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Modified extended median test for c matched samples

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Abstract: This work presented an extended median test for analyzing samples that are not independent but paired or matched given some criteria. Here, the data for analysis are presented in table form with the column corresponding to one factor with 'c' treatments or conditions considered as fixed, while the row as second factor with say 'k' subjects ,batches, blocks or levels which are considered random given that there is only one observation per cell. These observations themselves may be measurements on as low as the ordinal scale. The null hypothesis to be tested was that there is no difference between the 'c' treatments, thus having equal medians. This required the use of Friedman test and an alternative ties adjusted method. Although these methods lead to the same conclusions, the relative sizes of the calculated chi-square values suggest that the Friedman test is likely to lead to an acceptance of a false null hypothesis (Type II error) more frequently and hence likely to be less powerful than the ties adjusted modified extended median test. Nevertheless, the Friedman's two-way analysis of variance test by ranks is here at least shown to be still more powerful than the usual extended median test.

Keywords: Friedman Test, Treatment, Modified Extended Median Test, Measurements, Matched Samples, Ties Adjusted Method

1. Introduction

The extended median test may also be used to analyze samples that are not independent but which are paired or matched on the basis of some criteria. As before the sampled populations may be measurements on as low as the ordinal scale and need not be continuous or numeric (Gibbon, 1971).Also the data may be presented in the form of a table with say the column corresponding to one factor with 'c' treatments or conditions which are considered fixed, and the row corresponding to a second factor with say 'k' subjects, batches, blocks or levels which are considered random and there is only one observation per cell. The null hypothesis to be tested is that there is no difference between the 'c' treatments while the alternative hypothesis is that the treatments do in fact differ (Agresti, 1992). Let x_{ij} be the response or score of a randomly selected subject in the ith batch of subjects, plots, levels or treatment blocks to the jth treatment or condition for i=1,2,…,k and j=1,2,…,c. We assume that the 'k' batches or blocks of subjects constitute a random sample of subjects with each sample having 'c' members matched on certain characteristics and each randomly assigned to one of the 'c' treatments or experimental conditions which are considered fixed. The observations themselves may be measurements on as low as the ordinal scale. Interest is in determining whether these treatments have equal medians. Hence the data is appropriate for analysis using the Friedman test. But an alternative method also exists. Unlike is the case with samples drawn from independent populations where subjects are matched across treatments. The approach using the median test here is not to find the common median of a combined sample. Instead, we first find the median of each experimental plot or treatment block across the 'c' treatments. Subsequent analyses are then based on the medians of these batches or blocks of subjects across the 'c' treatments. Now for each treatment the number of observations that are greater than, that is above the various plot medians, and the number of observations that are less than, that is fall below the medians are determined and tallied. These are used to set up a 2xc contingency table for an extended median test of the null hypothesis of equal population medians (Agresti, 1992). This approach however assumes that no observations in a given batch or block are exactly equal to the median of that block, that is that there are no ties

between observations in a given median of that block of observations. A possible solution to this problem is to either ignore for each treatment observations in a block that are tied with that blocks median score or randomly assign such tied observation to either one or the other of the two portions of each treatment into which the observations for that treatment have been dichotomized by the median of that block (Munzel and Brunner, 2002). However, unless there are very few such tied observations, these approaches may lead to serious loss of information and erroneous conclusions. A more general solution to the problem of ties may be adopted that intrinsically and structurally adjust the test statistic to provide for the possibility of ties between observations or scores in a given batch or block of observations with the median score for that block (Oyeka and Okeh, 2012.). Thus if in particular too many ties occur, that is if many observations are exactly equal to the median of the ith block of subjects, for i=1,2,..,k then the test statistic for the equality of population or treatment medians should preferably be adjusted to provide for these ties. Specifically suppose that M_i is the median of the scores or responses by the ith batch or block of subjects to the c treatments for i=1,2,…,k. To adjust for the possibility of ties between M_i and observations for the ith batch or block of subjects, we may let

$$u_{ij}=\begin{cases}1, if\ x_{ij}\ is\ a\ higher(better, larger)score\ or\ observation\ than\ M_i; or\ x_{ij}<M_i\\ 0, if\ x_{ij}\ is\ the\ same\ score\ as\ (equal\ to)\ M_i; or\ x_{ij}=M_i\\ -1, if\ x_{ij}\ is\ a\ lower(worse, smaller)score\ or\ observation\ than\ M_i; or\ x_{ij}<M_i\end{cases} \quad (1)$$

A test statistic for the null hypothesis of equal population median may be constructed be determining the sampling distribution of W of 5 based on Equations 1 to 8 but this approach would here develop an equivalent test statistic based on the chi-square test for independence.

Now $\pi_j^+;\pi_j^0\ and\ \pi_j^-$ are respectively probabilities. Observations for the jth treatment are on the average higher (better, larger), the same as (equal to), or lower (worse, smaller) than the medians of all the 'k' blocks. These sample estimates are respectively.

$$for\ i=1,2,...,k;\ j=1,2,...,c$$

Let

$$\pi_j^+=P(u_{ij}=1);\pi_j^0=P(u_{ij}=0);\pi_j^-=P(u_{ij}=-1) \quad (2)$$

where

$$\pi_j^+ + \pi_j^0 + \pi_j^- = 1 \quad (3)$$

Let

$$W_j=\sum_{i=1}^{k} u_{ij} \quad (4)$$

And

$$W=\sum_{j=1}^{c} W_j=\sum_{j=1}^{c}\sum_{i=1}^{k} u_{ij} \quad (5)$$

Now

$$E(u_{ij})=\pi_j^+-\pi_j^-;Var(u_{ij})=\pi_j^++\pi_j^--(\pi_j^+-\pi_j^-)^2 \quad (6)$$

Also

$$E(W_j)=\sum_{i=1}^{k} E(u_{ij})=k(\pi_j^+-\pi_j^-) \quad (7)$$

And

$$Var(W_j)=\sum_{i=1}^{k} Var(u_{ij})=k\left(\pi_j^++\pi_j^--(\pi_j^+-\pi_j^-)^2\right) \quad (8)$$

$$P_j^+=\hat{\pi}_j^+=\frac{f_j^+}{k};P_j^0=\hat{\pi}_j^0=\frac{f_j^0}{k};P_j^-=\hat{\pi}_j^-=\frac{f_j^-}{k} \quad (9)$$

where $f_j^+, f_j^0\ and\ f_j^-$ are respectively the number of observations or scores in the jth treatment that are higher,(better, larger),the same as (equal to);or lower (worse, smaller) than the medians of all the k blocks. That is $f_j^+, f_j^0\ and\ f_j^-$ are respectively the number of 1s,0s and -1s in the frequency distribution of the 'k' values of these numbers in u_{ij} ;i=1,2,…,k; for each j=1,2,…,c. Note that

$$f_j^0=k-f_j^+-f_j^-;P_j^0=1-P_j^+-P_j^- \quad (10)$$

If we let

f_j^+ , f_j^0 and f_j^-

be respectively the total number of subjects in all batches or blocks whose scores are higher (better, larger) the same as (equal to), lower(worse, smaller) than the medians of the 'k' blocks for all the 'c' treatments; that is $f_j^+, f_j^0\ and\ f_j^-$ are respectively the total number of 1s,0s and -1s in the frequency distribution of the k-c values of these numbers in u_{ij} ,for j=1,2,..,k;and j=1,2,…,c, then we have that

$$f^+=\sum_{j=1}^{c} f_j^+;f^0=\sum_{j=1}^{c} f_j^0;f^-=\sum_{j=1}^{c} f_j^- \quad (11)$$

The corresponding sample proportions are

$$P^+=\frac{f^+}{kc};P^0=\frac{f^0}{kc};P^-=\frac{f^-}{kc} \quad (12)$$

Note that

$$f^0=kc-f^+-f^-;P^0=1-P^+-P^- \quad (13)$$

Note that the observed number of times scores in the jth treatment are higher (better, larger) than, lower (worse, smaller) than, or the same as (equal to) all the k block medians are respectively

$$O_{1j}=f_j^+;O_{2j}=f_j^-;O_{3j}=f_j^0=k-f_j^+-f_j^- \quad (14)$$

These results are summarized in Table 1

Table 1. *3xc contingency Table for the analysis of c matched samples with ties*

Relation to Batch median M_i	**1**	**2**	**C**	**Total(f)**	**Proportion(P)**
Above Median(1)	f_1^+	f_2^+	f_c^+	f^+	P^+
Below Median (-1)	f_1^-	f_2^-	f_c^-	f^-	P^-
Equal to median (0)	$f_1^0=k-f_1^+-f_1^-$	$f_2^0=k-f_2^+-f_2^-$	$f_c^0=k-f_c^+-f_c^-$	$f^0=kc-f^+-f^-$	$P_1^0=1-P_1^+-P_1^-$
Total (k)	k	k	K	Kc	
Proportion Above median P_j^+	P_1^+	P_2^+	P_c^+		P^+
Proportion below median P_j^-	P_1^-	P_2^-	P_c^-		P^-
Proportion equal median P_j^0	$P_1^0=1-P_1^+-P_1^-$	$P_2^0=1-P_2^+-P_1^-$	$P_c^0=1-P_c^+-P_c^-$		$1-P^+-P^-$

Now under the null hypothesis of equal population or treatment effects the expected number of observations in the jth treatment that are higher (better, larger) or lower (worse, smaller) than or the same as (equal to) all the k block medians are respectively

$$E_{1j}=\frac{kf^+}{kc}=\frac{f^+}{c};E_{2j}=\frac{kf^-}{kc}=\frac{f^-}{c};E_{3j}=\frac{k(kc-f^+-f^-)}{kc}=\frac{kc-f^+-f^-}{c} \quad (15)$$

Hence under the null hypothesis of equal treatment effects or equal population medians the test statistic

$$\chi^2=\sum_{j=1}^{c}\sum_{i=1}^{3}\frac{\left(o_{ij}-E_{ij}\right)^2}{E_{ij}} \quad (16)$$

Has approximately a chi-square distribution with (3-1)(c-1)=2(c-1) degree of freedom for sufficiently large k and c $(kc\geq 20)$ where o_{ij} *and* E_{ij} are given in equation

$$\chi^2=\frac{c}{f^+-f^-\left(kc-f^+-f^-\right)}\left((f^-(kc-f^-)\sum_{j=1}^{c}\left(f_j^+-\frac{f^+}{c}\right)^2+f^+(kc-f^+)\sum_{j=1}^{c}\left(f_j^--\frac{f^-}{c}\right)^2+2f^+f^-\sum_{j=1}^{c}\left(f_j^+-\frac{f^+}{c}\right)\left(f_j^--\frac{f^-}{c}\right)\right) \quad (17)$$

Which under H_0 has approximately the chi-square distribution with 2(c-1) degrees of freedom for sufficiently large 'k' and 'c', $(kc\geq 20)$ and may be used to test the null hypothesis of equal population medians. The null hypothesis is rejected at the α level of significance if

$$\chi^2\geq\chi^2_{1-\alpha;2(c-1)} \quad (18)$$

14 and 15 respectively. Using these Equations in Equation 16 we have

$$\chi^2=\frac{\sum_{j=1}^{c}\left(f_j^+-\frac{f^+}{c}\right)^2}{\frac{f^+}{c}}+\frac{\sum_{j=1}^{c}\left(f_j^--\frac{f^-}{c}\right)^2}{\frac{f^-}{c}}+\frac{\sum_{j=1}^{c}\left(\left(k-f_j^+-f_j^-\right)-\frac{\left(kc-f^+-f^-\right)}{c}\right)^2}{\frac{kc-f^+-f^-}{c}}$$

$$=\frac{c\left(\sum_{j=1}^{c}f_j^+-\frac{f^+}{c}\right)^2}{f^+}+\frac{\sum_{j=1}^{c}\left(f_j^--\frac{f^-}{c}\right)^2}{f^-}+\frac{\sum_{j=1}^{c}\left(\left(f_j^+-\frac{f^+}{c}\right)+\left(f_j^--\frac{f^-}{c}\right)\right)^2}{kc-f^+-f^-}$$

This expression when further simplified reduces to the test statistic

Otherwise H_0 is accepted.

An equivalent test statistic in terms of the sample proportions of Equations 9 and 12 is

An equivalent and easier to use computational formulae is

$$\chi^2 = \frac{k}{P^+P^-(1-P^+-P^-)}\left((P^-(1-P^-)\sum_{j=1}^{c}\left(P_j^+-P^+\right)^2 + P^+(1-P^+)\sum_{j=1}^{c}\left(P_j^--P^-\right)^2 + 2P^+P^-\sum_{j=1}^{c}\left(P_j^+-P^+\right)\left(P_j^--P^-\right)\right) \quad (19)$$

$$\chi^2 = \frac{k}{P^+P^-(1-P^+-P^-)}\left((P^-(1-P^-)\left(\sum_{j=1}^{c}\left(P_j^{+2}-cp^{+2}\right)\right) + P^+(1-P^+)\left(\sum_{j=1}^{c}P_j^{-2}-cp^{-2}\right) + 2P^+P^-\left(\sum_{j=1}^{c}P_j^+P_j^- - cp^+p^-\right)\right) \quad (20)$$

which also has a chi-square distribution with 2(c-1) degrees of freedom. If there are only two populations or treatments, that is 'c'=2, then we would have that $P_1^+ = P_2^-, P_1^- = P_2^+, P_1^0 = P_2^0, and\ P_1 = P_2$. In this case the test statistic of Equation 19 or 20 would simply become

$$\chi^2 = \frac{4k\left(P_1^+ - P_2^+\right)^2}{P_1^+ + P_2^+} = \frac{4k\left(P_1^- - P_2^-\right)^2}{P_1^- + P_2^-} \quad (21)$$

With 2 degrees of freedom.

If for the treatments no provision is made as in Equation 1-3 for the possibilities between observations or scores in some blocks and the median scores for those blocks, then to correct for the problem of ties these tied scores may be included in one of the two categories, say, the 'above the median' category, in which the observations for the treatments may have been dichotomized by those block medians. In this situation the test statistic of Equation 19 or Equation 20 simply reduces to

$$\chi^2 = \frac{k\sum_{j=1}^{c}\left(P_j^+ - P^-\right)^2}{\overline{pq}} = \frac{k\left(\sum_{j=1}^{c}P_j^{+2} - P^{-2}\right)}{\overline{pq}} \quad (22)$$

Which has the chi-square distribution with c-1 degrees of freedom where $P_j^+ = f^+/k$

is the population of subjects in the ith treatment whose scores are higher (better, lower) than proportion of subjects whose scores are higher (better, larger), than the medians for all blocks across all the 'c' treatments. Here f_j^+ is the number of subjects in the jth treatment whose scores are higher (better, larger) than the median scores for all blocks, j=1,2,…,c.

2. Example 2

Presented in Table 2 are the scores on a ten point scale given by each member of a panel of five judges to a random sample of twelve candidates who attended a job placement interview with 1 including the best performed and 10,the worst performed candidate according to the assessment by the judge.

Table 2. *Scores earned by a random sample of candidates fewer than 5 judges in a job placement interview.*

Candidates	Judges				
	1	2	3	4	5
1	4	5	5	2	5
2	7	9	2	4	5
3	9	7	4	2	10
4	9	3	8	1	9
5	4	1	10	4	1
6	2	3	9	10	2
7	1	8	10	4	3
8	2	4	9	10	10
9	3	10	5	10	2
10	6	7	8	2	2
11	7	5	1	9	5
12	7	1	7	1	6

Table 3. *Block Median, values of u_{ij} (Equation 1) and other statistics for the data of Table 2*

Candidates	Block medians(M_i)	1	2	3	4	5	
1	5	-1	0	0	-1	0	
2	5	1	1	-1	-1	0	
3	7	1	0	-1	-1	1	
4	8	1	-1	0	-1	1	
5	4	0	-1	1	0	-1	
6	3	-1	0	1	1	-1	
7	9	-1	1	1	0	-1	
8	9	-1	-1	0	1	1	
9	5	-1	1	0	1	-1	
10	6	0	1	1	-1	-1	
11	5	1	0	-1	1	0	
12	6	1	-1	1	-1	0	
Total(k)		12	12	12	12	12	60(=kc)
f_j^+		5	4	5	4	3	21(=f^+)
f_j^-		5	4	3	6	5	23(=f^-)
f_j^0		2	4	4	2	4	16(=f^0)
P_j^+		0.417	0.333	0.417	0.333	0.250	0.356(P^+)
P_j^-		0.417	0.333	0.250	0.500	0.417	0.383(P^-)
P_j^0		0.167	0.333	0.333	0.167	0.333	0.267(P^0)

To apply the method we first determine the median score for each block (candidate) and use them to determine the values of u_{ij} (Equation 1) which are presented in Table 3.

To test the null hypothesis of Equal population median, that is that judges do not differ in their assessment of candidate we have using the estimated proportions from Table 3 in Equation 20 we have the test statistic

$$\chi^2 = \frac{1}{(0.350)(0.383)(0.267)}\left(12((0.236)(0.633-0.615)+(0.228)(0.772-0.735)+2(0.268(0.660-0.670))\right)$$

$$=\frac{12}{0.036}(0.004+0.008-0.003)=\frac{12(0.009)}{0.036}=\frac{0.108}{0.036}=3.000$$

Which with 8 degrees of freedom is not statistically significant, leading to an acceptance of null hypothesis of no statistical difference between the judges in then assessment of these candidates. If the data of Table 2 had been analyzed using the usual Friedman test we would first rank the scores for each candidate across all the judges from say the each candidate across all the judges from say the highest assigned the rank 1,the next highest assigned the rank 2 and so on, until the lowest is assigned the rank 5.All tied scores for each candidate are assigned their mean rank. The results are shown in Table 4

Using the sums of the ranks in Table 4 we apply the Friedman test statistic. We have

$$\chi^2 = \frac{12\sum_{j=1}^{c} R_{.j}^2 - 3k(c+1)}{kc(c+1)}$$

$$= \frac{12\left(37^2+35^2+31.5^2+38.5+38^2\right)}{12(5)(6)} - 3(12)(6)$$

$$= 217.033 - 216 = 1.033$$

Which with 4 degrees of freedom is also not statistically significant leading to the same conclusion earlier reached above.

Table 4*. Ranks assigned by judges to a random sample of candidates (Table 2)*

Candidates Numbers	Ranks assigned by judges				
1	4	2	2	1	2
2	2	1	5	4	3
3	2	3	4	5	1
4	1.5	4	3	5	1.5
5	2.5	4.5	1	2.5	4.5
6	4.5	3	2	1	4.5
7	5	2	1	3	4
8	5	4	3	1.5	1.5
9	4	1.5	3	1.5	5
10	3	2	1	4.5	4.5
11	2	3.5	5	1	3.5
12	1.5	4.5	1.5	4.5	3
Total, $R_{.j}$	37	35	31.5	38.5	38

3. Summary and Conclusion

In conclusion, we say that even though the ties adjusted method and the Friedman test here lead to the same conclusions, the relative sizes of the calculated chi-square values suggest that the Friedman test is likely to lead to an acceptance of a false null hypothesis (Type II error) more frequently and hence likely to be less powerful than the ties adjusted modified extended median test. Nevertheless, the Friedman's two-way analysis of variance test by ranks is here at least shown to be still more powerful than the usual extended median test.

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