



# A Coupling Method of Regularization and Adomian Decomposition for Solving a Class of the Fredholm Integral Equations Within Local Fractional Operators

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**Abstract:** In this paper, we will apply the combined regularization-Adomian decomposition method within local fractional differential operators to handle local fractional Fredholm integral equation of the first kind. Theoretical considerations are being discussed. To illustrate the ability and simplicity of the method, some examples are provided. The iteration procedure is based on local fractional derivative. The obtained results reveal that the proposed methods are very efficient and simple tools for solving local fractional integral equations.

**Keywords:** Local Fractional Fredholm Integral Equation, Local Fractional Adomian Decomposition Method, Local Fractional Operator

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## 1. Introduction

Many initial and boundary value problems associated with ordinary differential equations and partial differential equation can be transformed into problems of solving some approximate integral equations [1]. Engineering problems can be mathematically described by differential equations, and thus differential equations play very important roles in the solution of practical problems. The differential equations can be transformed into problems of solving some approximate integral equations. However, some initial and boundary value domains are fractal curves, which are everywhere continuous but nowhere differentiable. As a result, we cannot employ the classical calculus, which requires that the defined functions should be differentiable, to process ordinary and partial differential equation with fractal conditions [2].

Integral equations occur naturally in many fields of science and engineering. A computational approach to solve integral equation is an essential work in scientific research. Integral equation is encountered in a variety of applications in many fields including continuum mechanics, potential theory, geophysics, electricity and magnetism, kinetic theory of gases, hereditary phenomena in physics and biology, renewal

theory, quantum mechanics, radiation, optimization, optimal control systems, communication theory, mathematical economics, population genetics, queuing theory, medicine, mathematical problems of radioactive equilibrium, the particle transport problems of astrophysics and reactor theory, acoustics, fluid mechanics, steady state heat conduction, fracture mechanics, and radioactive heat transfer problems. Fredholm integral equation is one of the most important integral equations [3].

Local fractional calculus was successfully applied in local fractional Fokker Planck equation [4], the fractal heat conduction equation [5], local fractional diffusion equation [6, 7], local fractional Laplace equation [8, 9], local fractional integral equations [10, 11], local fractional differential equations [12-14] and local fractional wave equation [15]. Several analytical and numerical techniques were successfully applied to deal with differential and integral equations within local fractional derivative operators such as local fractional Adomian decomposition method [16], local fractional variational iteration method [11, 17], local fractional Picards successive approximation method [18], local fractional Laplace decomposition method [19, 20], local fractional differential transform method [21, 22], local fractional series expansion method [23], local fractional

homotopy perturbation method [22, 25], local fractional similarity solution [26], local fractional Laplace variational iteration method [27, 28], and local fractional Fourier series [29-31]. This paper is organized as follows: In Section 2, the basic mathematical tools are reviewed. In section 3, we give analysis of the method used. In Section 4, we consider several illustrative examples. Finally, in Section 5, we present our conclusions.

## 2. Mathematical Fundamentals

In this section we present some basic definitions and notations of the local fractional differential operators (see [7-9, 32, 33]).

Definition 1. Suppose that there is the relation

$$|f(x) - f(x_0)| < \varepsilon^\alpha, 0 < \alpha \leq 1, \quad (1)$$

with  $|x - x_0| < \delta$ , for  $\varepsilon, \delta > 0$  and  $\varepsilon, \delta \in \mathbb{R}$ , then the function  $f(x)$  is called local fractional continuous at  $x = x_0$  and it is denoted by  $\lim_{x \rightarrow x_0} f(x) = f(x_0)$ .

Definition 3. In fractal space, let  $f(x) \in C_\alpha(a, b)$ , local fractional derivative of  $f(x)$  of order  $\alpha$  at  $x = x_0$  is given by

$$D_x^\alpha f(x_0) = \frac{d^\alpha}{dx^\alpha} f(x) \Big|_{x=x_0} = f^{(\alpha)}(x) = \lim_{x \rightarrow x_0} \frac{\Delta^\alpha(f(x) - f(x_0))}{(x - x_0)^\alpha} \quad (2)$$

where  $\Delta^\alpha(f(x) - f(x_0)) \equiv \Gamma(\alpha + 1)(f(x) - f(x_0))$ .

Local fractional derivative of high order is written in the form

$$D_x^{k\alpha} f(x) = f^{(k\alpha)}(x) = \overbrace{D_x^\alpha D_x^\alpha \dots D_x^\alpha}^{k \text{ times}} f(x). \quad (3)$$

Definition 4. A partition of the interval  $[a, b]$  is denoted as  $(t_j, t_{j+1})$ ,  $j = 0, \dots, N-1$ , and  $t_N = b$  with  $\Delta t_j = t_{j+1} - t_j$  and  $\Delta t = \max\{\Delta t_0, \Delta t_1, \dots\}$ . Local fractional integral of  $f(x)$  in the interval  $[a, b]$  is given by

$${}_a I_b^{(\alpha)} f(x) = \frac{1}{\Gamma(1 + \alpha)} \int_a^b f(t) (dt)^\alpha = \frac{1}{\Gamma(1 + \alpha)} \lim_{\Delta t \rightarrow 0} \sum_{j=0}^{N-1} f(t_j) (\Delta t_j)^\alpha. \quad (4)$$

Definition 5. In fractal space, the Mittag Leffler function, sine function and cosine function are defined as

$$E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(1 + k\alpha)}, 0 < \alpha \leq 1 \quad (5)$$

$$\sin_\alpha(x^\alpha) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{(2k+1)\alpha}}{\Gamma[1 + (2k+1)\alpha]}, 0 < \alpha \leq 1 \quad (6)$$

$$\cos_\alpha(x^\alpha) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k\alpha}}{\Gamma[1 + 2k\alpha]}, 0 < \alpha \leq 1 \quad (7)$$

Note that:

$$\frac{d^\alpha x^{k\alpha}}{dx^\alpha} = \frac{\Gamma(1 + k\alpha)}{\Gamma(1 + (k-1)\alpha)} x^{(k-1)\alpha} \quad (8)$$

$$\frac{d^\alpha E_\alpha(x^\alpha)}{dx^\alpha} = E_\alpha(x^\alpha) \quad (9)$$

$$\frac{d^\alpha E_\alpha(kx^\alpha)}{dx^\alpha} = k E_\alpha(kx^\alpha) \quad (10)$$

$$\frac{1}{\Gamma(1 + \alpha)} \int_a^b E_\alpha(x^\alpha) (dx)^\alpha = E_\alpha(b^\alpha) - E_\alpha(a^\alpha) \quad (11)$$

$$\frac{1}{\Gamma(1 + \alpha)} \int_a^b \sin_\alpha(x^\alpha) (dx)^\alpha = \cos_\alpha(a^\alpha) - \cos_\alpha(b^\alpha) \quad (12)$$

$$\frac{1}{\Gamma(1 + \alpha)} \int_a^b \frac{x^{k\alpha}}{\Gamma(1 + k\alpha)} (dx)^\alpha = \frac{1}{\Gamma(1 + (k+1)\alpha)} (b^{(k+1)\alpha} - a^{(k+1)\alpha}) \quad (13)$$

## 3. Analysis of the Method

The most standard form of Fredholm linear local fractional integral equations of the first kind is given by the form

$$f(x) = \frac{1}{\Gamma(1 + \alpha)} \int_a^b K(x, t) u(t) (dt)^\alpha, 0 < \alpha \leq 1 \quad (14)$$

The method of regularization transforms the linear local fractional Fredholm integral equation of the first kind (14) to the approximation local fractional Fredholm integral equation

$$\mu u_\mu(x) = f(x) - \frac{1}{\Gamma(1 + \alpha)} \int_a^b K(x, t) u_\mu(t) (dt)^\alpha \quad (15)$$

where  $\mu$  is a small positive parameter. It is clear that (15) is a local fractional Fredholm integral equation of the second kind that can be rewritten

$$u_\mu(x) = \frac{1}{\mu} f(x) - \frac{1}{\mu \Gamma(1 + \alpha)} \int_a^b K(x, t) u_\mu(t) (dt)^\alpha \quad (16)$$

We apply the local fractional Adomian decomposition method for handling (16). To achieve this goal, we represent the term  $\mu u(x)$  by an infinite series of components given by

$$u_\mu(x) = \sum_{n=0}^{\infty} u_{\mu_n}(x) \quad (17)$$

where the components  $u_{\mu_n}(x), n \geq 0$  will be recursively determined.

Substituting (17) into (16) leads to

$$\sum_{n=0}^{\infty} u_{\mu_n}(x) = \frac{1}{\mu} f(x) - \frac{1}{\mu \Gamma(1 + \alpha)} \int_a^b K(x, t) \left( \sum_{n=0}^{\infty} u_{\mu_n}(t) \right) (dt)^\alpha \quad (18)$$

The local fractional Adomian decomposition method admits the use of the following recursive relation

$$\begin{aligned} u_{\mu_0}(x) &= \frac{1}{\mu} f(x) \\ u_{\mu_{k+1}}(x) &= -\frac{1}{\mu} \frac{1}{\Gamma(1+\alpha)} \int_a^b K(x,t) u_{\mu_k}(t) (dt)^\alpha, k \geq 0 \end{aligned} \quad (19)$$

Finally, the exact solution  $u(x)$  of (14) can thus be obtained by

$$u(x) = \lim_{\mu \rightarrow 0} u_\mu(x) \quad (20)$$

## 4. Illustrative Examples

In this section two examples for the local fractional Fredholm integral equation from the first kind is presented in order to demonstrate the simplicity and the efficiency of the above method.

*Example 1.* we consider the local fractional Fredholm integral equation.

$$\frac{1}{3} E_\alpha(-x^\alpha) = \frac{1}{\Gamma(1+\alpha)} \int_0^{\frac{1}{3}} E_\alpha((t-x)^\alpha) u(t) (dt)^\alpha. \quad (21)$$

From Eq. (16), we get

$$u_\mu(x) = \frac{1}{3\mu} E_\alpha(-x^\alpha) - \frac{1}{\mu} \frac{1}{\Gamma(1+\alpha)} \int_0^{\frac{1}{3}} E_\alpha((t-x)^\alpha) u_\mu(t) (dt)^\alpha. \quad (22)$$

The local fractional Adomian decomposition method admits the use of

$$u_\mu(x) = \sum_{n=0}^{\infty} u_{\mu_n}(x), \quad (23)$$

and the recurrence relation

$$u_{\mu_0}(x) = \frac{1}{3\mu} E_\alpha(-x^\alpha), \quad (24)$$

$$u_{\mu_{k+1}}(x) = -\frac{1}{\mu} \frac{1}{\Gamma(1+\alpha)} \int_0^{\frac{1}{3}} E_\alpha((t-x)^\alpha) u_{\mu_k}(t) (dt)^\alpha, k \geq 0. \quad (25)$$

This in turn gives the components

$$u_{\mu_0}(x) = \frac{1}{3\mu} E_\alpha(-x^\alpha), \quad (26)$$

$$\begin{aligned} u_{\mu_1}(x) &= -\frac{1}{\mu} \frac{1}{\Gamma(1+\alpha)} \int_0^{\frac{1}{3}} E_\alpha((t-x)^\alpha) \left( \frac{1}{3\mu} E_\alpha(-t^\alpha) \right) (dt)^\alpha \\ &= -\frac{1}{9\mu^2} E_\alpha(-x^\alpha), \end{aligned} \quad (27)$$

$$\begin{aligned} u_{\mu_2}(x) &= -\frac{1}{\mu} \frac{1}{\Gamma(1+\alpha)} \int_0^{\frac{1}{3}} E_\alpha((t-x)^\alpha) \left( -\frac{1}{9\mu^2} E_\alpha(-t^\alpha) \right) (dt)^\alpha \\ &= \frac{1}{27\mu^3} E_\alpha(-x^\alpha), \end{aligned} \quad (28)$$

$$u_{\mu_3}(x) = -\frac{1}{81\mu^4} E_\alpha(-x^\alpha), \quad (29)$$

and so on. Substituting this result into (23) gives the approximate solution

$$u_\mu(x) = -\frac{1}{1+3\mu} E_\alpha(-x^\alpha). \quad (30)$$

The exact solution  $u(x)$  of (21) can be obtained by

$$u(x) = \lim_{\mu \rightarrow 0} u_\mu(x) = E_\alpha(-x^\alpha). \quad (31)$$

*Example 2.* we consider the local fractional Fredholm integral equation.

$$\frac{1}{4} \frac{x^\alpha}{\Gamma(1+\alpha)} = \frac{1}{\Gamma(1+\alpha)} \int_0^1 \frac{x^\alpha}{\Gamma(1+\alpha)} \frac{t^\alpha}{\Gamma(1+\alpha)} u(t) (dt)^\alpha. \quad (32)$$

From Eq. (16), we get

$$u_\mu(x) = \frac{1}{4\mu} \frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{1}{\mu} \frac{1}{\Gamma(1+\alpha)} \int_0^1 \frac{x^\alpha}{\Gamma(1+\alpha)} \frac{t^\alpha}{\Gamma(1+\alpha)} u_\mu(t) (dt)^\alpha. \quad (33)$$

The local fractional Adomian decomposition method admits the use of

$$u_\mu(x) = \sum_{n=0}^{\infty} u_{\mu_n}(x), \quad (34)$$

and the recurrence relation

$$u_{\mu_0}(x) = \frac{1}{4\mu} \frac{x^\alpha}{\Gamma(1+\alpha)}, \quad (35)$$

$$u_{\mu_{k+1}}(x) = -\frac{1}{\mu} \frac{1}{\Gamma(1+\alpha)} \int_0^1 \frac{x^\alpha}{\Gamma(1+\alpha)} \frac{t^\alpha}{\Gamma(1+\alpha)} u_{\mu_k}(t) (dt)^\alpha. \quad (36)$$

This in turn gives the components

$$u_{\mu_0}(x) = \frac{1}{4\mu} \frac{x^\alpha}{\Gamma(1+\alpha)}, \quad (37)$$

$$\begin{aligned} u_{\mu_1}(x) &= -\frac{1}{\mu} \frac{1}{\Gamma(1+\alpha)} \int_0^1 \frac{x^\alpha}{\Gamma(1+\alpha)} \frac{t^\alpha}{\Gamma(1+\alpha)} \left( \frac{1}{4\mu} \frac{t^\alpha}{\Gamma(1+\alpha)} \right) (dt)^\alpha \\ &= -\frac{1}{12\mu^2} \frac{x^\alpha}{\Gamma(1+\alpha)}, \end{aligned} \quad (38)$$

$$u_{\mu_2}(x) = -\frac{1}{\mu} \frac{1}{\Gamma(1+\alpha)} \int_0^1 \frac{x^\alpha}{\Gamma(1+\alpha)} \frac{t^\alpha}{\Gamma(1+\alpha)} \left( -\frac{1}{12\mu^2} \frac{t^\alpha}{\Gamma(1+\alpha)} \right) (dt)^\alpha$$

$$= \frac{1}{36\mu^3} \frac{x^\alpha}{\Gamma(1+\alpha)}, \quad (39)$$

$$u_{\mu_3}(x) = -\frac{1}{108\mu^4} \frac{x^\alpha}{\Gamma(1+\alpha)}, \quad (40)$$

and so on. Substituting this result into (34) gives the approximate solution

$$u_\mu(x) = \frac{3}{4(1+3\mu)} \frac{x^\alpha}{\Gamma(1+\alpha)}. \quad (41)$$

The exact solution  $u(x)$  of (32) can be obtained by

$$u(x) = \lim_{\mu \rightarrow 0} u_\mu(x) = \frac{3}{4} \frac{x^\alpha}{\Gamma(1+\alpha)}. \quad (42)$$

## 5. Conclusions

In this work, the analytical approximate solutions for the Fredholm integral equations of the second kind involving local fractional derivative operators are investigated by using the method of regularization with Adomian decomposition method. The obtained results demonstrate the reliability of the methodology and its wider applicability to local fractional integral equations and hence can be extended to other problems of diversified nonlinear nature. Our goal in the future is to apply this method to system of coupled PDEs within local fractional derivative operators.

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