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The Theorem of Cayley and Γ Matrices

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Abstract: In this article, the connections between symmetric groups and the matrix groups Γ_N are investigated for exploring the application of Cayley's theorem in finite group theory. The exact forms of the permutation groups isomorphic to the groups Γ_2, Γ_3, and Γ_4 are obtained within the frame of the group-theoretical approach. The results are analyzed in detail and compared with that from Cayley's theorem. It shows that the orders of the symmetric groups in present formulas are less than the latter. Various directions for future investigations on the research results have been pointed out.

Keywords: Permutation Group, Isomorphic, γ Matrices, Cayley's Theorem, Quaternion Group

1. Introduction

It is well known that Cayley's theorem is one of the most important results in group theory [1-3]. The theorem shows that if G is a finite group of order n, then G is isomorphic to a subgroup of S_n. This is a classic and intriguing result [4, 5]. With this in hand, if we can fully understand the structure and properties of S_n and its subgroups, then we will automatically understand the structure and properties of this finite group. However, the symmetric group S_n of all the permutations of n objects has order $n!$, trying to use S_n to answer any questions about G means working with a group that factorially larger. Due to the sheer size of S_n, this becomes problematic. Some questions may arise from the investigation for the further applications of Cayley's theorem [6, 7]. For instance, is it possible that G is isomorphic to a subgroup of S_k where k<n?

Consider the symmetry group of the equilateral triangle, D_3. The multiplication table shows that D_3 is a finite group of the 6 group elements. These elements may be represented as permutations of {1, 2, 3, 4, 5, 6} according to the Cayley's theorem. For example, the rotation through 2π/3 can be represented to

$$\begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 2\,3\,1\,5\,6\,4 \end{pmatrix} = (123)(456). \qquad (1)$$

On the other hand, if we label the vertices of the triangle with the numbers 1, 2, and 3, then the elements may be represented as permutations of {1, 2, 3}. With one certain labeling, we would get the rotation through 2π/3 to (123). We are acquainted with this fact that the dihedral group D_3 of order 6 is isomorphic to the symmetric group S_3 of all permutations of 3 objects. We now wonder whether one can find the symmetric group of S_k (k<n) corresponding to some other specific finite groups G of order n and have the explicit connections between them.

The groups of particular interest, discussed in this article, are Γ matrix groups. γ matrices, also known as the Dirac matrices, play a highly significant role in mathematics and physics [8-11]. As a set of matrices satisfying special anticommutation relations [12],

$$\{\gamma_a, \gamma_b\} = \gamma_a\gamma_b + \gamma_b\gamma_a = 2\delta_{ab}, 1 \le a, b \le N, \qquad (2)$$

N matrices γ_a and all its products generate a finite matrix group Γ_N. Analogous sets of γ matrices can be defined in any dimension and signature of the metric. In five space-time dimensions, the four γ matrices above together with the fifth γ matrix generate the Clifford algebra.

According to the familiar theorem of Cayley, the connections between Γ_N and S_n is fairly straightforward from the group table of the finite group Γ_N. However, it is not an easy task to find a symmetric group of S_k with k<n corresponding to the group Γ_N of order n.

In the following section, we briefly describe the comparison of different connections between the symmetric group and the matrix group Γ_N of N=2. Our method is then used for the analysis of the finite matrix group Γ_3 in Section 3. The explicit form of the permutation group corresponding to the group Γ_N of N=4 are obtained in Section 4. The results are discussed in detail in Section 5. We conclude in the final section after pointing out various directions for future investigations.

2. Matrix Group Γ_2

Two matrices γ_1 and γ_2 satisfying the anti-commutation relations Eq. (2) and all their possible products form the Γ matrix group, Γ_2={1, $\gamma_1\gamma_2$, -1, -$\gamma_1\gamma_2$, γ_1, -γ_2, -γ_1, γ_2}. The multiplication table of the group Γ_2 is calculated as follows.

Table 1. *The group table of Γ_2.*

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**
	1	$\gamma_1\gamma_2$	-1	-$\gamma_1\gamma_2$	γ_1	-γ_2	-γ_1	γ_2
1	1	$\gamma_1\gamma_2$	-1	-$\gamma_1\gamma_2$	γ_1	-γ_2	-γ_1	γ_2
$\gamma_1\gamma_2$	$\gamma_1\gamma_2$	-1	-$\gamma_1\gamma_2$	1	-γ_2	-γ_1	γ_2	γ_1
-1	-1	-$\gamma_1\gamma_2$	1	$\gamma_1\gamma_2$	-γ_1	γ_2	γ_1	-γ_2
-$\gamma_1\gamma_2$	-$\gamma_1\gamma_2$	1	$\gamma_1\gamma_2$	-1	γ_2	γ_1	-γ_2	-γ_1
γ_1	γ_1	γ_2	-γ_1	-γ_2	1	-$\gamma_1\gamma_2$	-1	$\gamma_1\gamma_2$
-γ_2	-γ_2	γ_1	γ_2	-γ_1	$\gamma_1\gamma_2$	1	-$\gamma_1\gamma_2$	-1
-γ_1	-γ_1	-γ_2	γ_1	γ_2	-1	$\gamma_1\gamma_2$	1	-$\gamma_1\gamma_2$
γ_2	γ_2	-γ_1	-γ_2	γ_1	-$\gamma_1\gamma_2$	-1	$\gamma_1\gamma_2$	1

Denote the eight elements {1, $\gamma_1\gamma_2$, -1, -$\gamma_1\gamma_2$, γ_1, -γ_2, -γ_1, γ_2}, in the Γ_2 group by the digits {1, 2, 3, 4, 5, 6, 7, 8} respectively as shown in the first row of the group table, then, in accordance with the Cayley's theorem, γ_1 and γ_2 appear

$$\gamma_1 = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8 \\ 5\,8\,7\,6\,1\,4\,3\,2 \end{pmatrix} = (15)(28)(37)(46),$$

$$\gamma_2 = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8 \\ 8\,7\,6\,5\,4\,3\,2\,1 \end{pmatrix} = (18)(27(36)(45). \quad (3)$$

The rest in the group can be got from the multiplication rule. The matrix group Γ_2 of order 8 is isomorphic to the subgroup of the permutation group S_8. These are results directly from the Cayley's theorem.

On the other hand, notice that the order of the elements in the group Γ_2 are respectively: 1, 4, 2, 4, 2, 2, 2, 2, which are the same as the elements in the dihedral group D_4 of the symmetry group of a square. This is a helpful hint. Label the vertices of the square with the number 1, 2, 3, and 4, then the elements in the group D_4 may be represented as permutations of 1, 2, 3, 4.

$$D_4 = \{E, R, R^2, R^3, S_0, S_1, S_2, S_3\}$$

$$= \{E, (1234), (13)(24), (1432), (13), (14)(23), (24), (12)(34)\}, \quad (4)$$

where R is a π/2 rotation about the center of the square and S_0, S_1, S_2, S_3 are the reflections about four symmetry axes respectively. It is not difficult to prove that there exists an isomorphism between these two groups, $\Gamma_2 \approx D_4$. Hence, this isomorphism gives a one-to-one correspondence of the elements from Γ_2 to D_4, the explicit form for γ_a (a=1, 2) is as follows

$$\gamma_1 = (13), \gamma_2 = (12)(34), \quad (5)$$

and all the remaining elements in the matrix group Γ_2 can be got from the explicit form (Eq. (5)) of γ_1 and γ_2. Then, Γ_2 becomes

$$\Gamma_2=\{1, \gamma_1\gamma_2, -1, -\gamma_1\gamma_2, \gamma_1, -\gamma_2, -\gamma_1, \gamma_2\}$$

$$\approx \{E, (1234), (13)(24), (1432), (13), (14)(23), (24), (12)(34)\}. \quad (6)$$

It means that the matrix group Γ_2 of order 8 is also isomorphic to a subgroup of the permutation group S_4. The results in Eq. (3) and Eq. (5) present different connections between the symmetric group S_k and the matrix group Γ_2. The comparison shows that the number of the different objects in the S_k in our results is only half of that from the theorem. We will continue to use our method to investigate the permutation group related to other Γ matrix groups in the following.

3. Matrix Group Γ_3

The Pauli matrices σ_1, σ_2 and σ_3, satisfying the relations,

$$\sigma_a\sigma_b = \delta_{ab}1 + i\sum_{c=1}^{3}\varepsilon_{abc}\,\sigma_c, \quad (7)$$

are a set of γ matrices in dimension 3 with the Euclidean metric signature. The matrix group Γ_3 is the set of three γ matrices and all their products,

$$\Gamma_3 = \{\pm 1, \pm\gamma_1, \pm\gamma_2, \pm\gamma_3, \pm\gamma_1\gamma_2, \pm\gamma_1\gamma_3, \pm\gamma_2\gamma_3, \pm\gamma_1\gamma_2\gamma_3\}. \quad (8)$$

With a similar analysis for the matrix group Γ_2, the Cayley's theorem states that the matrix group Γ_3 of order 16 is isomorphic to the subgroup of the permutation group S_{16} directly from the multiplication table.

What is interesting is that how to find a permutation group S_k (k<16) whose subgroup is isomorphic to the matrix group Γ_3 of order 16 through investigation. Choose the correspondence between the γ matrices and the permutations as follows,

$$\gamma_1 = (12)(34), \gamma_2 = (15)(26)(37)(48),$$

$$\gamma_3 = (18)(27)(35)(46), \quad (9)$$

the products of γ matrices can be written as

$$\gamma_1\gamma_2 = (1526)(3748), \gamma_1\gamma_3 = (1827)(3546),$$

$$\gamma_2\gamma_3 = (1423)(5768), \gamma_1\gamma_2\gamma_3 = (1324)(5768). \quad (10)$$

Notice that the square of $\gamma_a\gamma_b$(a, b=1, 2, 3 and a≠ b) is

$$(\gamma_1\gamma_2)^2 = (\gamma_1\gamma_3)^2 = (\gamma_2\gamma_3)^2 = (12)(34)(56)(78) = -1, \quad (11)$$

the explicit forms for -γ_a(a=1, 2, 3) are given by

$$-\gamma_1 = (56)(78), -\gamma_2 = (16)(25)(38)(47),$$

$$-\gamma_3 = (17)(28)(36)(45). \quad (12)$$

As noted, it is not difficult to come to the remaining products of the γ matrices,

$$-\gamma_1\gamma_2 = (1625)(3847), -\gamma_1\gamma_3 = (1728)(3645),$$

$$-\gamma_2\gamma_3 = (1324)(5867), -\gamma_1\gamma_2\gamma_3 = (1423)(5867). \quad (13)$$

In this way, all elements are accounted for. Hence, Γ_3 can be written as

$$\Gamma_3 = \{\pm 1, \pm\gamma_1, \pm\gamma_2, \pm\gamma_3, \pm\gamma_1\gamma_2, \pm\gamma_1\gamma_3, \pm\gamma_2\gamma_3, \pm\gamma_1\gamma_2\gamma_3\}$$
$$= \{\, E, (12)(34)(56)(78), (12)(34), (56)(78), (15)(26)(37)(48), (16)(25)(38)(47),$$
$$(18)(27)(35)(46), (17)(28)(36)(45), (1526)(3748), (1625)(3847), (1827)(3546),$$
$$(1728)(3645), (1423)(5768), (1324)(5867), (1324)(5768), (1423)(5867)\,\}. \quad (14)$$

The calculated result means that the matrix group Γ_3 of order 16 is isomorphic to a subgroup of the permutation group S_8. Though the correspondence chosen in Eq. (9) is not unique, for example,

$$\gamma_1 = (12)(34), \gamma_2 = (17)(28)(36)(45),$$
$$\gamma_3 = (16)(25)(38)(47), \quad (15)$$

or

$$\gamma_1 = (12)(56), \gamma_2 = (13)(24)(58)(67),$$
$$\gamma_3 = (18)(27)(54)(36), \quad (16)$$

or

$$\gamma_1 = (12)(78),\ \gamma_2 = (15)(26)(38)(47),$$
$$\gamma_3 = (14)(23)(58)(67), \quad (17)$$

it can be found that the procedures for calculating the products of γ_a matrices are almost the same.

It is worth mentioning that the case of the matrix group Γ_N of N=2 can be taken as that of N=4m-2 when m=1. Since $\gamma_1\gamma_2 = i\gamma_3$ leads to $\gamma_3 = -i\gamma_1\gamma_2$, and $i = \gamma_1\gamma_2\gamma_3$, Γ_3 can be written as

$$\Gamma_3 = \{\pm 1, \pm\gamma_1, \pm\gamma_2, \pm\gamma_1\gamma_2, \pm\gamma_1\gamma_2\gamma_3, \pm\gamma_2\gamma_3, \mp\gamma_1\gamma_3, \mp\gamma_3\}$$
$$=\{\pm 1, \pm\gamma_1, \pm\gamma_2, \pm\gamma_1\gamma_2, \pm \mathrm{i}, \pm \mathrm{i}\gamma_1, \pm \mathrm{i}\gamma_2, \pm \mathrm{i}\gamma_1\gamma_2\}. \quad (18)$$

The results in Eqs. (6), (14) and (18) can be used to check the relation $\Gamma_3 \approx \{\Gamma_2, \mathrm{i}\ \Gamma_2\}$ through direct calculations on the production of the permutations. This method might also be generalized to understand the general properties of the gamma matrix groups, $\Gamma_{4\mathrm{m}-1} \approx \{\Gamma_{4\mathrm{m}-2}, \mathrm{i}\ \Gamma_{4\mathrm{m}-2}\ \}$.

4. Matrix Group Γ_4

The set of products of the four γ matrices forms the matrix group Γ_4,

$$\Gamma_4 = \{\pm 1, \pm\gamma_1, \pm\gamma_2, \pm\gamma_3, \pm\gamma_4, \pm\gamma_1\gamma_2, \pm\gamma_1\gamma_3, \pm\gamma_1\gamma_4, \pm\gamma_2\gamma_3, \pm\gamma_2\gamma_4, \pm\gamma_3\gamma_4,$$
$$\pm\gamma_1\gamma_2\gamma_3, \pm\gamma_1\gamma_2\gamma_4, \pm\gamma_1\gamma_3\gamma_4, \pm\gamma_2\gamma_3\gamma_4, \pm\gamma_1\gamma_2\gamma_3\gamma_4\}. \quad (19)$$

In order to express implicitly the symmetric pattern of the formulas, the second half of the digits from 1 to 16 are denoted by the letters A, B, C, D, E, F, G and H respectively for convenience. Based on careful analysis, the explicit form of the four γ matrices are taken as follows,

$$\gamma_1 = (12)(34)(56)(78),$$
$$\gamma_2 = (1A)(2B)(3C)(4D)(5E)(6F)(7G)(8H),$$
$$\gamma_3 = (1F)(2E)(3G)(4H)(5A)(6B)(7D)(8C),$$
$$\gamma_4 = (1H)(2G)(3F)(4E)(5C)(6D)(7A)(8B). \quad (20)$$

This leads to the products of two γ matrices, $\gamma_a\gamma_b$ (a, b=1, 2, 3, 4 and a ≠ b)

$$\gamma_1\gamma_2 = (1A2B)(3C4D)(5E6F)(7G8H),$$
$$\gamma_1\gamma_3 = (1F2E)(3G4H)(5A6B)(7D8C),$$
$$\gamma_1\gamma_4 = (1H2G)(3F4E)(5C6D)(7A8B),$$
$$\gamma_2\gamma_3 = (1625)(3748)(AEBF)(CHDG),$$
$$\gamma_2\gamma_4 = (1827)(3645)(AGBH)(CEDF),$$
$$\gamma_3\gamma_4 = (1423)(5867)(ADBC)(EHFG), \quad (21)$$

and the products of three γ_a matrices

$$\gamma_1\gamma_2\gamma_3 = (1526)(3847)(AEBF)(CHDG),$$
$$\gamma_1\gamma_2\gamma_4 = (1728)(3546)(AGBH)(CEDF),$$
$$\gamma_1\gamma_3\gamma_4 = (1324)(5768)(ADBC)(EHFG),$$
$$\gamma_2\gamma_3\gamma_4 = (1D2C)(3A4B)(5H6G)(7E8F). \quad (22)$$

In consideration of the square of $\gamma_a\gamma_b$ (a, b=1, 2, 3, 4 and a ≠ b),

$$(\gamma_1\gamma_2)^2 = (\gamma_1\gamma_3)^2 = (\gamma_1\gamma_4)^2 = (\gamma_2\gamma_3)^2 = (\gamma_2\gamma_4)^2$$
$$= (\gamma_3\gamma_4)^2 = (12)(34)(56)(78)(AB)(CD)(EF)(GH) = -1, \quad (23)$$

the explicit form for -γ_a (a=1, 2, 3, 4) becomes

$$-\gamma_1 = (AB)(CD)(EF)(GH),$$
$$-\gamma_2 = (1B)(2A)(3D)(4C)(5F)(6E)(7H)(8G),$$
$$-\gamma_3 = (1E)(2F)(3H)(4G)(5B)(6A)(7C)(8D),$$
$$-\gamma_4 = (1G)(2H)(3E)(4F)(5D)(5C)(7B)(8A). \quad (24)$$

One may also check the results for the products of -$\gamma_a\gamma_b$ (a, b=1, 2, 3, 4 and a ≠ b),

$$-\gamma_1\gamma_2 = (1B2A)(3D4C)(5F6E)(7H8G),$$
$$-\gamma_1\gamma_3 = (1E2F)(3H4G)(5B6A)(7C8D),$$
$$-\gamma_1\gamma_4 = (1G2H)(3E4F)(5D6C)(7B8A),$$
$$-\gamma_2\gamma_3 = (1526)(3847)(AFBE)(CGDH),$$

$$-\gamma_2\gamma_4 = (1728)(3546)(AHBG)(CFDE),$$

$$-\gamma_3\gamma_4 = (1324)(5768)(ACBD)(EGFH), \quad (25)$$

and the products of -$\gamma_a\gamma_b\gamma_c$ (a, b, c=1, 2, 3, 4 and a ≠ b ≠ c),

$$-\gamma_1\gamma_2\gamma_3 = (1625)(3748)(AFBE)(CGDH),$$

$$-\gamma_1\gamma_2\gamma_4 = (1827)(3645)(AHBG)(CFDE),$$

$$-\gamma_1\gamma_3\gamma_4 = (1423)(5867)(ACBD)(EGFH),$$

$$-\gamma_2\gamma_3\gamma_4 = (1C2D)(3B4A)(5G6H)(7F8E). \quad (26)$$

Similar to the previous statements, there are also other options for the explicit form taken in Eq. (20) for the four γ matrices, such as

$$\gamma_1 = (12)(34)(AB)(CD),$$

$$\gamma_2 = (15)(26)(37)(48)(AE)(BF)(CG)(DH),$$

$$\gamma_3 = (1F)(2E)(3G)(4H)(5A)(6B)(7D)(8C),$$

$$\gamma_4 = (1H)(2G)(3F)(4E)(5C)(6D)(7A)(8B). \quad (27)$$

Different from the groups Γ_2 and Γ_3, it is interesting to find that the product of all γ_a is a two-order element in the group Γ_4,

$$\gamma_1\gamma_2\gamma_3\gamma_4 = -\gamma_1\gamma_2\gamma_3\gamma_4$$

$$= (1D)(2C)(3A)(4B)(5H)(6G)(7E)(8F), \quad (28)$$

while Eq. (6) and Eq. (10) show that the product of all γ_a are the elements of order four.

This formula is helpful in understanding the isomorphism relations between Γ_4 and Γ_5 ($\Gamma_4 \approx \Gamma_5$). Since this part can be viewed as the case of N=4m when m=1, this research might also be used to reveal the properties of the Γ matrix groups, such as, $\Gamma_{4m} \approx \Gamma_{4m+1}$.

Hence, the symmetric group whose subgroup corresponds to the gamma matrix group Γ_4 of order 32 is related to S_{16} in present results, while it is connected with S_{32} from the Cayley's theorem. The order of the symmetric group in our method is far less than the latter.

5. Discussions

The symmetric groups isomorphic to the matrix groups Γ_N have been discussed when the value of N is even or odd. It is found that the matrix group Γ_2 of order 8 is isomorphic to a subgroup of the permutation group S_4 and the matrix group Γ_3 of order 16 is isomorphic to a subgroup of the permutation group S_8. As is well known, up to isomorphism, there are five different finite groups of order 8. The first is the cyclic group C_8. The second is the dihedral group D_4, where two generators can be denoted by R and S_0, satisfying $R^4 = S_0^2 = E$. The third is an Abelian group, $C_{4h} = C_4 \otimes C_2$, where the generators satisfy $R^4 = S_0^2 = E$ and $RS_0 = S_0R$. The fourth is also a commutative group, $D_{2h} = V_4 \otimes C_2$, and the generators satisfy $R_1^2 = R_2^2 = S_0^2 = E$. The fifth is a quaternion group Q_8, the generators satisfy $R_1^4 = S_0^4 = E$. Since the symmetric groups corresponding to groups Γ_2 and Γ_3 have been found, one might try to think if Γ_3 can be represented as the product of finite group of order 8 and the cyclic group C_2. Since there should be elements of order 8 in the cyclic group C_8, it is natural that Γ_3 is not related with $C_8 \otimes C_2$. It is verified that Γ_3 is also not the direct product $D_4 \otimes C_2$ or $C_{4h} \otimes C_2$ or $D_{2h} \otimes C_2$ or $Q_8 \otimes C_2$ through multiplication of the permutations. In fact, it is finally found that Γ_3 can be represented as the semi-product of Q_8 and the group C_2, $\Gamma_3 = Q_8 \times C_2$. Due to the expression

$$\Gamma_3 = \{\pm 1, \pm\gamma_1, \pm\gamma_2, \pm\gamma_3, \pm\gamma_1\gamma_2, \pm\gamma_1\gamma_3, \pm\gamma_2\gamma_3, \pm\gamma_1\gamma_2\gamma_3\}$$

$$= \{\pm 1, \pm\gamma_1, \pm\gamma_2, \pm\gamma_1\gamma_2, \pm i, \pm i\gamma_1, \pm i\gamma_2, \pm i\gamma_1\gamma_2\}$$

$$= \{1, \gamma_2\gamma_3, -\gamma_1\gamma_3, \gamma_1\gamma_2, -1, -\gamma_2\gamma_3, \gamma_1\gamma_3, -\gamma_1\gamma_2,$$

$$\gamma_1, \gamma_1\gamma_2\gamma_3, \gamma_3, -\gamma_2, -\gamma_1, -\gamma_1\gamma_2\gamma_3, -\gamma_3, \gamma_2\}$$

$$= \{1, i\gamma_1, i\gamma_2, \gamma_1\gamma_2, -1, -i\gamma_1, -i\gamma_2, -\gamma_1\gamma_2,$$

$$\gamma_1, i, -i\gamma_1\gamma_2, -\gamma_2, -\gamma_1, -i, i\gamma_1\gamma_2, \gamma_2\} \quad (29)$$

therefore, Γ_3 can be rewritten in the form

$$\Gamma_3 = \{1, i\gamma_1, i\gamma_2, \gamma_1\gamma_2, -1, -i\gamma_1, -i\gamma_2, -\gamma_1\gamma_2\} \times \{1, \gamma_1\}$$

$$= \{E, (1423)(5768), (1728)(3645), (1526)(3748), (12)(34)(56)(78),$$

$$(1324)(5867), (1827)(3546), (1625)(3847)\} \times \{E, (12)(34)\},$$

$$= \{E, (1423)(5768), (1728)(3645), (1526)(3748),$$

$$(12)(34)(56)(78), (1324)(5867), (1827)(3546), (1625)(3847),$$

$$(12)(34), (1324)(5768), (18)(27)(35)(46), (16)(25)(38)(47),$$

$$(56)(78), (1423)(5867), (17)(28)(36)(45), (15)(26)(37)(48)\}$$

$$\simeq \{E, (1526)(3748), (1728)(3645), (1324)(5867),$$

$$(12)(34)(56)(78), (1625)(3847), (1827)(3546), (1423)(5768),$$

$$(12)(56), (1625)(3748), (18)(27)(54)(36), (14)(23)(57)(68),$$

$$(34)(78),(1526)(3847),(17)(28)(35)(46),(13)(24)(58)(67)\}$$
$$\simeq \{E,(1728)(3645),(1324)(5867),(1526)(3748),$$
$$(12)(34)(56)(78),(1827)(3546),(1423)(5768),(1625)(3847),$$
$$(12)(78),(1827)(3645),(14)(23)(58)(67),(16)(25)(37)(48),$$
$$(34)(56),(1728)(3546),(13)(24)(57)(68),(15)(26)(38)(47)\}. \quad (30)$$

It can be verified that the set of $\{1, \mathrm{i}\gamma_1, \mathrm{i}\gamma_2, \gamma_1\gamma_2, -1, -\mathrm{i}\gamma_1, -\mathrm{i}\gamma_2, -\gamma_1\gamma_2\}$ is a group isomorphic to the quaternion group Q_8, which is an invariant subgroup of index two of Γ_3. The cyclic group of $\{1, \gamma_1\}$ is a subgroup of Γ_3 and it is not a normal subgroup. That is, Γ_3 is the semi-product of these two subgroups.

It can also be found that though γ_1, γ_2, and γ_3 have various forms in Eqs. (15)-(17), all the elements of order 4 in the group Q_8 remain unchanged. They are (1423)(5768), (1728)(3645), (1526)(3748), (1324)(5867), (1827)(3546) and (1625)(3847) respectively. Further, it is noticed that Eq. (11) leads to

$$(\gamma_1\gamma_2)^2 = (\gamma_2\gamma_3)^2 = (\gamma_1\gamma_3)^2$$
$$= (\gamma_1\gamma_2)(\gamma_2\gamma_3)(\gamma_1\gamma_3) = (12)(34)(56)(78) = -1. \quad (31)$$

If the corresponding relations between the permutations and fundamental quaternion units are

$$\gamma_1\gamma_2 \leftrightarrow i, \gamma_2\gamma_3 \leftrightarrow j, \gamma_1\gamma_3 \leftrightarrow k, \quad (32)$$

Equation (31) will immediately remind us of the famous formula of quaternion algebra

$$\mathrm{i}^2 = j^2 = k^2 = ijk = -1. \quad (33)$$

This indicates that there is an isomorphism between the quaternion group and an invariant subgroup of index two in the group Γ_3. This conclusion could be meaningful in understanding the properties of the quaternion group. It is known that the quaternion group play an important role in mathematics and physics. Equations (31)-(33) might be an interesting starting point to study the quaternions and octonions.

6. Conclusions

In this article, we find the symmetric group S_k (k=n/2) corresponding to the matrix group Γ_N of order n and provide the exact relations between them. This specific finite group is investigated for the value of N that is odd or even. Especially, when N is even, we studied separately the cases when N=4n-2 and when N=4n. This research indicates that the order of the symmetric group in our approach is less than that of the group directly from the Cayley's theorem.

The generalization of this method to matrix group Γ_N when N is arbitrary number, even or odd, is straightforward. The properties of the symmetric groups can be used to check and understand the properties of the Γ matrix groups which will widen the application of the familiar theorem of Cayley. It is interesting to find that the study of Γ matrix groups is meaningful to understand the properties of the quaternion group. These conclusions might be useful to study the Clifford algebra, Lorentz group and its representations. It is also hoped to stimulate one to apply these results to other interesting fields.

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