Biomedical Statistics and Informatics
2017; 2(3): 128-130
http://www.sciencepublishinggroup.com/j/bsi
doi: 10.11648/j.bsi.20170203.17

Some Results on the Bounded Nadir's Operator

Mostefa Nadir

Department of Mathematics, Faculty of Mathematics and Informatics University of Msila, Msila, Algeria

Email address:
mostefanadir@yahoo.fr

To cite this article:
Mostefa Nadir. Some Results on the Bounded Nadir's Operator. *Biomedical Statistics and Informatics*. Vol. 2, No. 3, 2017, pp. 128-130. doi: 10.11648/j.bsi.20170203.17

Received: July 24, 2017; **Accepted**: August 15, 2017; **Published**: September 4, 2017

Abstract: In this paper, we present some new results for the Nadir's operator such the normality, the skew normality and the compactness of this operator and study its invertibility in the algebra of all bounded linear operators on a complex separable Hilbert space.

Keywords: Skew Operator, Compact Operator, Normal Operator

1. Introduction

We call Nadir's operator all operators of the form $N = AB^* - BA^*$ where the two operators A and B are bounded or unbounded operators acting on a Hilbert space is a central concept in quantum mechanics, since it quantifies how well the two observables described by these operators can be measured simultaneously. From this definition, we can see that if the operator N is zero, then $AB^* = BA^*$ represents in a sense the degree to which $\left(AB^*\right)$ is equal to its adjoint $\left(AB^*\right)^*$ so, the order of which the two corresponding measurements are applied to the physical system does not matter. On the contrary, if it is non-zero, then the order does matter also we reveal that the identity of any such algebra is never a Nadir's operator. This translates in the Banach algebra $B(H)$ with unit element that the operator $I+K$ where K is a compact operator is non Nadir's operator.

2. Main Results

Theorem 2.1

Let $B(H)$ be a Banach algebra with unit element I, then for all operators A and B in $B(H)$. The operator $N = AB^* - BA^*$ is a Skew self-adjoint operator and never equal to the identity I. In other words

$$N = AB^* - BA^* \neq I$$

Proof

Indeed, we have

$$\begin{aligned} N^* &= \left(AB^* - BA^*\right)^* \\ &= \left(AB^*\right)^* - \left(BA^*\right)^* \\ &= BA^* - AB^* \\ &= -\left(AB^* - BA^*\right) \\ &= -N. \end{aligned}$$

Assume that, $N = AB^* - BA^* = I$, it follows $N^* = \left(AB^* - BA^*\right)^* = I^* = I.$ Hence, from the relation $N^* = -N$, we get

$$I = -I.$$

Contradiction

Proposition 2.1

Let A and B be two normal operators. If A commutes with B^*, then operator sum $A+B$ is normal.

Indeed, we have

$$(A+B)^*(A+B) = (A^*+B^*)(A+B) = A^*A + A^*B + B^*A + B^*B$$

and

$$(A+B)(A+B)^* = AA^* + AB^* + BA^* + BB^*$$

Since $AB^* = B^*A$, $BA^* = A^*B$. Combining these two equations with the normality of A and B yield the equality of the first and the second expressions above and hence establishing the normality of $A+B$.

Remark 2.1

The converse of the previous proposition is not generally true.

Remark 2.2

The sum of two normal operators is not generally a normal operator

Example 2.1

Consider the matrices A and B defined as

$$A=\begin{pmatrix}-2 & 3\\ 3 & 2\end{pmatrix},\quad B=\begin{pmatrix}2 & -3\\ 3 & 2\end{pmatrix},\quad A+B=\begin{pmatrix}0 & 0\\ 6 & 4\end{pmatrix}$$

$$AA^*=A^*A=\begin{pmatrix}13 & 0\\ 0 & 13\end{pmatrix},\quad BB^*=B^*B=\begin{pmatrix}13 & 0\\ 0 & 13\end{pmatrix}$$

$$(A+B)(A+B)^*=\begin{pmatrix}0 & 0\\ 0 & 52\end{pmatrix},\quad (A+B)^*(A+B)=\begin{pmatrix}36 & 24\\ 24 & 16\end{pmatrix}$$

So,

$$(A+B)(A+B)^*\neq(A+B)^*(A+B)$$

Proposition 2.2

For all bounded operators A and B, the operator $N=AB^*-BA^*$ is normal

Indeed, it follows from the theorem 2.1

$$\begin{aligned}NN^*&=(AB^*-BA^*)(AB^*-BA^*)^*\\&=N(-N)\\&=(-N)N\\&=N^*N\end{aligned}$$

Corollary

The operator N^2 is negative, that is to say $\langle N^2x,x\rangle\leq 0$ for all non-zero vectors x in H.

Indeed, it is known that the operator NN^* is always positive and from the proposition 2.1

$$NN^*=(-N)N=-N^2$$

Theorem 2.2

Let A, B be a bounded operator on the Hilbert space H where one of the operators A and B is compact, then the operator $N=AB^*-BA^*$ is compact with the operator $I-N$ is invertible.

1. First case A is compact B bounded

A compact $\Rightarrow A^*$ compact $\Rightarrow AB^*$ and BA^* are compacts. Hence N is compact.

2. Second case B is compact A bounded

B compact $\Rightarrow B^*$ compact $\Rightarrow AB^*$ and BA^* are compacts. Hence N is compact.

Besides, it is known that, the operator N is never equal to the identity I, then $N-I\neq 0$ and so, $N-I$ is injective. Hence $N-I$ is bijective.

Theorem 2.3

Let A, B be a bounded operator on the Hilbert space H where one of the operators A and B is compact, then

1. the operator $N=AB^*-BA^*$ has a eigenvalue satisfying $\lambda_1=\|N\|$
2. If λ is scalar and x in H then $Nx=\lambda x$ if and only if $N^*x=\bar{\lambda}x$
3. Every eigenvalue of N has an index one

Proof

Indeed, the spectral radius of N is $\|N\|$, so there exists an element λ_1 in $\sigma(N)$ satisfying

$$\lambda_1=\|N\| \quad (1)$$

By the compactness of the operator N, we have $\|Nx\|=\|N^*x\|$ for each $x\in H$. Replacing the operator N by the normal operator $N-I$ we obtain

$$\|Nx-\lambda x\|=\|N^*x-\bar{\lambda}x\|$$

which proves (2)

Suppose that λ is a eigenvalue of the operator N, and $(N-\lambda)^2x=0$, for $x\in H$, by the relation (2) we can write

$$(N^*-\bar{\lambda}I)(N-\lambda I)x=0.$$

Hence

$$\begin{aligned}\|(N-\lambda I)x\|^2&=\langle(N-\lambda I)x,(N-\lambda I)x\rangle\\&=\langle(N-\lambda I)^*(N-\lambda I)x,x\rangle=0,\end{aligned}$$

and therefore $(N-\lambda)x=0.$ This proves (3).

3. Conclusion

There are various uses of Nadir's operator, but the most obvious one is this. Given two operators AB^* and BA^* that correspond to physically observable quantities, if their difference is 0, it means we can simultaneously measure both physical quantities. If their difference is non-zero, it means that we cannot simultaneously know the values of both those observables, and that there is an uncertainty relationship between those observables, with the Nadir's operator telling you how much your knowledge of one observable limits how well you can measure the other.

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