G_{n\beta_2}$ can be seen as G, G_1 and G_2, respectively. Thus, we have

$$ME(G_n) \leq ME(G_{n;\beta_1,\beta_2} \cup G_{n\beta_1} \cup G_{n\beta_2}),$$

$$= ME(G_{n;\beta_1,\beta_2}) + ME(G_{n\beta_1}) + ME(G_{n\beta_2}).$$

From above, we can get

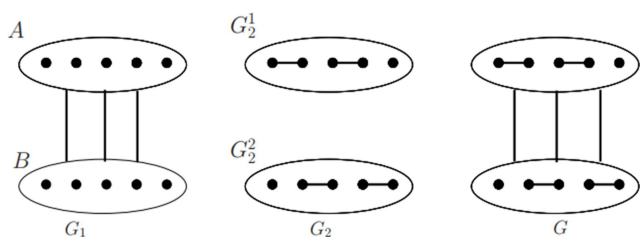$$ME(G_n) - ME(G_{n\beta_1}) - ME(G_{n\beta_2}) \leq ME(G_{n;\beta_1,\beta_2}).$$

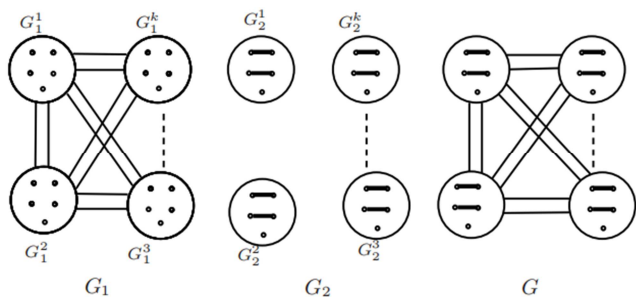

Figure 1. *The separation on random bipartite graph.*

Figure 2. *The separation on random multipartite graph.*

From Theorem 15, it follows that

$$n^{\frac{3}{2}} p^{\frac{1}{2}} (\frac{8}{3\pi} + o(1))(1 - \beta_1^{\frac{3}{2}} - \beta_2^{\frac{3}{2}}) \leq ME(G_{n;\beta_1,\beta_2}).$$

3. The Matching Energy of Random Multipartite Graphs

Next, we will give a lower bound for the matching energy of random multipartite graphs.

Theorem 18. *Let* G_1 *and* G_2 *be two graphs with the same number of vertices (see Figure 2), and let* G_1 *be connected and* k-*partite with* $V(G_1) = V(G_1^1) \cup V(G_1^2) \cup \cdots \cup V(G_1^k)$ *and* G_2 *be disconnected with* k *connected components* $G_2^1, G_2^2, \cdots, G_2^k$, *i.e.,* $G_2 = G_2^1 \cup G_2^2 \cup \cdots \cup G_2^k$. *Let* $|V(G_1^i)| = |V(G_2^i)|$, $i = 1, \cdots k$. *If* G *is a graph such that* $V(G) = V(G_1) = V(G_2)$ *and* $E(G) = E(G_1) \cup E(G_2)$, *then* $ME(G) \leq ME(G_1 \cup G_2)$.

Proof. Similar to the proof of Theorem 16, for any non-negative integer k, we claim that $m(G, k) \leq m(G_1 \cup G_2, k)$. We also need to consider the following cases.

Case 1. All the k independent edges in $M(G, k)$ appear in G_1, that is, $M(G, k) = M(G_1, k)$.

Case 2. All the k independent edges in $M(G, k)$ appear in G_2, that is, $M(G, k) = M(G_2, k)$.

Case 3. Some of the k independent edges in $M(G, k)$ appear in G_1, and the others appear in G_2.

Then we can see that a set of k-matchings of G is also a one of $G_1 \cup G_2$. Thus, we have $m(G, k) \leq m(G_1 \cup G_2, k)$. From Lemma 4, it follows that $ME(G) \leq ME(G_1 \cup G_2)$.

Theorem 19. *Let* $G_{n;\beta_1,\beta_2,\cdots,\beta_k} \in G_{n;\beta_1,\beta_2,\cdots,\beta_k}(P)$ *be a random multipartite graph. Then*

$$n^{\frac{3}{2}} p^{\frac{1}{2}} (\frac{8}{3\pi} + o(1))(1 - \beta_1^{\frac{3}{2}} - \beta_2^{\frac{3}{2}} - \cdots - \beta_k^{\frac{3}{2}}) \leq ME(G_{n;\beta_1,\beta_2,\cdots,\beta_k}). (6)$$

Proof. It is easy to see that $n = n\beta_1 + n\beta_2 + \cdots + n\beta_k$. Let $G_n \in G_n(p)$ be a random graph, and set $G_{n;\beta_1,\beta_2,\cdots,\beta_k} \subseteq G_n$, that is, $G_{n;\beta_1,\beta_2,\cdots,\beta_k}$ is a spanning k-bipartite subgraph of G_n. Then by Theorem 18, G_n, $G_{n;\beta_1,\beta_2,\cdots,\beta_k}$ and $G_{n\beta_1} \cup G_{n\beta_2} \cup \cdots \cup G_{n\beta_k}$ can be seen as G, G_1 and G_2, respectively. Thus, we have

$$ME(G_n) \leq ME(G_{n;\beta_1,\beta_2,\cdots,\beta_k} \cup G_{n\beta_1} \cup G_{n\beta_2} \cup \cdots \cup G_{n\beta_k}),$$

$$= ME(G_{n;\beta_1,\beta_2,\cdots,\beta_k}) + ME(G_{n\beta_1}) + ME(G_{n\beta_2}) + \cdots + ME(G_{n\beta_k}).$$

From above, we can get

$$ME(G_n) - ME(G_{n\beta_1}) - ME(G_{n\beta_2}) - \cdots - ME(G_{n\beta_k})$$
$$\leq ME(G_{n;\beta_1,\beta_2,\cdots,\beta_k})$$

From Theorem 15, it follows that

$$n^{\frac{3}{2}} p^{\frac{1}{2}} (\frac{8}{3\pi} + o(1))(1 - \beta_1^{\frac{3}{2}} - \beta_2^{\frac{3}{2}} - \cdots - \beta_k^{\frac{3}{2}})$$

$$\leq ME(G_{n;\beta_1,\beta_2,\cdots,\beta_k})$$

4. Conclusion

From the above discussion, we obtain the lower bounds for the matching energy of random bipartite graphs, that is, if $G_{n;\beta_1,\beta_2} \in G_{n;\beta_1,\beta_2}(p)$ be a random bipartite graph, then we have

$$n^{\frac{3}{2}} p^{\frac{1}{2}} (\frac{8}{3\pi} + o(1))(1 - \beta_1^{\frac{3}{2}} - \beta_2^{\frac{3}{2}}) \le ME(G_{n;\beta_1,\beta_2}).$$

We use the same method to generalize the conclusion to the random multipartite graphs, and we give the lower bounds for the matching energy of the random multipartite graphs, which is for any *random multipartite graph* $G_{n;\beta_1,\beta_2,\cdots,\beta_k} \in G_{n;\beta_1,\beta_2,\cdots,\beta_k}(P)$. *Then we obtain*

$$n^{\frac{3}{2}} p^{\frac{1}{2}} (\frac{8}{3\pi} + o(1))(1 - \beta_1^{\frac{3}{2}} - \beta_2^{\frac{3}{2}} - \cdots - \beta_k^{\frac{3}{2}}) \le ME(G_{n;\beta_1,\beta_2,\cdots,\beta_k}).$$

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