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Prediction Intervals for Progressive Type-II Right-Censored Order Statistics from Two Independent Sequences

M. M. Mohie El-Din[1], M. S. Kotb[1], W. S. Emam[2]

[1]Department of Mathematics, Faculty of Science, Al-Azhar, University, Nasr City, Cairo, Egypt
[2]Department of Basic Science, Faculty of Engineering, British, University in Egypt, Al-Shorouq City, Cairo, Egypt

Email address:
mmmmoheeldin@yahoo.com (M. M. M. El-Din), kottp5_mas@yahoo.com (M. S. Kotb), walid_emam42@yahoo.com (W. S. Emam)

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Abstract: This article discusses the problem of predicting future progressive Type-II right censored order statistics based on progressive Type-II right-censored, ordered statistics, record values and current records that observed from the past X-sequence. Such coverage probabilities of the prediction intervals are exact and don't depend on the sampling distribution *F*. Finally, a real life time data were given to breakdown the insulating fluid between electrodes which is used to illustrate the derived results.

Keywords: Prediction Intervals, Progressive Censoring, Order Statistics, Records, Coverage Probability, Prediction Coefficient

1. Introduction

Statistical prediction is the derivation issue of the unknown values of future sample based on available observations. Prediction problems can be generally categorized as One-sample prediction, two-sample prediction and multi-sample prediction. Type II-sample prediction is governed here. In this type, predicting of future variables based on another independent observed sample. Also, the predictor can be either point or interval. The prediction (confidence) intervals can be one-sided or two-sided. Moreover, the predictor can be parametric (if it depends on the distribution parameters) or nonparametric (distribution-free prediction type). Distribution-free two-sided prediction intervals (DFPIs) are interest of natural in this article. Many contexts have taken place in the DFPIs direction in recent years by using several assumption; see, for example [1-11]. Recently, DFPIs for future progressively Type-II right-censored order statistics sample (PCOs) (which it's cdf purposed by Kamps and Cramer [12]) based on k-records [13]. Here, the DFPIs derivation for a future PCOs (that developed by Balakrishnan et al. [16]) based on different observed samples (PCOs; order statistics; upper records and largest current records).

This paper is organized as follows, section 2 contains some preliminaries. In section 3, the DFPIs discussion of a future PCO based on PCOs, order statistics, upper records and current records. In section 4, a real life time data are considered and numerical computations are given to illustrate most the results which are derived. Finally, the conclusion of this study is given in Section 5.

2. Preliminaries

Suppose $X_1, X_2, ..., X_n$ denote the observed lifetimes of identical units of size n that placed on a reliability experiment of life-test. Assume that, these units are independent and identically distributed (iid) from a continuous population with cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. Further, independently of X-sample, let $Y_1, Y_2, ..., Y_m$ another unobserved sample of size m that withdrawn from the same population, consider we intend to study k units-out-of-m and remove from the life-test at different time points (progressively) the remaining $m-k$ units from these unobserved iid random variables (r.v's). Specifically, immediately following the first observed failure $Y^*_{1:k:m}$, remove from right at random R_1 surviving units from the test; next, immediately observe following the second failure $Y^*_{2:k:m}$, remove at random R_2 surviving units from the test, and so on;

finally, after the k^{th} observed failure $Y^*_{k:k:m}$, remove all the remaining surviving units (that did not failed yet) $R_k = n-k-(R_{k-1}+...+R_1)$. Such k the number of failures (that were already observed) and the progressive censoring scheme $\overline{R}=(R_1,...,R_k)$ are pre-fixed, the scheme $Y^*_{1:k:m},...,Y^*_{k:k:m}$ is said to be progressive Type-II right-censoring, and the ordered values $Y_{1:k:m} < Y_{2:k:m} < ... < Y_{k:k:m}$ were obtained from these observations are referred to PCOs. For more details concerning a PCOs, see, for example, Balakrishnan and Aggarwala [14], Balakrishnan [15]. The marginal pdf of the r^{th} PCO, $Y_{r:k:m}$ $(1 \le r \le k \le m)$ that developed by Balakrishnan et al. [16], given by

$$f_{Y_{r:k:m}}(y) = c_r^{'} \sum_{\ell=0}^{r-1} c_{\ell,r-1} \quad f(y)\left\{\overline{F}(y)\right\}^{R^{''}_{\ell,r}-1}, \tag{1}$$

where $-\infty < y < \infty$, $c_r{}' = m(m-R_1-1)\ldots(m-R_1-\ldots-R_{r-1}-r+1)$, $R^{''}_{\ell,r} = (R^*_r+1)+\sum_{j=r-\ell}^{r-1}(R_j+1)$, $R^*_r = m-r-(R_1+\ldots+R_{r-1})$, $c_{\ell,r} = c_{\ell,r}(R_1+1,\ldots,R_r+1)$ and $\left(\prod_{j=1}^{\ell}\sum_{k=r-\ell+1}^{r-\ell+j}(R_k+1)\right)\left(\prod_{j=1}^{r-\ell}\sum_{k=j}^{r-\ell}(R_k+1)\right)c_{\ell,r} = (-1)^{\ell}$. The survival function of $Y_{r:k:m}$ $(1 \le r \le k \le m)$ can be easily obtained as

$$\overline{F}_{Y_{r:k:m}}(y) = c_r^{'} \sum_{\ell=0}^{r-1} \frac{c_{\ell,r-1}}{R^{''}_{\ell,r}} \quad \left\{\overline{F}(y)\right\}^{R^{''}_{\ell,r}} \tag{2}$$

It remains to refer that, not only the prediction coefficients φ and π which are stated here depend on the subscripts but also depend on observed sample size n, future PCOs size $k \le m$ and the progressive censoring scheme $\overline{R}$, i.e.

$$\varphi(.) = \varphi\left(.\ ,n;k,m,\overline{R}\right), \qquad \pi(.) = \pi\left(.\ ,n;k,m,\overline{R}\right).$$

3. DFPIs for Single PCO

In this section, we intend to construct $100(1-\alpha)\%$ DFPIs for the r^{th} PCO, $Y_{r:k:m}(1 \le r \le k \le m)$ from the future PCOs of size k out of m from Y-sequence of the form (X_i, X_j), such that, the lower X_i and the upper X_j bounded are observed from the following four schemes: PCOs, order statistics, upper records and current records, each case has separately discussion in special subsection, such that the corresponding coverage probabilities are free of the parent distribution F.

3.1. Based on PCOs

Table 1. *values of* $\pi_1(i,j,\mu;r)$ *for* $n:\mu = 20:10$ *and* $\overline{R}^*_1 = (10,0,...,0)$ *for some choices of* $m,k,\overline{R},i,j$ *and* r.

			j				j			
			4	6	8	10	4	6	8	10
$m:k$	r	i	$(m-k,0,...,0)$				$(0,...,0,m-k)$			
$20:10$	3	1	0.659	0.885	0.974	0.995	0.781	0.875	0.884	0.885
		2	0.444	0.670	0.733	0.741	0.424	0.517	0.527	0.527
		3	0.202	0.429	0.492	0.500	0.152	0.245	0.255	0.255
	5	1	0.312	0.676	0.918	0.993	0.688	0.926	0.973	0.976
		2	0.265	0.629	0.872	0.946	0.494	0.731	0.778	0.782
		3	0.159	0.523	0.766	0.840	0.232	0.469	0.516	0.520
20:15	9	1	0.116	0.435	0.811	0.985	0.284	0.698	0.945	0.996
		2	0.108	0.427	0.804	0.978	0.255	0.669	0.916	0.968
		3	0.077	0.396	0.772	0.946	0.164	0.577	0.825	0.877
	11	1	0.031	0.198	0.581	0.940	0.139	0.501	0.864	0.993
		2	0.030	0.197	0.580	0.939	0.131	0.492	0.855	0.984
		3	0.023	0.191	0.574	0.932	0.092	0.454	0.817	0.946
15:8	4	1	0.345	0.691	0.916	0.989	0.655	0.906	0.968	0.974
		2	0.284	0.629	0.824	0.928	0.469	0.720	0.781	0.788
		3	0.163	0.509	0.734	0.807	0.223	0.474	0.536	0.542
	6	1	0.070	0.277	0.624	0.934	0.398	0.780	0.962	0.996
		2	0.065	0.272	0.619	0.929	0.336	0.718	0.900	0.934
		3	0.045	0.252	0.600	0.910	0.197	0.579	0.760	0.794
25:15	4	1	0.690	0.913	0.962	0.966	0.800	0.905	0.915	0.915
		2	0.477	0.700	0.748	0.753	0.459	0.563	0.573	0.574
		3	0.218	0.441	0.489	0.494	0.169	0.274	0.284	0.284
	6	1	0.432	0.804	0.967	0.995	0.710	0.937	0.976	0.978
		2	0.359	0.732	0.895	0.923	0.510	0.737	0.776	0.778
		3	0.206	0.578	0.741	0.769	0.237	0.464	0.503	0.505

Let $X_{1:\mu:n} < X_{2:\mu:n} < ... < X_{\mu:\mu:n}$ be an observed sequence of PCOs of size μ from the X-sequence with the progressive censoring scheme $\overline{R}^* = (R_1^*,...,R_\mu^*)$. Suppose we are interested in obtaining $100(1-\alpha)\%$ DFPIs for $Y_{r:k:m}$ of the form $\left(X_{i:\mu:n}, X_{j:\mu:n}\right)$, such that the coverage probability $p\left(X_{i:\mu:n} \le Y_{r:k:m} \le X_{j:\mu:n}\right) = 1-\alpha$ being free of the parent distribution F.

Lemma 1. Let $\{X_{i:\mu:n}, 1 \le i \le \mu \le n\}$ and $\{Y_{j:k:m}, 1 \le j \le k \le m\}$ be two independent PCOs from continuous cdf F. then $(-\infty, X_{i:\mu:n})$, $1 \le i \le \mu \le n$, is DF one-sided PI for the future $Y_{r:k:m}$, with the corresponding prediction coefficient $\varphi_1(i,\mu;r)$, that does not depend on the sampling distribution F, and is given by

$$\varphi_1\left(i,\mu,\overline{R}^*;r\right) = c_r' c_i^{*'} \sum_{\nu=0}^{r-1}\sum_{\ell=0}^{i-1} \frac{c_{\nu,r-1} c^*_{\ell,i-1}}{R^{*''}_{\ell,i}\left(R^{*''}_{\ell,i} + R''_{\nu,r}\right)}. \tag{3}$$

Proof: Upon (1) and (2), we can write

$$\varphi_1\left(i,\mu,\overline{R}^*;r\right) = p\left(X_{i:\mu:n} \ge Y_{r:k:m}\right) = \int_{-\infty}^{\infty} p\left(X_{i:\mu:n} \ge y\right) f_{Y_{r:k:m}}(y)dy = c_r' c_i^{*'} \sum_{\nu=0}^{r-1}\sum_{\ell=0}^{i-1} \frac{c_{\nu,r-1} c^*_{\ell,i-1}}{R^{*''}_{\ell,i}} \int_0^1 y^{R^{*''}_{\ell,i} + R''_{\nu,r} - 1} dy. \tag{4}$$

Table 2. *values of* $\pi_1(i,j,\mu;r)$ *for* $n:\mu = 30:25$ *and* $\overline{R}^* = (0,0,2,3,0,...,0)$ *for some choices of* $m,k,\overline{R},i,j$ *and* r.

			j				j			
			r+3	**r+4**	**r+5**	**r+6**	**r+3**	**r+4**	**r+5**	**r+6**
$m:k$	r	i	$(0,0,2,3,0,...,0)$				$(0,...,0,5)$			
15:10	4	r-3	0.472	0.584	0.684	0.768	0.504	0.616	0.714	0.794
		r-2	0.449	0.561	0.660	0.744	0.477	0.589	0.687	0.768
		r-1	0.405	0.417	0.617	0.701	0.428	0.541	0.638	0.719
	6	r-3	0.199	0.273	0.356	0.444	0.372	0.477	0.579	0.674
		r-2	0.192	0.267	0.350	0.438	0.355	0.459	0.562	0.657
		r-1	0.177	0.252	0.335	0.423	0.319	0.423	0.526	0.621
	8	r-3	0.057	0.088	0.127	0.179	0.267	0.358	0.457	0.557
		r-2	0.056	0.086	0.126	0.177	0.254	0.345	0.444	0.544
		r-1	0.052	0.082	0.123	0.174	0.231	0.322	0.421	0.521
25:20	10	r-3	0.525	0.627	0.716	0.789	0.636	0.713	0.770	0.808
		r-2	0.475	0.577	0.666	0.739	0.549	0.625	0.682	0.721
		r-1	0.406	0.508	0.597	0.670	0.442	0.519	0.576	0.615
	14	r-3	0.430	0.545	0.656	0.755	0.619	0.697	0.754	0.792
		r-2	0.398	0.513	0.624	0.723	0.535	0.613	0.670	0.708
		r-1	0.352	0.466	0.577	0.677	0.434	0.512	0.569	0.607
	18	r-3	0.305	0.442	0.603	0.769	0.659	0.740	0.792	0.820
		r-2	0.293	0.430	0.591	0.757	0.576	0.656	0.708	0.736
		r-1	0.272	0.409	0.570	0.736	0.472	0.552	0.604	0.632

Theorem 1. Under the assumption of lemma 1, then $(X_{i:\mu:n}, X_{j:\mu:n})$, $1 \le i < j \le \mu \le n$, is a DFPIs for $Y_{r:k:m}(1 \le j \le k \le m)$, whose coverage probability is free of the parent distribution F.

Proof: Based on lemma 1, and by assuming that the $X_{i:\mu:n}(1 \le i \le \mu \le n)$, are continuous r.v's, we obtain

$$\pi_1\left(i,j,\mu,\overline{R}^*;r\right) = p\left(X_{i:\mu:n} \le Y_{r:k:m} \le X_{j:\mu:n}\right) = p\left(X_{j:\mu:n} \ge Y_{r:k:m}\right) - p\left(X_{i:\mu:n} \ge Y_{r:k:m}\right) = \varphi_1\left(j,\mu,\overline{R}^*;r\right) - \varphi_1\left(i,\mu,\overline{R}^*;r\right). \tag{5}$$

Under the assumptions of theorem 1, we can choose i and j so that $\pi_1(i,j,\mu;r)$ exceeds the desired confidence level π_0, tables (1 and 2) presents values of the prediction coefficient $\pi_1(i,j,\mu;r)$ for two different censoring schemes of the past PCOs $\overline{R}^* = (10,0,...,0)$; $\overline{R}^* = (0,0,2,3,0,...,0)$, respectively, for some choices of $\mu,n,\overline{R},i,j$ and r.

3.2. Based on Order Statistics

In the following, the prediction discussion of future PCOs based on observed order statistics. Let $X_1, X_2, ..., X_n$ denote sequences of the lifetimes of reliability experiment units, we shall assume that these variables are iid from an absolutely continuous population with cdf F. Suppose $X_{1:n} \le X_{2:n} \le ... \le X_{n:n}$ are usual order statistics that obtained from these iidr.v's, for more details about order statistics, see, Arnold et al. [17]. The marginal pdf and the survival function of $X_{i:n}$ are easily expressed in terms of F and f, respectively as.

$$f_{X_{i:n}}(x) = i\binom{n}{i}[F(x)]^{i-1}[\overline{F}(x)]^{n-i} f(x), \tag{6}$$

$$\overline{F}_{X_{i:n}}(x) = \sum_{\ell=0}^{i-1}\binom{n}{i}\left[F(y)\right]^{\ell}\left[\overline{F}(y)\right]^{n-\ell}. \tag{7}$$

Lemma 2.

Based on X-sample observations, suppose we are interested in obtaining $(1-\alpha)100\%$ DFPIs for Y_r from a future Y-sample of the form $(X_i, X_j), 1 \le i < j$, such that, the coverage probability $P\left(X_i \le Y_r \le X_j\right) = 1-\alpha$. We refer to the interval (X_i, X_j) as $(1-\alpha)100\%$ PI for Y_r. For a given y, if X_i and X_j are continuous r.v's, we get

$$\begin{aligned} p\left(X_i \le y \le X_j\right) &= P\left(X_j \ge y\right) - P\left(X_i \ge y\right) \\ &= \overline{F}_{X_j}(y) - \overline{F}_{X_i}(y). \end{aligned} \tag{8}$$

Upon the conditioning argument, in the non-parametric prediction procedure by assuming that X_i and X_j are continuous r.v's also, we then have

$$\pi(i,j;r) = P\left(X_i \le Y_r \le X_j\right) = \int_{-\infty}^{\infty} P\left(X_i \le Y_r \le X_j \mid Y_r = y\right) dF_{Y_r}(y) = \int_{-\infty}^{\infty} p\left(X_i \le y \le X_j\right) f_{Y_r}(y) dy. \tag{9}$$

Such that, $\pi(i,j;r) = 1-\alpha$ represent the prediction coefficient which does not depend on the parameters of the parent distribution F, wich it's depends only on the r.v's positions (the indices i, j and r). Here, X_i and X_j are the lower and upper bounds of the prediction interval for Y_r respectively.

Theorem 2. Let $X_{i:n}(1 \le i \le n)$ be i^{th} order statistic from an observed random sample of size n with continuous cdf F. If $Y_{r:k:m}$, $1 \le r \le k \le m$ is the r th PCOs from a future unobserved random sample of size k -out-of m with the same cdf F, then $(X_{i:n}, X_{j:n})$, $1 \le i < j \le n$, is a DFPI for $Y_{r:k:m}$, whose coverage probability is free of F and the corresponding prediction coefficient π_2 is given by

$$\pi_2(i,j;r) = c_r' \sum_{\ell=i}^{j-1}\sum_{\nu=0}^{r-1}\binom{n}{\ell} c_{\nu,r-1}\left\{(n-\ell+R''_{\nu,r})\binom{n+R''_{\nu,r}}{\ell}\right\}^{-1} \tag{10}$$

Proof: Such $X_{i:n}$ is continuous, it is known from (7) and (8) that

$$P\left(X_{i:n} \le y \le X_{j:n}\right) = \sum_{\ell=i}^{j-1}\binom{n}{\ell}\left(F(y)\right)^{\ell}\left(1-F(y)\right)^{n-\ell}. \tag{11}$$

By conditioning on $Y_{r:k:m} = y$, and from (6) and (9), it follows that

$$P\left(X_{i:n} \le Y_{r:k:m} \le X_{j:n}\right) = c_r' \sum_{\ell=i}^{j-1}\sum_{\nu=0}^{r-1}\binom{n}{\ell} c_{\nu,r-1}\int_{-\infty}^{\infty}\left(F(y)\right)^{\ell}\left(1-F(y)\right)^{n+R''_{r,\nu}-\ell-1} f(y) dy = c_r' \sum_{\ell=i}^{j-1}\sum_{\nu=0}^{r-1}\binom{n}{\ell} c_{\nu,r-1} B\left(\ell+1, n-\ell+R''_{\nu,r}\right).$$

Equation (10) can be obtained directly by simplifying the previous beta constants, we have presented the values of $\pi_2(i,j;r)$ that given by (10) for $m:k = 20:10$ and for some selected values of the integers $n, i, j, \overline{R}$ and r in table 3. Thus, $(X_{i:n}, X_{j:n})$ is a $(\pi_2)100\%$ DFPI for the future r^{th} PCOs from a future unobserved random sample of size k, with $\pi_2(i,j;r)$ given by (10) which does not depend on F.

3.3. Based on Upper Records

Here, The prediction discussion of future PCOs based on observed upper records. Let $X_i, (i \ge 1)$ be a sequence of iid continuous r.v's with cdf F. An observation X_j is defined to be an upper record if $X_j > X_i$ for every $i < j$, one may refer to the books by Arnold et al.[17], Nevzorov[18], Gulati and Padgett[19] and Ahsanullah[20]. Let us denote the j^{th} upper record value by U_j. Then the survival function of of the U_j is given by

$$\overline{F}_{U_j}(u) = \overline{F}(u)\sum_{\ell=0}^{j-1}\frac{\{-\log \overline{F}(u)\}^{\ell}}{\ell!}. \tag{12}$$

Table 3. *The values of* $\pi_2(i,j;r)$ *for* $m:k=20:10$ *and for some selected choices of* $n,i,j,\overline{R}$ *and* r .

			j							j			
n	r	i	**4**	**6**	**8**	**10**	n	r	i	**4**	**6**	**8**	**10**
			$(0,\ldots,0,10)$							$(10,0,\ldots,0)$			
12	3	1	0.3547	0.6120	0.7927	0.8922	12	3	1	0.6749	0.8952	0.9495	0.9555
		2	0.2629	0.5203	0.7009	0.8004			2	0.4548	0.6751	0.7294	0.7354
		3	0.1384	0.3957	0.5764	0.6758			3	0.2064	0.4266	0.4810	0.4870
	6	1	0.0423	0.1445	0.3189	0.5345		6	1	0.2486	0.6218	0.9035	0.9934
		2	0.0376	0.1398	0.3142	0.5298			2	0.2195	0.5927	0.8744	0.9643
		3	0.0250	0.1272	0.3017	0.5173			3	0.1392	0.5124	0.7941	0.8840
	9	1	0.0012	0.0079	0.0329	0.0981		9	1	0.0325	0.1915	0.5537	0.9254
		2	0.0011	0.0079	0.0329	0.0980			2	0.0311	0.1902	0.5523	0.9241
		3	0.0008	0.0076	0.0326	0.0977			3	0.0237	0.1828	0.5449	0.9166
30	3	1	0.6285	0.7569	0.7870	0.7922	30	3	1	0.7547	0.7910	0.7928	0.7929
		2	0.3641	0.4925	0.5226	0.5277			2	0.3236	0.3599	0.3617	0.3618
		3	0.1456	0.2740	0.3041	0.3093			3	0.0916	0.1279	0.1297	0.1298
	6	1	0.4396	0.7421	0.8968	0.9487		9	1	0.5836	0.9105	0.9882	0.9942
		2	0.3377	0.6402	0.7948	0.8467			2	0.4713	0.7982	0.8758	0.8819
		3	0.1801	0.4826	0.6372	0.6891			3	0.2517	0.5786	0.6563	0.6623
	7	1	0.3446	0.6643	0.8682	0.9517		18	1	0.0712	0.3740	0.8006	0.9881
		2	0.2782	0.5979	0.8018	0.8854			2	0.0687	0.3715	0.7981	0.9855
		3	0.1582	0.4779	0.6817	0.7653			3	0.0527	0.3555	0.7822	0.9696
	8	1	0.2581	0.5711	0.8169	0.9387		19	1	0.0493	0.3056	0.7455	0.9817
		2	0.2165	0.5296	0.7753	0.8972			2	0.0478	0.3042	0.7441	0.9803
		3	0.1299	0.4430	0.6887	0.8106			3	0.0376	0.2939	0.7339	0.9700
	20	1	0.0003	0.0043	0.0276	0.1098		20	1	0.0329	0.2423	0.6822	0.9725
		2	0.0003	0.0043	0.0276	0.1098			2	0.0321	0.2415	0.6814	0.9717
		3	0.0003	0.0042	0.0276	0.1097			3	0.0258	0.2352	0.6751	0.9654
			$(0,\ldots,0,5,5)$							$(5,5,0,\ldots,0)$			
12	3	1	0.3547	0.6120	0.7927	0.9082	12	3	1	0.6298	0.8841	0.9483	0.9555
		2	0.2629	0.5203	0.7009	0.8164			2	0.4956	0.7499	0.8141	0.8213
		3	0.1384	0.3957	0.5764	0.6919			3	0.2307	0.4850	0.5492	0.5564
	6	1	0.0423	0.1445	0.3189	0.6066		6	1	0.2053	0.5868	0.8921	0.9927
		2	0.0376	0.1398	0.3142	0.6019			2	0.1953	0.5769	0.8821	0.9827
		3	0.0250	0.1272	0.3017	0.5894			3	0.1299	0.5114	0.8167	0.9172
	9	1	0.0012	0.0079	0.0329	0.1441		9	1	0.0233	0.1686	0.5301	0.9197
		2	0.0011	0.0079	0.0329	0.1441			2	0.0231	0.1684	0.5298	0.9195
		3	0.0008	0.0076	0.0326	0.1438			3	0.0185	0.1638	0.5253	0.9149
30	3	1	0.6285	0.7569	0.7870	0.7924	30	3	1	0.7427	0.7904	0.7928	0.7929
		2	0.3641	0.4925	0.5226	0.5280			2	0.4113	0.4590	0.4614	0.4615
		3	0.1456	0.2740	0.3041	0.3095			3	0.1184	0.1662	0.1686	0.1686
	6	1	0.4396	0.7421	0.8968	0.9532		9	1	0.5209	0.8934	0.9867	0.9942
		2	0.3377	0.6402	0.7948	0.8513			2	0.4727	0.8451	0.9385	0.9460
		3	0.1801	0.4826	0.6372	0.6937			3	0.2661	0.6386	0.7319	0.7394
	7	1	0.3446	0.6643	0.8682	0.9604		18	1	0.0507	0.3348	0.7793	0.9863
		2	0.2782	0.5979	0.8018	0.8940			2	0.0505	0.3346	0.7790	0.9861
		3	0.1582	0.4779	0.6817	0.7740			3	0.0411	0.3252	0.7697	0.9767
	8	1	0.2581	0.5711	0.8169	0.9537		20	1	0.0222	0.2112	0.6561	0.9692
		2	0.2165	0.5296	0.7753	0.9122			2	0.0222	0.2111	0.6561	0.9691
		3	0.1299	0.4430	0.6887	0.8256			3	0.0188	0.2078	0.6527	0.9658

Table 4. *The values of* $\pi_2(i,j;r)$ *for* $m:k=20:10$ *and for some selected choices of* $n,i,j,\overline{R}$ *and* r.

n	r	i	j: 4	6	8	10	n	r	i	j: 4	6	8	10
			$(5,0,\ldots,0,5)$							$(0,0,0,0,5,5,0,0,0,0)$			
12	2	1	0.6042	0.7845	0.8475	0.8638	12	3	1	0.3547	0.6548	0.9206	0.9548
		2	0.3708	0.5511	0.6142	0.6303			2	0.2629	0.5630	0.8289	0.8631
		3	0.1606	0.3409	0.4040	0.4202			3	0.1384	0.4385	0.7043	0.7385
	3	1	0.4826	0.7542	0.8927	0.9421		4	1	0.2043	0.4922	0.9018	0.9834
		2	0.3484	0.6200	0.7585	0.8079			2	0.1650	0.4529	0.8625	0.9440
		3	0.1753	0.4470	0.5855	0.6349			3	0.0958	0.3837	0.7933	0.8748
	4	1	0.3207	0.6223	0.8397	0.9456		5	1	0.1002	0.3210	0.8310	0.9879
		2	0.2562	0.5578	0.7752	0.8811			2	0.0856	0.3064	0.8164	0.9734
		3	0.1444	0.4460	0.6634	0.7692			3	0.0537	0.2745	0.7845	0.9414
	5	1	0.1848	0.4540	0.7227	0.8995		6	1	0.0423	0.1827	0.7218	0.9793
		2	0.1578	0.4270	0.6957	0.8725			2	0.0376	0.1780	0.7171	0.9746
		3	0.0972	0.3664	0.6351	0.8119			3	0.0250	0.1655	0.7046	0.9621
30	7	1	0.5024	0.8221	0.9485	0.9761	30	10	1	0.1281	0.4423	0.9279	0.9959
		2	0.3933	0.7130	0.8393	0.8670			2	0.1134	0.4277	0.9133	0.9812
		3	0.2096	0.5292	0.6556	0.6832			3	0.0743	0.3885	0.8741	0.9420
	8	1	0.4118	0.7629	0.9357	0.9820		11	1	0.0852	0.3530	0.8990	0.9964
		2	0.3380	0.6891	0.8619	0.9082			2	0.0769	0.3447	0.8907	0.9881
		3	0.1920	0.5430	0.7159	0.7622			3	0.0522	0.3200	0.8660	0.9634
	9	1	0.3260	0.6891	0.9093	0.9808		12	1	0.0546	0.2726	0.8621	0.9956
		2	0.2778	0.6408	0.8610	0.9326			2	0.0501	0.2681	0.8576	0.9911
		3	0.1667	0.5297	0.7499	0.8215			3	0.0351	0.2531	0.8425	0.9761
			$(0,1,0,1,3,0,1,1,0,3)$							$(2,3,0,4,1,0,\ldots,0)$			
12	3	1	0.3698	0.6748	0.8707	0.9437	12	3	1	0.4593	0.8300	0.9422	0.9554
		2	0.2780	0.5830	0.7790	0.8519			2	0.3536	0.7243	0.8365	0.8497
		3	0.1458	0.4508	0.6467	0.7197			3	0.1799	0.5506	0.6628	0.6759
	6	1	0.0469	0.2010	0.5012	0.8312		6	1	0.0824	0.4469	0.8425	0.9891
		2	0.0422	0.1963	0.4965	0.8265			2	0.0762	0.4408	0.8364	0.9830
		3	0.0283	0.1824	0.4825	0.8126			3	0.0515	0.4160	0.8116	0.9582
	9	1	0.0014	0.0158	0.0980	0.3857		9	1	0.0039	0.0932	0.4416	0.8964
		2	0.0013	0.0157	0.0980	0.3856			2	0.0038	0.0931	0.4415	0.8963
		3	0.0010	0.0154	0.0976	0.3853			3	0.0030	0.0922	0.4407	0.8955
30	3	1	0.6379	0.7685	0.7910	0.7928	30	3	1	0.6845	0.7866	0.7927	0.7929
		2	0.3735	0.5040	0.5265	0.5284			2	0.3962	0.4983	0.5045	0.5046
		3	0.1469	0.2775	0.3000	0.3018			3	0.1419	0.2440	0.2502	0.2503
	6	1	0.4583	0.7979	0.9365	0.9612		10	1	0.2255	0.7496	0.9722	0.9967
		2	0.3564	0.6960	0.8346	0.8593			2	0.2063	0.7304	0.9531	0.9775
		3	0.1888	0.5284	0.6670	0.6917			3	0.1331	0.6572	0.8798	0.9043
	7	1	0.3633	0.7364	0.9324	0.9764		11	1	0.1649	0.6820	0.9601	0.9978
		2	0.2969	0.6700	0.8660	0.9100			2	0.1537	0.6709	0.9489	0.9867
		3	0.1680	0.5412	0.7372	0.7812			3	0.1036	0.6207	0.8988	0.9365
	8	1	0.2754	0.6565	0.9110	0.9826		12	1	0.1168	0.6090	0.9422	0.9980
		2	0.2338	0.6149	0.8695	0.9411			2	0.1105	0.6027	0.9359	0.9917
		3	0.1400	0.5211	0.7756	0.8473			3	0.0774	0.5696	0.9028	0.9586
	9	1	0.2004	0.5663	0.8743	0.9822		13	1	0.0801	0.5333	0.9182	0.9974
		2	0.1753	0.5413	0.8492	0.9571			2	0.0767	0.5299	0.9148	0.9940
		3	0.1102	0.4761	0.7841	0.8920			3	0.0556	0.5088	0.8937	0.9729

Theorem 3. Let $U_j,(j\geq 1)$ be j^{th} upper record from an observed random sample of size n with continuous cdf F. If $Y_{r:k:m}$, $1\leq r\leq k\leq m$ is the r th PCOs from a future unobserved sample of size k with the same cdf F, then (U_i,U_j), $1\leq i<j$, is DFPI for the future r^{th} PCOs, $X_{r:k:m}$, whose coverage probability is free from F, and is given by

$$\pi_3(i,j;r)=c_r'\sum_{\ell=i}^{j-1}\sum_{\nu=0}^{r-1}\frac{c_{\nu,r-1}}{\left(R_{\nu,r}''+1\right)^{\ell+1}}. \tag{13}$$

Proof: For a given y, by assuming that the records $U_j,(j\geq 1)$ are continuous r.v's, and from (12) and (8) it is now easy to write (see,[6])

$$p\left(U_i\leq y\leq U_j\right)=\overline{F}(y)\sum_{\ell=i}^{j-1}\frac{\{-\log\overline{F}(y)\}^{\ell}}{\ell!}. \tag{14}$$

The same method as a proof of Theorem 1; and from (14) and (9), we readily obtain

$$p\left(U_i\leq Y_{r:k:m}\leq U_j\right)=\int_0^1 y\sum_{\ell=i}^{j-1}\frac{\{-\log y\}^{\ell}}{\ell!}c_r'\sum_{\nu=0}^{r-1}c_{\nu,r-1}y^{R_{\nu,r}''-1}dy=c_r'\sum_{\ell=i}^{j-1}\sum_{\nu=0}^{r-1}c_{\nu,r-1}\int_0^1\frac{\{-\log y\}^{\ell}}{\ell!}y^{R_{\nu,r}''}dy. \tag{15}$$

The values of $\pi_3(i,j;r)$ are presented in table 5 for $m:k=30:25;50:45$ and for some selected choices of

$i, j, \overline{R}$ and r. (U_i, U_j) is a $(\pi_3)100\%$ PIs for the future r^{th} PCOs, such $\pi_3(i, j; r)$ given by (13).

3.4. Based on Current Records

Suppose $X_i, (i \geq 1)$ is a sequence of iid continuous r.v's with cdf F. Let us denote the i^{th} upper records by U_i (with $U_1 \equiv R_0^\ell \equiv X_1$). Now, let R_m^ℓ be the largest observation, at the time when the i^{th} upper record occurs in the X-sequence. Then, the marginal density and the survival functions of R_i^ℓ (see Arnold et al. [21], p. 276), are given respectively by

$$f_{R_i^\ell}(y) = 2^i f(y)\left(1 - \overline{F}(y)\sum_{\lambda=0}^{i-1}\frac{\{-log\overline{F}(y)\}^\lambda}{\lambda!}\right), \tag{16}$$

$$\overline{F}_{R_i^\ell}(u) = \overline{F}(u)\left[2^i - \overline{F}(u)\sum_{\lambda=0}^{i-1}(2^i - 2^\lambda)\frac{\{-log\overline{F}(u)\}^\lambda}{\lambda!}\right]. \tag{17}$$

DFPI of future PCOs based on observed upper records are discussed as the same method of Theorem 1 In the following, let $Y_{r:k:m}$ be a future r^{th} PCOs from the Y-sequence that we are interested in obtaining $100(1-\alpha)\%$ PIs for it of the form $\left(R_i^l, R_j^l\right)$ such that R_j^l is the j^{th} largest record and $p\left(R_i^l \leq y \leq R_j^l\right) \geq 1-\alpha$.

Lemma 3.

Let $R_j^l, (j \geq 1)$ be j^{th} largest current record from an observed random sample with continues cdf F. If $Y_{r:k:m}$, $1 \leq r \leq k \leq m$ is the r th PCOs from a future unobserved random sample of size k with the same cdf F, then $\left(-\infty, R_i^l\right)$, $1 \leq i \leq j$, is DF one-sided PI for the future $Y_{r:k:m}$, with the corresponding prediction coefficient $\varphi_2(i;r)$, that does not depend on F, and is given by

$$\varphi_2(i;r) = c_r^{'}\sum_{\nu=0}^{r-1}c_{\nu,r-1}\left[\frac{2^i}{R_\nu^{''}+1} - \sum_{\lambda=0}^{i-1}\frac{(2^i - 2^\lambda)}{\{R_\nu^{''}+2\}^{\lambda+1}}\right]. \tag{18}$$

Proof: From (17) and (1), we get

$$\varphi_2(i;r) = p\left(R_i^l \geq Y_{r:k:m}\right) = \int_{-\infty}^{\infty} p\left(R_i^l \geq y\right) f_{Y_{r:k:m}}(y)dy = c_r^{'}\sum_{\nu=0}^{r-1}c_{\nu,r-1}\left[2^i\int_0^1 y^{R_\nu^{''}}dy - \sum_{\lambda=0}^{i-1}(2^i - 2^\lambda)\int_0^1 y^{R_\nu^{''}+1}\frac{\{-\log y\}^\lambda}{\lambda!}dy\right]. \tag{19}$$

Table 5. The values of $\pi_3(i, j; r)$ for $m:k = 30:25; 50:45$ and for some selected choices of $i, j, \overline{R}$ and r.

$m:k$	r	i	j 15	j 25	j 35	$m:k$	r	i	j 15	j 25	j 35
			(2,2,1,0,...,0)						(5,0,...,0)		
:25	10	1	0.3700	0.3700	0.3700	50:45	30	1	0.6514	0.6514	0.6514
		2	0.0858	0.0858	0.0858			2	0.2910	0.2910	0.2910
		3	0.0150	0.0150	0.0150			3	0.0980	0.0980	0.0980
	15	1	0.5669	0.5669	0.5669		40	1	0.8693	0.8693	0.8693
		2	0.2152	0.2152	0.2152			2	0.6125	0.6125	0.6125
		3	0.0622	0.0622	0.0622			3	0.3516	0.3516	0.3516
	20	1	0.7938	0.7938	0.7938		43	1	0.9346	0.9346	0.9346
		2	0.4373	0.4373	0.4373			2	0.7659	0.7659	0.7659
		3	0.1982	0.1982	0.1982			3	0.5396	0.5396	0.5396
	25	1	0.9599	0.9599	0.9599		45	1	0.9769	0.9769	0.9769
		2	0.8484	0.8491	0.8484			2	0.9025	0.9038	0.9025
		3	0.7606	0.7606	0.7606			3	0.7687	0.7687	0.7700

Theorem 4. Under the assumption of lemma 3, then $\left(R_i^l, R_j^l\right)$, $1 \leq i \leq j$, is DFPI for the future $Y_{r:k:m}$, with the corresponding prediction coefficient $\pi_4(i, j; r)$ does not depend on F.

Proof: Based on lemma 3, and by assuming that the current records $R_j^l, (j \geq 1)$ are continuous r.v's, we get

$$\pi_4(i, j; r) = p\left(R_i^l \leq Y_{r:k:m} \leq R_j^l\right) = p\left(R_j^l \geq Y_{r:k:m}\right) - p\left(R_i^l \geq Y_{r:k:m}\right) = \varphi_2(j;r) - \varphi_2(i;r). \tag{20}$$

The values of $\pi_4(i, j; r)$ are presented in table 6, for $m:k = 30:25; 50:45$ and for some selected choices of

$i, j, \overline{R}$ and r. Such $\pi_4(i, j; r)$ is given by (20) which does not depend on the parent distribution F.

Table 6. *The values of* $\pi_4(i, j; r)$ *for* $m : k = 30 : 25; 50 : 45$ *and for some selected choices of* $i, j, \overline{R}$ *and* r.

			j				j			
			15	**25**	**35**	**45**	**15**	**25**	**35**	**45**
$m : k$	r	i	$(0,2,0,3,0,...,0)$				$(5,0,...,0)$			
30:25	10	1	0.145	0.145	0.145	0.145	0.154	0.154	0.154	0.154
		2	0.046	0.046	0.046	0.046	0.050	0.810	0.870	0.932
		3	0.012	0.012	0.012	0.012	0.014	0.014	0.014	0.014
	15	1	0.331	0.331	0.331	0.331	0.339	0.339	0.339	0.339
		2	0.160	0.160	0.160	0.160	0.166	0.166	0.166	0.166
		3	0.066	0.066	0.066	0.066	0.070	0.070	0.070	0.070
	20	1	0.590	0.590	0.590	0.590	0.596	0.596	0.596	0.596
		2	0.403	0.403	0.403	0.403	0.409	0.409	0.409	0.409
		3	0.244	0.244	0.244	0.244	0.250	0.250	0.250	0.250
	25	1	0.899	0.924	0.924	0.924	0.900	0.925	0.925	0.925
		2	0.840	0.865	0.866	0.866	0.842	0.867	0.868	0.868
		3	0.759	0.783	0.784	0.784	0.761	0.786	0.787	0.787
50:45	30	1	0.427	0.427	0.427	0.427	0.429	0.429	0.429	0.429
		2	0.234	0.234	0.234	0.234	0.236	0.236	0.236	0.236
		3	0.110	0.110	0.110	0.110	0.111	0.111	0.111	0.111
	35	1	0.580	0.580	0.580	0.580	0.582	0.582	0.582	0.582
		2	0.387	0.387	0.387	0.387	0.389	0.389	0.389	0.389
		3	0.226	0.226	0.226	0.226	0.227	0.227	0.227	0.227
	40	1	0.757	0.757	0.757	0.757	0.758	0.758	0.758	0.758
		2	0.606	0.606	0.606	0.606	0.608	0.608	0.608	0.608
		3	0.442	0.442	0.442	0.442	0.443	0.443	0.443	0.443
	45	1	0.914	0.956	0.957	0.957	0.914	0.956	0.957	0.957
		2	0.877	0.920	0.921	0.921	0.877	0.920	0.921	0.921
		3	0.821	0.864	0.865	0.865	0.821	0.864	0.865	0.865

4. Real Life Data Example

The real life-observations that given in Nelson [22] are considered in this section to illustrate the derived results. These data which was also used in Lawless ([23], p. 185), The observation is data time to breakdown of an insulating fluid between electrodes at a voltage of 34 kV (minutes). The 19 times to breakdown are contained in the following table (*)

0.96	4.15	0.19	0.78	8.01	31.75	7.35	6.50	8.27	33.91
32.52	3.16	4.85	2.78	4.67	1.31	12.06	36.71	72.89	

The observed PCOs from the data (*) as $n : \mu = 19 : 14$ with the progressive censoring scheme $(0,...,0,5)$ can be:

0.19	0.78	0.96	2.78	3.16	4.15	4.85
6.50	7.35	8.01	8.27	31.75	32.52	33.91

Also, the usual order statistics values that obtained from the sample (*) will be:

0.19	0.78	0.96	1.31	2.78	3.16	4.15	4.67	4.85	6.50
7.35	8.01	8.27	12.06	31.75	32.52	33.91	36.71	72.89	

Moreover, we can obtain from the sample (*) the following eight upper records:

0.19	0.96	4.15	8.01	31.75	33.91	36.71	72.89

Based on these fourteen PCOs with $\overline{R}^{*} = (0,...,0,5)$, the results of DFPI of future r th PCOs $X_{r:10:15}$, $1 \le r \le 10$ from unobserved PCOs of size 10 with $\overline{R} = (0,...,0,5)$ from a future Y-sample of size 15 are displayed in table 7. Also, based on the above order statistics obtained, DFPI of future r th PCOs $X_{r:10:15}$, $1 \le r \le 10 \le 15$ from a future unobserved random sample of size 10 with $\overline{R} = (0,...,0,5)$ from a future Y-sample of size 15, with corresponding prediction coefficient of at least 0.80 are displayed in table 8. In table 9, we have presented the values of $\pi_3(1,5;r)$ for $m:k = 100:90$, for some selected choices of r based on the real observed upper records. Such $\pi_3(i,j;r)$ is given by (13).

Table 7. *PIs for future* r^{th} *PCOs* $Y_{r:10:15}, 1 \le r \le 10 \le 15$ *from a future unobserved random sample of size* 10 *from Y-sample, based on the real observed PCOs.*

r	(i,j)	$(X_{i:14:19}, X_{j:14:19})$	π_1	r	(i,j)	$(X_{i:14:19}, X_{j:14:19})$	π_1
	(1,5)	(0.19,3.16)	0.5170	6	(3,13)	(0.96,32.52)	0.7318
	(1,5)	(0.19,3.16)	0.6251	7	(5,14)	(3.16,33.91)	0.6423
	(1,7)	(0.19,4.85)	0.6987	8	(5,14)	(3.16,33.91)	0.4364
	(1,9)	(0.19,7.35)	0.7203	9	(5,14)	(3.16,33.91)	0.2186
	(2,12)	(0.78,31.75)	0.8012	10	(5,14)	(3.16,33.91)	0.0614

Table 8. *DFPI for future* r^{th} *PCOs* $Y_{r:10:15}, 1 \le r \le 10$ *, based on the real observed ordered statistics sample .*

r	(i,j)	$(X_{i:19}, X_{j:19})$	π_2	r	(i,j)	$(X_{i:19}, X_{j:19})$	π_2
1				6	(2,14)	(0.96,8.27)	0.8042
2	(1,8)	(0.19,2.78)	0.8045	7	(5,16)	(2.78,12.06)	0.8246
3	(1,9)	(0.19,2.78)	0.8181	8	(7,18)	(2.78,12.06)	0.8778
4	(1,12)	(0.19,2.78)	0.8186	9	(3,19)	(2.78,12.06)	0.8891
5	(3,13)	(0.78,8.01)	0.8295	10			

Table 9. *DFPI for some future* r^{th} *PCOs* $Y_{r:90:100}, 1 \le r \le 90 \le 100$ *from a future unobserved random sample of size* 90 *from Y-sample, based on above upper records.*

r	(i,j)	(U_1,U_5)	π_3	r	(i,j)	(U_1,U_5)	π_3
10	(1,5)	(0.19,31.75)	0.0990	60	(1,5)	(0.19,31.75)	0.5913
20			0.1980	70			0.6848
30			0.2970	80			0.7674
40			0.3958	90			0.8085
50			0.4942				

5. Conclusion

In this article, based on progressive Type-II right-censored, ordered statistics, record values and current records, distribution-free PIs for future progressively Type-II right censored order statistics are derived. The obtained results in all tables show that the DFPI decreases with decreasing PI length. Also, the corresponding coefficient increases with decreasing the past sample size. Moreover it may be noted from the results, the prediction coefficients for a future r^{th} PCOs $Y_{r:k:m}, 1 \le r \le k \le m$ increase as k increase, even if the censored observations size $m-k$ increase. Based on results of tables (5,6 and 8), DFPI for PCOs based on upper (current) records may be advised only for higher r^{th} PCOs.

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