

Method of principal factors estimation of optimal number of factors: an information criteria approach

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Abstract: The issue of the number of factors to be retained in a factor analysis has been an undefined. Be that as it may, this paper tries to x-ray the number of factors (k) to be retained in a factor analysis for different sample sizes using the method of Principal Factor estimation when the number of variables are ten (10). Stimulated data were used for sample sizes of 30, 50 and 70 and the Akaike Information Criterion (AIC), the Schwarz Information Criterion (SIC) and the Hannan Quinne Information Criterion (HQIC) values were obtained when the number of factors(k) are two, three, and five (2,3 and 5). It was discovered that the AIC, SIC, and HQIC values are smallest when k = 5, and highest when k = 2 for the sample sizes of 30 and 70. But, for a sample of 50, the values of these information criteria is smallest for k = 3, highest for k=5. Hence, conclusion is drawn that for the sample sizes of 30 and 70, the optimal number of factors to retain is 5 and 3 for the sample size of 70. This implies that, the number of factors to retain is a function of the sample size of the data.

Keywords: Factor Analysis, Factor Rotation, Principal Factors Estimation Method, Hannan Quinne Information Criteria, Akaike, Schwarz

1. Introduction

Factor analysis originated from the pioneering work on the subject by Charles Spearman (1904). It was developed extensively by L.L. Thurstone (1931, 1947). It is the most familiar multivariate procedure used in behavioural sciences. Some factors such as arguments over psychological interpretations of several early studies and lack of high speed computing facilities retarded its initial development as a statistical procedure. The purpose of factor analysis is to describe if possible the covariance relationships among variables in terms of a few underlying but unobservable random quantities called factors. Factor analysis can be considered as an extension of principal component analysis. Both are attempts to approximate the covariance matrix Σ , but the approximation based on factor analysis is more elaborate.

Lawley and Maxwell (1963) explained the differences between factor analysis and principal component analysis. Factor analysis is covariance (or correlation) oriented. In principal component analysis, all components are needed to reproduce an intercorrelation (covariance) exactly. In factor analysis, a few factors will reproduce the intercorrelation

(covariances) exactly and model is fitted to data while in principal component analysis, a data is fitted to a model.

Factor analysis attempts to simplify complex and diverse relationships that exist among a set of observed variables by uncovering common dimensions or factors that link together the seemingly unrelated variables and consequently provides insight into the underlying structure of the data. The goal of factor analysis is to reduce a large number of variables to a smaller number of factors, to concisely describe the relationship among observed variables or to test theory about underlying processes(Onyeagu;2003).

1.1. The Mathematical Model for Factor Structure

Suppose that the multivariate system consists of p responses described by the observable random variables X_1, \dots, X_p . The X_i have a non-singular multinormal distribution. Since only the covariance structure will be of interest, we can assume without loss of generality that the population means of X_i are zero(Harman;1976).

Let,

$$Y_1 - \mu_1 = \lambda_{11}X_1 + \dots + \lambda_{1m}X_m + e_1$$

⋮

$$Y_p - \mu_p = \lambda_{p1}X_1 + \dots + \lambda_{pm}X_m + e_p$$

where,

X_j = j-th common-factor variates

λ_{ij} = parameter reflecting importance of the j-th factor in composition of ith response

e_i = ith specific factor variates.

In the language of factor analysis, λ_{ij} is called the loading of the ith response on the jth common factor.

For matrix version of the model;

Let,

$$X^I = (x_1, x_2, \dots, x_m), \quad Y^I = (y_1, y_2, \dots, y_p)$$

$$\varepsilon^I = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_p), \quad \mu^I = (\mu_1, \mu_2, \dots, \mu_p),$$

$$\text{and } \Lambda = \begin{bmatrix} \lambda_{11} & \dots & \lambda_{1m} \\ \vdots & & \vdots \\ \lambda_{p1} & \dots & \lambda_{pm} \end{bmatrix}.$$

Then, the factor model can be written as

$$(y_i - \mu_y) = \Lambda X_i + \varepsilon_i, i = 1, \dots, q.$$

where q is the number of observations. Y_i is an observed vector with p components with mean μ_y and Λ is a $p \times q$ matrix called the factor loadings. X_i is an observed vector with q components, $q < p$, the components of which are called common factors and ε_i is an unobserved error with mean 0_(px1) and is called the specific factor. X_i is assumed to follow a normal distribution with

$E(X_i) = 0_{(qx1)}$, $\text{Var}(X_i) = E(X_i X_i^I) = I_{(qxq)}$, that is the orthogonal case.

Then, Y_i/X_i follows $N(\mu_y + \Lambda X_i, \text{Cov}(\varepsilon_i))$, where $\text{Cov}(\varepsilon_i) = \psi^2 = \text{diag}(\psi_1^2, \dots, \psi_p^2)$.

ψ^2 is called uniqueness or specific variance (Bai and Serena; 2002).

Consequently, a covariance structure for Y is

$$\begin{aligned} \Sigma &= \text{Cov}(Y_i) = E(Y_i - \mu_y)(Y_i - \mu_y)^I \\ &= E(\Lambda X_i + \varepsilon_i)(\Lambda X_i + \varepsilon_i)^I \\ &= E(\Lambda X_i X_i^I \Lambda^I + \varepsilon_i X_i^I \Lambda^I + \Lambda X_i \varepsilon_i^I + \varepsilon_i \varepsilon_i^I) \\ &= E(\Lambda X_i X_i^I \Lambda^I) + E(\varepsilon_i X_i^I \Lambda^I) + E(\Lambda X_i \varepsilon_i^I) + E(\varepsilon_i \varepsilon_i^I) \end{aligned}$$

Recall that, $E(X_i) = 0_{(qx1)}$, $E(X_i X_i^I) = I_{(qxq)}$, $\text{Cov}(\varepsilon_i) = E(\varepsilon_i \varepsilon_i^I) = \psi^2$,

Then,

$$\begin{aligned} &E(\Lambda X_i X_i^I \Lambda^I) + E(\varepsilon_i X_i^I \Lambda^I) + E(\Lambda X_i \varepsilon_i^I) + E(\varepsilon_i \varepsilon_i^I) \\ &= \Lambda \Lambda^I + \psi^2, \text{ and} \end{aligned}$$

$$\text{Cov}(Y_i, X_i) = E(Y_i - \mu_y)X_i^I = E(\Lambda X_i + \varepsilon_i)X_i^I.$$

Introducing communality, $\text{Cov}(Y_i) = \Lambda \Lambda^I + \psi^2$ can be written as

$$\begin{aligned} \text{Var}(Y_{ij}) &= \Lambda_{1j}^2 + \dots + \Lambda_{qj}^2 + \Psi_j^2; \text{Cov}(Y_{ij}, Y_{ik}) = \\ &\Lambda_{1j} \Lambda_{1k} + \dots + \Lambda_{qj} \Lambda_{qk} \text{ and } \text{Cov}(Y_{ik}, X_{jk}) = \Lambda_{jk}. \end{aligned}$$

Communality is the portion of the variance of the variable contributed by the q common factors.

Suppose the jth communality is h_j^2 , then

$$\begin{aligned} \text{Var}(Y_{ij}) &= \sigma_{ij} = \text{communality} + \text{specific variance} = \\ h_j^2 + \psi_j &= (\lambda_{1j}^2 + \lambda_{2j}^2 + \dots + \lambda_{qj}^2) + \psi_j, j = 1, \dots, p. \end{aligned}$$

The j-th communality is the sum of square of the loading of the jth variable on the q common factor. When the number of factors $q > 1$, there are multiple factor loadings that generate the same covariance matrix.

The loading in the model above can be multiplied by an orthogonal matrix without impairing their ability to reproduce the covariance matrix in $\Sigma = \Lambda \Lambda^I + \psi^2$.

Let β be any $q \times q$ orthogonal matrix. If we let $\Lambda^* = \Lambda \beta$ and $X_i^* = \beta^I X_i$, then X_i^* has the same statistical properties as X_i since

$$E(X_i^*) = E(\beta^I X_i) = \beta^I E(X_i) = 0.$$

$$\text{Cov}(X_i^*) = \text{Cov}(\beta^I X_i) = \beta^I \text{Cov}(X_i) \beta = \beta^I \beta = I_{mxm}$$

Λ and Λ^I yield the same covariance because $\Lambda \Lambda^I = \Lambda^* \Lambda^{*I}$. The factor model

$$(Y_i - \mu_y) = \Lambda X_i + \varepsilon_i = \Lambda \beta \beta^I X_i + \varepsilon_i = \Lambda^* X_i^* + \varepsilon_i,$$

produces the same covariance matrix Σ , since $\Sigma = \Lambda \Lambda^I + \psi^2 = \Lambda \beta \beta^I \Lambda + \psi^2 = \Lambda^* \Lambda^{*I} + \psi^2$.

A particular set of loadings needs to be chosen. A good set is one that is easily interpreted. This means sparse solution with many zero. Factor rotation is one approval to finding the solution, as it rotates the coordinates system for Y on X (Johnson and Wichern; 2002).

In this work, we consider four kinds of orthogonal factor rotations namely, varimax, equimax, quartimax and orthomax.

2. The Principal Factors Method

The principal factors method (also called the principal axis method) uses an initial estimate $\hat{\Psi}$ and factors $S - \hat{\Psi}$ or $R - \hat{\Psi}$ to obtain

$$S - \hat{\Psi} \cong \hat{\Lambda} \hat{\Lambda}^I, \quad (2.1)$$

$$R - \hat{\Psi} \cong \hat{\Lambda} \hat{\Lambda}^I. \quad (2.2)$$

where $\hat{\Lambda}$ is $p \times m$ and is calculated using the eigenvalues and eigenvectors of $S - \hat{\Psi}$ or $R - \hat{\Psi}$.

The i-th diagonal element of $S - \hat{\Psi}$ is given by $S_{ii} - \hat{\Psi}_i$, which is the i-th communality $\hat{h}_i^2 = S_{ii} - \hat{\Psi}_i$. Likewise, the diagonal elements of $R - \hat{\Psi}$ are the communalities $\hat{h}_i^2 = 1 - \hat{\Psi}_i$. (Clearly, $\hat{\Psi}_i$ and \hat{h}_i^2 have different values for S than for R). With these diagonal values, $S - \hat{\Psi}$ and $R - \hat{\Psi}$ have the form

$$S - \hat{\Psi} = \begin{bmatrix} \hat{h}_1^2 & S_{12} & \cdots & S_{1p} \\ S_{21} & \hat{h}_2^2 & \cdots & S_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ S_{p1} & S_{p2} & \cdots & \hat{h}_p^2 \end{bmatrix} \quad (2.3)$$

$$R - \hat{\Psi} = \begin{bmatrix} \hat{h}_1^2 & r_{12} & \cdots & r_{1p} \\ r_{21} & \hat{h}_2^2 & \cdots & r_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ r_{p1} & r_{p2} & \cdots & \hat{h}_p^2 \end{bmatrix} \quad (2.4)$$

A popular initial estimate for a communality in $R - \hat{\Psi}$ is $\hat{h}_i^2 = R_i^2$, the squared multiple correlation between y_i and the other $p - 1$ variables. This can be found as

$$\hat{h}_i^2 = R_i^2 = 1 - \frac{1}{r^{ii}}, \quad (2.5)$$

where, r^{ii} is the i -th diagonal element of R^{-1} .

For $S - \hat{\Psi}$, an initial estimate of the communality is

$$\hat{h}_i^2 = S_{ii} - \frac{1}{s^{ii}} \quad (2.6)$$

where S_{ii} is the i -th diagonal element of S and s^{ii} is the i -th diagonal element of S^{-1} .

It can be shown that equation (2.6) is equivalent to

$$\hat{h}_i^2 = S_{ii} - \frac{1}{s^{ii}} = S_{ii} R_i^2 \quad (2.7)$$

which is a reasonable estimate of the amount of variance that y_i has in common with the other y 's to use equation (2.5) or (2.6), R or S must be non-singular. If R is singular, we can use the absolute value or the square of the largest correlation in the i -th row of R as an estimate of communality.

After obtaining communality estimates, we calculate eigenvalues and eigenvectors of $S - \hat{\Psi}$ or $R - \hat{\Psi}$ and use it to obtain estimates of factor loadings, $\hat{\Lambda}$. Then, the columns and rows of $\hat{\Lambda}$ can be used to obtain new eigenvalues (variance explained) and communalities, respectively. The sum of squares of the j -th column of $\hat{\Lambda}$ is the j -th eigenvalues of $S - \hat{\Psi}$ or $R - \hat{\Psi}$, and the sum of squares of the i -th row of $\hat{\Lambda}$ is the communality of y_i . The proportion of variance explained by the j -th factor is

$$\frac{\sum_{i=1}^p \hat{\lambda}_{ij}^2}{tr(S - \hat{\Psi})} = \frac{\theta_j}{\sum_{i=1}^p \theta_i}, \text{ or } \frac{\sum_{i=1}^p \hat{\lambda}_{ij}^2}{tr(R - \hat{\Psi})} = \frac{\theta_j}{\sum_{i=1}^p \theta_i}, \quad (2.8)$$

where θ_j is the j^{th} eigenvalue of $S - \hat{\Psi}$ or $R - \hat{\Psi}$. The matrices $S - \hat{\Psi}$ and $R - \hat{\Psi}$ are not necessarily positive semi-definite and will often have some small negative eigenvalues. In such a case, the cumulative proportion of variance will exceed 1 and then decline to 1 as the negative eigenvalues are added.

2.1. Information Criteria

The necessity of introducing the concept of model

evaluation has been recognized as one of the important technical areas and the problem is posed on the choice of the best approximating model among a class of competing models by a suitable model evaluation criterion given a data set. Model evaluation criteria are figures of merit, or performance measures for competing models. Factor analysis can be characterized as multivariate technique for analyzing the internal relationship among a set of variables. Based on the usual factor analysis model, choosing a model with too few parameters can involve making unrealistically simple assumptions and lead to high bias, poor prediction, and missed opportunities for insight. Such models are not flexible enough to describe the sample or the population well. A model with too many parameters can fit the observed data very well, but be too closely tailored to it; such models may generalize poorly. Penalized-likelihood information criteria, such as Akaike's information criterion (AIC), the Schwarz's information criterion (SIC), the Hannan-Quinn information criterion (HQIC) and so on are widely used for model selection. The comparison of the models using information criterion can be viewed as equivalent to a likelihood ratio test and understanding the differences among the criteria may make it easier to compare the results and to use them to make informed decisions (Akaike; 1973).

AKAIKE'S information criterion is probably the most relevant and famous as for the comparison and selection between different models and is constructed on log likelihood

$$AIC = -2 \log \max L + 2k$$

where L denotes the likelihood function of the factor model and k is the number of the model's parameter/factors. $\log \max L(k) = -\frac{1}{2} N [\log |\Sigma_k| + tr \Sigma_k^{-1} S]$, where S denotes the sample covariance matrix of Y and $\Sigma_k = \Lambda_k \Lambda_k^{-1} + \Psi^2$; Λ_k is the matrix factor of factor loading. The first term can be interpreted as a goodness-of-fit measurement, while the second gives a growing penalty to increasing numbers of parameters according to the parsimony principle. In the choice of model, a minimization rule is used to select the model with the minimum Akaike information criterion value.

Still in the likelihood based procedures, Schwarz (1978) proposed the alternative information criterion given by

$$SIC = -\log \max L + \frac{1}{2} k \log N.$$

Unlike the AIC, SIC considers the number of N of observations and is therefore less favourable to factors inclusion.

Finally, the third criteria is the Hannan-Quinn information criterion (HQIC); it is a criterion for model selection. It is an alternative to Akaike information criterion (AIC) and Bayesian information criterion (BIC). It is given as

$HQIC = -\log \max L + 2k \log \log N$ where k is the number of parameters, N is the number of observations.

3. Comparison of AIC and SIC after Rotation at Different Sample Sizes and Different Retained Number of Factors (k)

3.1. For $n=30$, $p=10$ and $k=2$

3.1.1. Rotated Factor Loadings

Varimax Loadings		Equamax Loadings		Quartimax Loadings		Orthomax Loadings	
I	II	I	II	I	II	I	II
-0.0642	-0.0629	-0.0674	-0.0595	-0.635	-0.0636	-0.0830	0.0344
0.0653	0.5438	0.0938	0.5396	0.0595	0.5445	0.2639	-0.4800
-0.2413	0.2374	-0.2285	0.2498	-0.2438	0.2348	-0.1350	-0.3104
0.4135	0.0049	0.4131	-0.0169	0.4134	0.0093	0.3853	0.1501
0.3448	-0.4505	0.3206	-0.4680	0.3496	-0.4468	0.1513	0.5468
-0.7154	-0.0663	-0.7179	-0.0285	-0.7147	-0.0739	-0.6883	-0.2060
0.2980	0.3957	0.3185	0.3795	0.2938	0.3989	0.4244	-0.2555
-0.0381	-0.3340	-0.0556	-0.3316	-0.0345	-0.3344	-0.1603	0.2956
0.0809	0.1253	0.0874	0.1208	0.0795	0.1261	0.1219	-0.0859
0.5046	-0.1780	0.4945	-0.2043	0.5064	-0.1726	0.4014	0.3537

3.1.2. Factor Rotation Matrix

Varimax.

	Factor I	Factor II
Factor I	0.9798	-0.2002
Factor II	0.2002	0.9798

Equamax.

	Factor I	Factor II
Factor I	0.9679	-0.2515
Factor II	0.2515	0.9679

Quartimax.

	Factor I	Factor II
Factor I	0.9818	-0.1897
Factor II	0.1897	0.9818

Orthomax

	Factor I	Factor II
Factor I	0.8338	0.5520
Factor II	0.5520	0.9679

3.1.3. Information Criteria

Information Criteria	Values
Log Likelihood	-118.6680
Akaike	241.3360
Schwarz	120.1451
Hannan Quinne	119.3457

3.2. For $n=30$, $p=10$ and $k=3$

3.2.1. Rotated Factor Loadings

Varimax Loadings			Equamax Loadings			Quartimax Loadings			Orthomax Loadings		
Factor I	Factor II	Factor III	Factor I	Factor II	Factor III	Factor I	Factor II	Factor III	Factor I	Factor II	Factor III
0.0957	-0.1784	0.2864	0.1452	-0.2688	-0.1723	0.0910	-0.1776	0.2884	0.1452	-0.2688	-0.1723
-0.0268	0.5787	0.0116	-0.0437	-0.0126	0.5776	-0.0256	0.5788	0.0092	-0.0437	-0.0126	0.5776
-0.0066	0.0331	0.5721	0.0820	-0.5659	0.0388	-0.0150	0.0351	0.5718	0.0820	-0.5659	0.0388
0.2283	0.1722	-0.4317	0.1521	0.4631	0.1749	0.2351	0.1702	-0.4289	0.1521	0.4631	0.1749
0.2336	-0.3113	-0.4441	0.1713	0.4736	-0.3083	0.2395	-0.3133	-0.4395	0.1713	0.436	-0.3083
-0.7385	-0.1067	0.0927	-0.7110	-0.2079	-0.1291	-0.7401	-0.1046	0.0820	-0.7110	-0.2079	-0.1291
0.2666	0.4172	0.0182	0.2522	0.0261	0.4256	0.2672	0.4166	0.0207	0.2522	0.0261	0.4256
-0.0190	-0.3286	-0.0766	-0.0199	0.0709	-0.3298	-0.0186	-0.3288	-0.0757	-0.0199	0.0709	-0.3298
0.0704	0.1313	0.0078	0.0664	0.0040	0.1335	0.0706	0.1312	0.0084	0.0664	0.0041	0.1335
0.6056	-0.1949	0.0099	0.6058	0.0841	-0.1755	0.64049	-0.1963	0.0196	0.6058	0.0841	-0.1755

3.2.2. Factor Rotation Matrix

Varimax.

	Factor I	Factor II	Factor III
Factor I	0.8742	-0.0274	-0.4848
Factor II	0.1791	0.9462	0.2694
Factor III	0.4513	-0.3224	0.8321

Equamax.

	Factor I	Factor II	Factor III
Factor I	0.8483	-0.0142	-0.5293
Factor II	0.1827	0.9461	0.2674
Factor III	0.4970	-0.3235	0.8052

Quartimax.

	Factor I	Factor II	Factor III
Factor I	0.8813	-0.0311	-0.4715
Factor II	0.1772	0.9467	0.2689
Factor III	0.4380	-0.3205	0.8399

Orthomax

	Factor I	Factor II	Factor III
Factor I	0.7880	0.6156	-0.0047
Factor II	0.1877	-0.2330	0.9542
Factor III	0.5863	-0.7528	-0.2992

3.2.3. Information Criteria

Information Criteria	Values
Log Likelihood	-115.2820
Akaike	236.5640
Schwarz	117.4977
Hannan Quinne	116.2985

3.3. For $n=30$, $p=10$, and $k=5$

3.3.1. Rotated Factor Loadings

Varimax Loadings					Equamax Loadings				
1	2	3	4	5	1	2	3	4	5
0.0291	-0.2467	0.3084	0.0620	0.2325	0.0477	0.1511	-0.1059	-0.1428	0.3975
0.0234	-0.0180	-0.6165	0.0890	0.0394	0.0330	0.1354	-0.2761	0.0460	-0.5410
-0.0010	-0.5785	-0.0341	0.0792	-0.0162	0.0512	0.0799	-0.2646	-0.5112	0.0488
0.2259	0.4472	-0.1390	-0.0010	0.1133	0.1857	0.0571	0.1099	0.4582	-0.1530
0.1903	0.4352	0.3914	0.1877	-0.2053	0.1227	0.0440	0.6111	0.2002	0.1600
-0.7466	-0.0988	0.0735	-0.0342	-0.0851	-0.7327	-0.1018	-0.0455	-0.1659	0.0636
0.2088	0.0397	-0.2198	0.2673	0.3851	0.1995	0.4346	-0.2040	0.1871	-0.0892
0.0375	0.0550	0.3189	-0.5019	-0.0406	0.0588	-0.4707	-0.0673	0.1305	0.1752
-0.0117	0.0244	0.0899	0.4600	0.0840	-0.0421	0.4389	0.1693	-0.0548	0.0382
0.6235	-0.0267	0.1618	-0.0660	-0.0882	0.6206	-0.0825	0.1496	-0.0291	0.1121
Quartimax Loadings					Orthomax Loadings				
1	2	3	4	5	1	2	3	4	5
0.0297	0.2638	-0.2609	0.0756	0.2653	0.0240	-0.2256	0.3701	0.0538	0.1527
0.0282	-0.6139	0.0174	0.1016	-0.0465	0.0099	-0.0751	-0.5901	0.0568	0.1823
-0.0109	-0.0656	-0.5769	0.0696	-0.0181	0.0186	-0.5746	0.0084	0.1026	-0.0351
0.2374	-0.1219	0.4528	0.0167	0.0814	0.1988	0.4317	-0.1460	-0.0393	0.1858
0.1906	0.4459	0.3998	0.1655	-0.1837	0.1925	0.4915	0.2963	0.2133	-0.2102
-0.7508	0.0701	-0.0906	-0.0391	-0.0507	-0.7323	-0.1100	0.0621	-0.0190	-0.1685
0.2225	-0.2530	0.0526	0.3057	0.3216	0.1698	0.0120	-0.1353	0.2029	0.4741
0.0342	0.1329	0.0539	-0.5051	0.0226	0.0493	0.0564	0.1293	-0.4871	-0.1330
-0.0073	0.0925	0.0131	0.4649	0.0507	-0.0253	0.0414	0.0968	0.4467	0.1274
0.6190	0.1724	-0.0483	-0.0819	-0.0763	0.6324	0.0103	0.1397	-0.0423	-0.0803

3.3.2. Factor Rotation Matrix

Varimax.

	Factor 1	Factor 2	Factor 3	Factor 4	Factor 5
Factor 1	0.8580	0.4939	0.1058	0.0858	0.0374
Factor 2	0.1716	-0.2264	-0.7907	0.4076	0.3579
Factor 3	0.4338	-0.8274	0.3469	-0.0354	0.0751
Factor 4	-0.1857	0.0665	0.4480	0.8605	0.1412
Factor 5	-0.1086	0.1255	0.2064	-0.2914	0.9192

Equamax.

	Factor 1	Factor 2	Factor 3	Factor 4	Factor 5
Factor 1	0.7969	0.1087	0.3833	0.4540	0.0018
Factor 2	0.1911	0.5859	-0.5280	-0.0276	-0.5837
Factor 3	0.5083	0.0221	-0.2768	-0.6656	0.4706
Factor 4	-0.2496	0.7894	0.4476	-0.1300	0.3119
Factor 5	-0.0877	0.1459	-0.5453	0.5771	0.5836

Quartimax.

	Factor 1	Factor 2	Factor 3	Factor 4	Factor 5
Factor 1	0.8669	0.1400	0.4702	0.0877	0.0117
Factor 2	0.1839	-0.8282	-0.1816	0.4480	0.2159
Factor 3	0.4191	0.2856	-0.8516	-0.0485	0.1231
Factor 4	-0.1791	0.4521	0.0313	0.8653	0.1178
Factor 5	-0.0836	0.0923	0.1402	-0.2015	0.9614

Orthomax.

	Factor 1	Factor 2	Factor 3	Factor 4	Factor 5
Factor 1	0.6376	0.4172	0.6419	0.0820	0.0246
Factor 2	0.1114	-0.4246	0.2291	-0.6661	0.5578
Factor 3	0.5581	-0.5949	-0.2381	0.5081	0.1402
Factor 4	-0.2275	0.3484	-0.0820	0.3943	0.8153
Factor 5	-0.4666	-0.4127	0.6871	0.3686	-0.0630

3.3.3. Information Criteria

Information Criteria	Values
Log Likelihood	-111.6615
Akaike	233.3230
Schwarz	115.3543
Hannan Quinne	113.3557

3.4. For $n=50$, $p=10$, and $k=2$ **3.4.1. Rotated Factor Loadings**

Varimax Loadings		Equamax Loadings		Quartimax Loadings		Orthomax Loadings	
I	II	I	II	I	II	I	II
0.4485	0.4337	0.3588	0.5103	0.4600	0.4213	-0.1449	0.6068
-0.6198	0.0535	-0.6188	-0.0641	-0.6182	0.0703	0.5513	-0.2883
-0.0483	0.2891	-0.1019	0.2748	-0.0405	0.2903	0.1962	0.2177
-0.0766	-0.0221	-0.0711	-0.0361	-0.0772	-0.0200	0.0527	-0.0598
-0.1126	-0.1679	-0.0790	-0.1861	-0.1171	-0.1648	0.0046	-0.2021
0.2855	-0.0914	0.2976	-0.0360	0.2830	-0.0991	-0.2899	0.0765
0.5097	0.0521	0.4908	0.1471	0.5110	0.0383	-0.4017	0.3181
0.1060	-0.4813	0.1947	-0.4527	0.0929	-0.4840	-0.3482	-0.3488
-0.2930	0.2081	-0.3270	0.1492	-0.2873	0.2159	0.3590	0.0178
0.0519	0.3253	-0.0102	0.3292	0.0607	0.3237	0.1311	0.3021

3.4.2. Factor Rotation Matrix*Varimax.*

	Factor I	Factor II
Factor I	0.9806	0.1960
Factor II	-0.1960	0.9806

Equamax.

	Factor I	Factor II
Factor I	0.9262	0.3771
Factor II	-0.3771	0.9262

Quartimax.

	Factor I	Factor II
Factor I	0.9855	0.1694
Factor II	-0.1694	0.9855

Orthomax

	Factor I	Factor II
Factor I	-0.7213	0.6926
Factor II	0.6926	0.7213

3.4.3. Information Criteria

Information Criteria	Values
Log Likelihood	-239.8450
Akaike	483.6900
Schwarz	241.5440
Hannan Quinne	240.7657

3.5. For $n=50$, $p=10$, and $k=3$ **3.5.1. Rotated Factor Loadings**

Varimax Loadings			Equamax Loadings			Quartimax Loadings			Orthomax Loadings		
Factor I	Factor II	Factor III	Factor I	Factor II	Factor III	Factor I	Factor II	Factor III	Factor I	Factor II	Factor III
0.6151	0.2322	0.0738	0.6213	-0.0074	0.2273	0.6091	0.2376	0.1010	0.5339	-0.0567	0.3865
-0.6020	0.2385	0.1382	-0.5759	0.2211	0.2406	-0.6097	0.2349	0.1073	-0.5917	0.2883	0.0714
-0.0674	0.2570	0.1277	0.0861	0.1204	0.2539	0.0593	0.2585	0.1285	0.0277	0.1242	0.2659
0.0799	-0.1549	0.3391	0.1237	0.3234	-0.1549	0.0647	-0.1514	0.3439	0.1959	0.3012	-0.1275
-0.0866	-0.1962	0.1414	-0.0683	0.1495	-0.1974	-0.0918	-0.1957	0.1388	0.0032	0.1453	-0.2118
0.1547	-0.0731	-0.2839	0.1144	-0.3031	-0.0705	0.1687	-0.0742	-0.2756	0.0956	-0.3160	-0.0302
0.3851	0.0279	-0.3733	0.3313	-0.4216	0.0302	0.4022	0.0278	-0.3548	0.2539	-0.4500	0.1278
0.0137	-0.5326	-0.0158	0.0072	-0.0236	-0.5325	0.0184	-0.5326	-0.0104	0.1458	-0.0511	-0.5101
-0.0598	0.1192	0.4453	0.0019	0.4506	0.1140	-0.0819	0.1225	0.4409	0.0201	0.4535	0.1002
0.1197	0.3169	-0.0032	0.1207	-0.0157	0.3161	0.1173	0.3177	-0.0003	0.0299	-0.0115	0.3372

3.5.2. Factor Rotation Matrix

Varimax.

	Factor I	Factor II	Factor III
Factor I	0.8989	0.0180	-0.4378
Factor II	0.1582	0.9184	0.3626
Factor III	0.4087	-0.3952	0.8227

Equamax.

	Factor I	Factor II	Factor III
Factor I	0.8316	-0.5552	0.0178
Factor II	0.2131	0.3484	0.9128
Factor III	0.5129	0.7552	-0.4081

Quartimax.

	Factor I	Factor II	Factor III
Factor I	0.9186	0.0214	-0.3946
Factor II	0.1339	0.9227	0.3616
Factor III	0.3718	-0.3850	0.8447

Orthomax

	Factor I	Factor II	Factor III
Factor I	0.7320	-0.6323	0.2538
Factor II	-0.0009	0.3716	0.9284
Factor III	0.6813	0.6798	-0.2714

3.5.3. Information Criteria

Information Criteria	Values
Log Likelihood	-236.4975
Akaike	478.9950
Schwarz	239.0460
Hannan Quinne	237.8786

3.6. For $n=30$, $p=10$, and $k=5$

3.6.1. Rotated Factor Loadings

Varimax Loadings					Equamax Loadings				
1	2	3	4	5	1	2	3	4	5
0.5901	0.2836	0.0242	0.1131	0.0711	0.5819	0.2639	-0.0608	0.0645	0.1762
-0.6315	0.2194	0.0819	0.0653	0.0445	-0.6066	0.2276	0.1530	0.1275	-0.0223
0.0172	0.1732	0.0989	0.1316	0.3041	0.0007	0.1458	0.0919	0.1382	0.3180
0.0288	-0.0887	0.0930	0.4435	0.0077	0.0685	-0.0911	0.0309	0.4473	0.0114
-0.0492	-0.0470	0.0735	0.0489	-0.4135	0.0194	-0.0106	0.0536	0.0647	-0.4194
0.1370	0.0345	-0.4148	0.0481	-0.0581	0.0877	0.0364	-0.4335	-0.0129	-0.0245
0.3890	-0.0033	-0.2925	-0.1813	0.1472	0.3089	-0.0233	-0.3060	-0.2483	0.2033
0.0364	-0.5667	-0.0207	0.0549	-0.0385	0.0362	-0.5621	-0.0343	0.0486	-0.0809
-0.0668	0.1290	0.4106	0.1767	-0.0671	0.0133	0.1350	0.3862	0.2300	-0.0700
0.1335	0.2413	0.1637	-0.2005	0.1088	0.1281	0.2289	0.1751	-0.1908	0.1378
Quartimax Loadings					Orthomax Loadings				
1	2	3	4	5	1	2	3	4	5
0.5883	0.2846	0.0478	0.1236	0.0489	0.2976	0.5461	-0.1804	0.0347	-0.1629
-0.6340	0.2212	0.0561	0.0558	0.0527	-0.5633	-0.2496	-0.1453	0.1811	0.1625
0.0183	0.1818	0.1008	0.1321	0.2982	-0.1145	0.1416	-0.0763	0.2698	-0.1953
0.0182	-0.0907	0.1028	0.4415	0.0085	-0.0910	0.2516	0.3001	0.1996	0.1125
-0.0625	-0.0585	0.0725	0.0452	-0.4108	0.0481	-0.0025	0.0394	-0.1220	0.4058
0.1513	0.0292	-0.4083	0.0581	-0.0624	-0.0590	0.1861	0.0674	-0.3681	-0.1399
0.4065	-0.0004	-0.2798	-0.1694	0.1385	0.2466	0.1424	-0.0472	-0.2612	-0.3740
0.0353	-0.5681	-0.0140	0.0518	-0.0231	0.2099	-0.1933	0.4938	-0.0440	0.0159
-0.0874	0.1291	0.4097	0.1685	-0.0695	-0.0181	0.0835	-0.0813	0.3664	0.2779
0.1326	0.2468	0.1634	-0.1995	0.0992	0.1319	0.0308	-0.3346	0.1247	-0.0949

3.6.2. Factor Rotation Matrix

Varimax.

	Factor 1	Factor 2	Factor 3	Factor 4	Factor 5
Factor 1	0.9023	0.0291	-0.3708	-0.1568	0.1513
Factor 2	0.1000	0.8444	0.4161	0.0547	0.3175
Factor 3	0.3858	-0.2759	0.5520	0.6485	-0.2229
Factor 4	-0.1486	-0.1712	-0.2775	0.5186	0.7764
Factor 5	-0.0699	0.4250	-0.5547	0.5319	-0.4731

Equamax.

	Factor 1	Factor 2	Factor 3	Factor 4	Factor 5
Factor 1	0.8020	-0.0020	-0.4499	-0.2768	0.2787
Factor 2	0.1250	0.8117	0.4044	0.0933	0.3915
Factor 3	0.5241	-0.2659	0.4007	0.6763	-0.1913
Factor 4	-0.2540	-0.2355	-0.2865	0.4860	0.7495
Factor 5	-0.0439	0.4637	-0.6256	0.4701	-0.4132

Quartimax.

	Factor 1	Factor 2	Factor 3	Factor 4	Factor 5
Factor 1	0.9220	0.0311	-0.3372	-0.1355	0.12975
Factor 2	0.0899	0.8559	0.4144	0.0550	0.2906
Factor 3	0.3479	-0.2825	0.5808	0.6412	-0.2253
Factor 4	-0.1275	-0.1544	-0.2719	0.5226	0.7829
Factor 5	-0.0671	0.4034	-0.5507	0.5426	-0.4848

Orthomax

	Factor 1	Factor 2	Factor 3	Factor 4	Factor 5
Factor 1	0.5874	0.4895	-0.0547	-0.3691	-0.5255
Factor 2	-0.1591	0.4020	-0.7224	0.5301	-0.1005
Factor 3	0.3569	0.4681	0.4711	0.4792	0.4494
Factor 4	-0.4676	0.1567	0.5023	0.3035	-0.6421
Factor 5	-0.5326	0.5959	0.0302	-0.5108	0.3154

3.6.3. Information Criteria

Information Criteria	Values
Log Likelihood	-237.0900
Akaike	484.1800
Schwarz	241.3374
Hannan Quinne	239.3919

3.7. For $n=70$, $p=10$ and $k=2$ **3.7.1. Rotated Factor Loadings**

Varimax Loadings		Equamax Loadings		Quartimax Loadings		Orthomax Loadings	
I	II	I	II	I	II	I	II
0.4699	0.1452	0.4705	0.1433	0.4698	0.1456	0.4434	-0.2128
-0.2739	0.1140	-0.2735	0.1151	-0.2740	0.1138	-0.1235	0.2698
0.1437	-0.0249	0.1436	-0.0254	0.1437	-0.0248	0.0885	-0.1159
-0.0327	-0.3481	-0.0341	-0.3480	-0.0324	-0.3482	-0.2606	-0.2332
-0.4369	0.0296	-0.4368	0.0313	-0.4369	0.0292	-0.3005	0.3186
-0.1192	0.0560	-0.1189	0.0564	-0.1192	0.0559	-0.0494	0.1220
-0.0112	0.3385	-0.0098	0.3385	-0.0114	0.3385	0.2218	0.2559
0.2034	-0.1007	0.2030	-0.1015	0.2035	-0.1005	0.0808	-0.2121
0.0872	0.3954	0.0888	0.3950	0.0869	0.3955	0.3327	0.2308
0.2853	-0.1387	0.2847	-0.1399	0.2854	-0.1385	0.1150	-0.2956

3.7.2. Factor Rotation Matrix*Varimax.*

	Factor I	Factor II
Factor I	0.9998	-0.0213
Factor II	0.0213	0.9998

Equamax.

	Factor I	Factor II
Factor I	0.9997	-0.0253
Factor II	0.0253	0.9997

Quartimax.

	Factor I	Factor II
Factor I	0.9998	-0.0205
Factor II	0.0205	0.9998

Orthomax

	Factor I	Factor II
Factor I	0.7191	-0.6950
Factor II	0.6950	0.7191

3.7.3. Information Criteria

Information Criteria	Values
Log Likelihood	-328.3934
Akaike	660.7868
Schwarz	330.2385
Hannan Quinne	329.4575

3.8. For $n=70$, $p=10$, and $k=3$ **3.8.1 Rotated Factor Loadings**

Varimax Loadings			Equamax Loadings			Quartimax Loadings			Orthomax Loadings		
Factor I	Factor II	Factor III	Factor I	Factor II	Factor III	Factor I	Factor II	Factor III	Factor I	Factor II	Factor III
0.5157	0.0368	-0.0268	0.5142	0.0452	0.0401	0.5145	0.0353	-0.0464	0.4769	0.1988	0.0326
-0.2715	0.1597	-0.0664	-0.2615	-0.0967	0.1609	-0.2735	0.1596	-0.0579	-0.2436	-0.1108	0.1789
0.0644	0.0974	0.2220	0.0327	0.2322	0.0890	0.0729	0.0995	0.2184	-0.0390	0.2477	0.0070
-0.0462	-0.3815	0.0461	-0.0483	0.0249	-0.3832	-0.0451	-0.3809	0.0517	-0.0100	-0.1118	-0.3704
-0.4095	0.0333	-0.1545	-0.3849	-0.2075	0.0375	-0.4149	0.0327	-0.1395	-0.3194	-0.2850	0.0972
-0.0771	0.0033	-0.1287	-0.0589	-0.1378	0.0079	-0.0819	0.0021	-0.1258	-0.0215	-0.1390	0.0523
-0.0272	0.4278	0.0172	-0.0334	0.0295	0.4267	-0.0257	0.4281	0.0138	-0.0874	0.1518	0.3917
0.1354	-0.0185	0.2108	0.1057	0.2264	-0.0261	0.1432	-0.0167	0.2058	0.0453	0.2268	-0.0981
0.2221	0.2129	-0.3325	0.2632	-0.2908	0.2267	0.2099	0.2090	-0.3427	0.3027	-0.1177	0.3158
0.1512	0.0466	0.3894	0.0964	0.4078	0.0320	0.1658	0.0502	0.3830	-0.0173	0.4068	-0.1041

3.8.2. Factor Rotation Matrix*Varimax.*

	Factor I	Factor II	Factor III
Factor I	0.9246	0.0005	0.3809
Factor II	0.2148	0.8250	-0.5227
Factor III	-0.3145	0.5651	0.7627

Equamax.

	Factor I	Factor II	Factor III
Factor I	0.8642	0.5031	-0.0102
Factor II	0.2759	-0.4568	0.8457
Factor III	-0.4208	0.7336	0.5336

Quartimax.

	Factor I	Factor II	Factor III
Factor I	0.9382	0.0022	0.3460
Factor II	0.1968	0.8192	-0.5387
Factor III	-0.2846	0.5735	0.7681

Orthomax.

	Factor I	Factor II	Factor III
Factor I	0.6990	0.6968	-0.1609
Factor II	0.2890	-0.0695	0.9548
Factor III	-0.6541	0.7139	0.2499

3.8.3. Information Criteria.

Information Criteria	Values
Log Likelihood	-326.2455
Akaike	656.4910
Schwarz	329.0131
Hannan Quinne	327.8416

3.9. For $n=70$, $p=10$, and $k=5$.**3.9.1. Rotated Factor Loadings**

Varimax Loadings					Equamax Loadings				
1	2	3	4	5	1	2	3	4	5
0.5468	0.0205	0.0155	-0.0389	-0.0246	0.5373	0.0112	0.0645	-0.0857	-0.0387
-0.1516	0.2139	-0.2445	0.1682	0.0821	-0.1097	0.2162	-0.2405	0.1487	0.1575
0.0075	0.0856	0.2493	0.0238	-0.1009	-0.0119	0.0957	0.2290	0.0377	-0.1306
-0.0621	-0.3719	0.0065	0.0877	-0.0337	-0.0578	-0.3627	-0.0060	0.1194	-0.0437
-0.3644	0.0607	-0.2408	0.0082	-0.0061	-0.3363	0.0649	-0.2727	0.0312	0.0434
-0.0695	-0.0030	-0.0635	-0.0350	0.3470	-0.0711	-0.0276	-0.0120	-0.0744	0.3450
-0.0196	0.4228	0.0350	-0.0633	-0.0276	-0.0233	0.4205	0.0289	-0.0820	-0.0153
0.2292	0.0287	0.0506	0.3203	0.1609	0.2501	0.0311	0.0991	0.2738	0.1901
0.1608	0.1440	-0.0744	-0.4111	0.0819	0.1286	0.1134	-0.0441	-0.4412	0.0430
0.0658	0.0316	0.4322	0.1046	-0.0202	0.0340	0.0415	0.4275	0.1104	-0.0750
Quartimax Loadings					Orthomax Loadings				
1	2	3	4	5	1	2	3	4	5
0.5476	0.0230	-0.0007	-0.0258	-0.0269	0.4868	0.0012	0.2146	0.1232	-0.0596
-0.1636	0.2113	-0.2390	0.1755	0.0648	-0.0673	0.2093	-0.1250	-0.1125	0.2946
0.0137	0.0828	0.2530	0.0186	-0.0942	-0.0382	0.1173	0.1011	-0.0457	-0.2296
-0.0626	-0.3746	0.0092	0.0761	-0.0312	-0.0370	-0.3455	-0.0376	-0.1641	-0.0445
-0.3720	0.0599	-0.2289	0.0032	-0.0129	-0.2473	0.0566	-0.3020	-0.0428	0.1928
-0.0687	0.0024	-0.0738	-0.0218	0.3462	-0.1523	-0.0552	0.1455	0.1037	0.2689
-0.0189	0.4236	0.0374	-0.0557	-0.0297	-0.0277	0.4122	-0.0035	0.1198	-0.0108
0.2234	0.0241	0.0430	0.3322	0.1476	0.1722	0.0462	0.2897	-0.2264	0.1320
0.1684	0.1555	-0.0874	-0.3989	0.0922	0.0990	0.0645	-0.0001	0.4618	0.0248
0.0762	0.0276	0.4321	0.1005	-0.0102	-0.0598	0.0756	0.3040	-0.1153	-0.2975

3.9.2. Factor Rotation Matrix*Varimax.*

	Factor 1	Factor 2	Factor 3	Factor 4	Factor 5
Factor 1	0.8368	-0.0542	0.5388	0.0030	-0.0807
Factor 2	0.2288	0.7574	-0.2605	-0.5434	0.1039
Factor 3	-0.3057	0.6295	0.5173	0.4775	-0.1207
Factor 4	0.3408	0.1378	-0.4437	0.6444	0.5027
Factor 5	-0.1944	-0.0902	0.4210	-0.2478	0.8458

Equamax.

	Factor 1	Factor 2	Factor 3	Factor 4	Factor 5
Factor 1	0.7781	-0.0567	0.5971	-0.0437	-0.1817
Factor 2	0.2084	0.7131	-0.2152	-0.6236	0.1129
Factor 3	-0.2992	0.6702	0.4603	0.4903	-0.0952
Factor 4	0.2694	-0.1519	0.5329	-0.3171	0.7210
Factor 5	-0.2694	-0.1519	0.5329	-0.3171	0.7210

Quartimax.

	Factor 1	Factor 2	Factor 3	Factor 4	Factor 5
Factor 1	0.8524	-0.0541	0.5155	0.0126	-0.0675
Factor 2	0.2314	0.7730	-0.2755	-0.5122	0.1037
Factor 3	-0.3048	0.6136	0.5400	0.4719	-0.1281
Factor 4	0.3130	0.1323	-0.4607	0.6813	0.4563
Factor 5	-0.1702	-0.0752	0.3931	-0.2253	0.8718

Orthomax.

	Factor 1	Factor 2	Factor 3	Factor 4	Factor 5
Factor 1	0.6035	-0.0321	0.6309	0.0717	-0.4812
Factor 2	0.1959	0.6292	-0.0807	0.7191	0.2050
Factor 3	-0.3507	0.7426	0.2256	-0.4544	-0.2612
Factor 4	0.3547	0.1139	0.3093	-0.3855	0.7854
Factor 5	-0.5904	-0.1966	0.6700	0.3501	0.2032

3.9.3. Information Criteria

Information Criteria	Values
Log Likelihood	-323.6940
Akaike	651.388
Schwarz	328.3067
Hannan Quinne	326.3542

4. Discussion of Results and Conclusion

In this work, the number of variables considered was ten (10). Hence, given a sample size (n) of 30, the values of Akaike's Information Criterion (AIC), the Schwarz Information Criterion (SIC) and the Hannan Quinne Information Criterion (HQIC) for the different number of retained factors considered are as follows; for $k = 2$, the AIC, SIC, and HQIC values are 241.3360, 120.1451 and 119.3457 respectively. When $k = 3$, AIC is 236.5600, SIC is 117.4977 and HQIC is 116.2985; and for $k = 5$, it shows that AIC = 233.3230, SIC = 115.3543 and HQIC = 113.3557.

When the sample size is increased to 50, the AIC, SIC, and HQIC are 483.6900, 241.5440 and 240.7657; 478.9950, 239.0460, and 237.8786; 484.1800, 241.3374 and 239.3919 for $k = 2, 3$, and 5 respectively.

Finally, at the sample size of 70, the values are AIC = 660.7868, SIC = 330.2385 and HQIC = 329.4575 for $k = 2$. When k is 3, AIC = 656.4910, SIC = 329.0131 and HQIC = 327.8416; and finally for $k = 5$, the values are 651.3880, 328.3067, and 326.3542 for AIC, SIC and HQIC respectively.

When the sample sizes are 30, and 70, the AIC, SIC, and HQIC are smallest for $k = 5$ followed by $k = 3$ and highest

for $k = 2$. But for the sample size of 50, the values are smallest for $k = 3$ followed by for $k = 2$ and highest for $k = 5$.

In conclusion, an insight into the results above shows that the optimal number of factors to retain using Principal Factors method of estimation is five (5) from the sample sizes of thirty (30) and Seventy (70); whereas for the sample size of Fifty (50), the number of factors is three(3). This conclusion is made based on the fact that in competing sets of models, the model with the smallest value of information criteria is chosen as the best model.

Also, from the vales of AIC, SIC, and HQIC obtained above, the Hannan Quinne information criterion performs best for all the three criteria considered. This is followed by the SIC and AIC respectively. Finally, we observed that the higher the sample size, the higher the value of the information criteria.

The factor rotation matrix for all the sample sizes and the number of parameters retained as considered are almost the same for all the four methods of rotation considered here.

Results above show that the values of AIC and SIC increased for all the sample sizes considered.

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