Double Solutions of the Euler and Navier-Stokes Equations Process of Origination the Vorticity and Turbulence

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Abstract: The Euler and Navier-Stokes equations, which describe flow of fluids and gases, possess solutions of two types, namely, the solutions that are not functions (they depends not only on the variables) and the solutions that are discrete functions. The solutions of the first type describe a non-equilibrium state of a gas dynamic system. And the solutions of the second type describe a locally-equilibrium state of a gas dynamic system. The transition from the solutions of the first type to ones of the second type describe a transition of gas dynamic system from a non-equilibrium state to a locally-equilibrium state, and this process is accompanied by emergence of vorticity or turbulence.

Keywords: Skew-Symmetric Forms, Solutions of Two Types, Non-equilibrium State and Locally-Equilibrium, Emergence of Vorticity or Turbulence

1. Introduction

As it is known, the Euler and Navier-Stokes equations describe a flow of a fluid or a gas. They are comprised of the conservation law equations for energy, angular momentum and mass.

It turns out that the solutions to the Euler and Navier-Stokes equations possess some specific properties that enable to describe not only the change of physical quantities but also processes such as a nonequilibrium, transitions to the state of locally equilibrium, origination of various structures and formations such as waves, vortices, turbulent pulsations and so on.

From the Euler and Navier-Stokes equations one can get a relation for entropy as a state functional, from which it follows the fact that the Euler and Navier-Stokes equations have double solutions, namely, the solutions that are not functions (they depends not only on the variables, their derivatives do not made up a differential) and the solutions that are discrete functions. The solutions of the first type are defined on initial coordinate space and describe the non-equilibrium state of a flow. And the solutions of the second type are defined on integrable structures and describe the locally-equilibrium state of a flow. The transition from the solutions of the first type to ones of the second type describes

the process of origination of the vorticity and turbulence.

Such properties of the solution were revealed when studying the problem of consistency the conservation law equations involved into the set of the Euler and Navier-Stokes equations.

[The peculiar properties of the solutions to the Euler and Navier-Stokes equations are properties that are typical for the solutions to the mathematical physics equations that describe such material systems (media) like the thermodynamical, gas-dynamical and cosmologic systems, the system of charged particles, and others. Such equations consist of the conservation law equations for energy, linear momentum, angular momentum, and mass (which made up the set of equations of mathematical physics). When studying the consistency of the conservation law equations, from the mathematical physics equations one obtains the evolutionary relations in skew-symmetrical differential forms for state functionals (such as the action functional, entropy, Pointing's vector, Einstein's tensor, wave function, and others). From the evolutionary relation it follows that the equations of mathematical physics have double solutions, which enable one to describe the processes of emerging various structures and formations (waves, vortices and so on).]

When studying the consistency of conservation law equations involved into the set of the Euler and Navier-Stokes equations the properties of physical quantities of gas-

dynamical system (which commonly do not accounted when solving the Euler and Navier-Stokes equations) are taken into account. Since the physical quantities (like temperature, energy, pressure or density) relates to a single material medium (gas-dynamical system), a connection between them should exist. Such a connection is described by the state functional. For gas-dynamical system the entropy is such state functional. The evolutionary relation obtained from the Euler and Navier-Stokes equations is a relation for entropy as a state functional.

These results are obtained due to the skew-symmetric forms. In this case, the skew-symmetric forms the basis of which are nonintegrable manifolds were used in addition to the closed exterior forms [1].

2. Peculiar Properties of the Solution to the Euler and Navier-Stokes Equations

It is known that the Euler and Navier-Stokes equations are a set of the conservation laws equations for energy, linear momentum and mass [2].

Peculiar properties of the solution to the Euler and Navier-Stokes equations are revealed when studying the problem of consistency of the conservation law equations.

2.1. Analysis of a Consistency of the Conservation Law Equations

Evolutionary Relation

The peculiar properties of solutions to the Euler and Navier-Stokes equations will be investigated for the case of gas-dynamic system, namely, a flow of ideal (nonviscous) gas described by the Euler equations and a flow of a viscid heat-conducting gas described by the Navier-Stokes equations. [The solutions of the Euler equations and those of the Navier-Stokes equations possess common specific features, however, flows of the gas (or fluid) described by the Euler equations and the Navier-Stokes equations are distinguished. This fact relates to the properties of a gas (or fluid).]

Assume that the gas is a thermodynamic system in the state of local equilibrium (whenever the gas -dynamic system itself may be in nonequilibrium state), that is, it is satisfied the relation [3]

$$T\,ds = de + p\,dV \qquad (1)$$

where T, p and V are the temperature, the pressure and the gas volume, and s, e are the entropy and the internal energy per unit volume.

[Relation (1) determines the entropy s as a thermodynamical state function. For a gas-dynamical system the thermodynamical state function describes only the state of the gas-dynamical element (a gas particle). For a gas-dynamical system the entropy is also a state function. But in this case the entropy is a function of space-time coordinates rather then that of thermodynamic variables.]

Let us now analyze the consistency of the conservation laws equations for energy and linear momentum.

The problem of consistency of the equations can be investigated only by introducing two nonequivalent frames of reference.

We introduce two frames of reference: the first is an inertial system and the second is an accompanying one that is connected with the manifold made up by the trajectories of elements of a gas-dynamic system. (The Euler and Lagrangian coordinate systems can be regarded as examples of such frames of reference.)

In the inertial frame of reference the energy equation can be reduced to the form:

$$\frac{Dh}{Dt} - \frac{1}{\rho}\frac{Dp}{dt} = A_1 \qquad (2)$$

where $\rho = 1/V$ is the total derivativewith respect to time, $\rho = 1/V$, h are respectively the density and enthalpy of the gas. A_1 is an expression that depends on the flow characteristic and energetic actions. In the case of ideal gas described by the Euler equations we have $A_1 = 0$. In the case of viscous heat-conducting gas described by the Navier-Stokes equations the expression A_1 can be written as (see [2], Chapter 6, formula (6.2.4))

$$A_1 = \frac{1}{\rho}\frac{\partial}{\partial x_i}\left(-\frac{q_i}{T}\right) - \frac{q_i}{\rho T}\frac{\partial T}{\partial x_i} + \frac{\tau_{ki}}{\rho}\frac{\partial u_i}{\partial x_k} \qquad (3)$$

Here q_i is the heat flux and τ_{ki} is the viscous stress tensor.

Expressing the enthalpy in terms of internal energy e with the help of formula $h = e + p/\rho$ and using the thermodynamic relation (1), equation (2) of the conservation law for energy can be reduced to the form

$$\frac{Ds}{Dt} = A_1 \qquad (4)$$

Here s is the entropy.

Since the total derivative with respect to time is that along the trajectory, in the accompanying frame of reference the equation of the conservation law for energy takes the form:

$$\frac{\partial s}{\partial \xi^1} = A_1 \qquad (5)$$

where ξ^1 is the coordinate along the trajectory.

In the accompanying frame of reference the equation of conservation law for linear momentum can be presented as

$$\frac{\partial s}{\partial \xi^\nu} = A_\nu \qquad (6)$$

where ξ^ν is the coordinate in the direction normal to the trajectory. In the case of two-dimensional flow of ideal gas one can obtain the following expression for the coefficient

A_ν (see [2], Chapter 6, formula (6.7.12)):

$$A_\nu = \frac{\partial h_0}{\partial \xi^\nu} + (u_1^2 + u_2^2)^{1/2} \varsigma - F_\nu + \frac{\partial U_\nu}{\partial t} \qquad (7)$$

where $\varsigma = \partial u_2 / \partial x - \partial u_1 / \partial y$. In the case of viscous gas the expression A_ν includes additional terms related to viscosity.

One can see that in the accompanying frame of reference the equations for energy and linear momentum are reduced to the equations for derivatives of entropy s. In this case equation (5) obtained from the energy equation defines the derivative of entropy along the trajectory, and equation (6), assigned to the equation for linear momentum, defines the derivatives of entropy in the direction normal to trajectory. [Here the attention should be attracted to the fact that the entropy has to obey two equations simultaneously. Such a peculiarity of the entropy just discloses the hidden properties of the Euler and Navier-Stokes equations.].

Equations (5) and (6) can be convoluted into the relation

$$ds = \omega \qquad (8)$$

where $\omega = A_\mu d\xi^\mu$ is the first degree skew-symmetric differential form and $\mu = 1, \nu$. (A summing over repeated indices is carried out.)

Since the conservation law equations are evolutionary ones, the relation obtained is also an evolutionary relation. In this case the skew-symmetric form ω is evolutionary one as well.

Relation (8) has been obtained from the conservation law equation for energy and linear momentum. In this relation the form ω is that of the first degree. Taking into account the conservation law equations for angular momentum and mass, the evolutionary relation may be written as

$$ds = \omega^p \qquad (9)$$

where the degree of form p takes the values $p = 1, 2, 3$. [Skew-symmetric forms (such as ω, for example), which are obtained from differential equations, are defined on nonintegrable (accompanying) manifolds as opposed to exterior forms, which are defined on integrable manifolds or structures. Such skew-symmetric forms which are evolutionary ones, possess the properties that enable one to investigate differential equations. From those one can obtain closed inexact exterior forms, which are invariants and describe physical structures. This gives a possibility to understand the mechanism of origination of various physical structures].

Peculiarities of the evolutionary relation

Evolutionary relation (8) (as well as relation (9)) has a certain peculiarity. This relation appears to be nonidentical. This relates to the fact that this relation involves the skew-symmetric differential form ω, which is unclosed and cannot be a differential like the left-hand side of this relation. The evolutionary form ω is not closed since the differential of evolutionary form ω and its commutator are nonzero.

The differential of evolutionary form ω is expressed as

$d\omega = \sum K_{1\nu} d\xi^1 d\xi^\nu$, where $K_{1\nu}$ are components of the form commutator. Without accounting for terms that are connected with the deformation of the manifold made up by the trajectories, the commutator can be written as

$$K_{1\nu} = \frac{\partial A_\nu}{\partial \xi^1} - \frac{\partial A_1}{\partial \xi^\nu} \qquad (10)$$

The coefficients A_μ of the form ω have been obtained either from the equation of the conservation law for energy or from that for linear momentum. This means that in the first case the coefficients depend on the energetic action and in the second case they depend on the force action. In actual processes energetic and force actions have different nature and appear to be inconsistent. The commutator of the form ω constructed of the derivatives of such coefficients is nonzero. Since the commutator of the form ω is nonzero, this means that the differential of the form ω is nonzero as well. Thus, the form ω proves to be unclosed and is not a differential. In the left-hand side of relation (8) it stands a differential, whereas in the right-hand side it stands an unclosed form that is not a differential. Such a relation cannot be an identical one. [The form ω is not a closed form due to the fact that it is defined on deforming (nonintegrable) manifold made up by the trajectories of the elements of the described material system. The commutator of such a system will include an additional term, namely, a commutator of metric form which is nonzero. This emphasizes once more that the differential of the form ω is nonzero, that is, the form ω cannot be a closed one and therefore the evolutionary relation cannot be identical.]

It is evident that the nonidentity of the evolutionary relation relates to inconsistence of the conservation law equations followed from the noncommutativity of the conservation law equations.

The evolutionary relation possesses one more peculiarity, namely, this relation is a selfvarying relation. The nonidentical evolutionary relation is a selfvarying one, because, firstly, it is a nonidentical, namely, it contains two objects one of which appears to be unmeasurable (since it is not differential), and, secondly, it is an evolutionary relation, that is, the variation of any object of the relation in some process leads to a variation of another object; and, in turn, the variation of the latter leads to variation of the former. Since one of the objects is an unmeasurable quantity, the other cannot be compared with the first one, and hence, the process of mutual variation cannot stop.

2.2. Double Solutions of Euler and Navier-Stokes Equations

Since from the Euler and Navier-Stokes equations it follows the evolutionary relation which is not integrated directly because the second term of the evolutionary relation is not a differential, this points out to the fact that the Euler and Navier-Stokes equations turn out to be nonintegrable.

This means that the solutions to equations are not functions, i.e. they depend not only on variables. They will

depend on the commutator of the form ω that enters into the evolutionary relation. (If the commutator be equal to zero, the evolutionary relation would be identical and the equations would be integrated directly). [Hereafter these solutions will be referred to as the solutions of the first type or "inexact solutions". However, one has to keep in mind that these solutions are not approximate ones. Inexactness is related to the fact that they cannot be represented analytically because they are not functions. Inexact solutions describe the quantities that are not the inherent quantities of a gas-dynamic system. The inconsistency of these quantities, as it will be said below, brings a system into a non-equilibrium state.]

Nonidentical evolutionary relation possesses a certain peculiarity. From this relation it follows that, under additional conditions the Euler and Navier-Stokes equations can have discrete solutions that are exact ones, i.e. they are functions.

The Euler and Navier-Stokes equations can have exact solutions only in the case if from the evolutionary skew-symmetric form ω in the right-hand side of nonidentical evolutionary relation it is realized a closed skew-symmetric form, which is a differential. (In this case the identical relation is obtained from the nonidentical relation, and this will point out to a consistency of the conservation law equations and an integrability of the Euler and Navier-Stokes equations.)

But here there is some delicate point.

From the evolutionary unclosed skew-symmetric form, whose differential is nonzero, one can obtain a closed exterior form with a differential being equal to zero only under degenerate transformation, namely, under a transformation that does not conserve differential. (The Legendre transformation is an example of such a transformation.)

Degenerate transformations can take place under additional conditions, which are associated with degrees of freedom.

The vanishing of such functional expressions as determinants, Jacobians, Poisson's brackets, residues and others corresponds to the additional conditions.

These conditions specify the integral surfaces (pseudostructures): the characteristics (the determinant of coefficients at the normal derivatives vanishes), the singular points (Jacobian is equal to zero), the envelopes of characteristics of Euler's equations and so on.

The conditions of degenerate transformation can be realized (if there are degrees of freedom) under change of nonidentical evolutionary relation, which, as it was noted, appears to be a selfvarying relation.

If the conditions of degenerate transformation are realized, from the unclosed evolutionary form ω (see evolutionary relation (8)) with nonvanishing differential $d\omega \neq 0$, one can obtain the differential form closed on pseudostructure. The differential of this form equals zero. That is, it is realized the transition

$$d\omega \neq 0 \rightarrow (\text{degenerate transformation}) \rightarrow$$
$$d_\pi \omega = 0, \; d_\pi{}^*\omega = 0$$

The realization of the conditions $d_\pi{}^*\omega = 0$ and $d_\pi \omega = 0$ means that it is realized a closed dual form $^*\omega$, which describes some integrable structure π, and it is obtained the closed exterior form ω_π with an integrable structure as a basis. (It should be noted that the integrable structure is a pseudostructure with respect to its metric properties.) In the present case closed exterior forms ω_π will be inexact exterior forms since they are defined only on integrable structures.

Thus, it appears that under degenerate transformation the closed inexact (defined only on pseudostructure) exterior form (with the differential being equal to zero) is realized. Such a form is a differential. (It should be emphasized that such a differential is an interior one: it asserts only on pseudostructure, which is defined by the condition of degenerate transformation).

On the pseudostructure π, which is an integrable structure, from nonidentical evolutionary relation (8) it is obtained the relation

$$d_\pi(s) = \omega_\pi \qquad (11)$$

which occurs to be an identical one, since the form ω_π is a differential.

Thus, on the pseudostructure, which is an integrable structure, from the evolutionary relation $ds = \omega$ it is obtained the identical relation $d_\pi(s) = \omega_\pi$.

The identity of the relation obtained from the evolutionary relation, firstly, means that on the realized integrable structure the equations of conservation laws become consistent (this points out that the conservation laws for energy and momentum become to be locally commutative ones). And, secondly, the identical relation composed of differentials can be directly integrated. This means that the Euler or Navier-Stokes equations considered become locally integrable (only on integrable structure).

Thus it appears that on integrable structures the desired quantities of gas-dynamic system (such as the temperature, pressure, density) become functions, that is, they depend only on variables and do not depend on the commutator (and on the path of integrating). These are generalized solutions, which are the discrete functions, since they are realized only under additional conditions (on the integrable structures). Such solutions may be found by means of integrating the Euler equations or the Navier-Stokes equations on integrable structures. Since generalized solutions are defined only on realized integrable structures, the gas-dynamic functions or their derivatives have discontinuities in the direction normal to integrable structure. When going through integral surfaces the gas-dynamic functions or their derivatives suffer breaks (contact breaks). (In the paper [4] the expressions for calculation of such brakes of derivatives in the direction normal to characteristics and to trajectories are presented.)

Thus, one can see that the Euler and Navier-Stokes equations can have the solutions of two types:

(1) the inexact solutions that are not functions, i.e., they depend on not only variables (such solutions are obtained on initial coordinate space)
and

(2) the generalized solutions, which are the discrete functions (such solutions are obtained on integral structures).

The specific feature is the fact that the solutions to the Euler and Navier-Stokes equations are defined on different spatial objects. The degenerate transformations, under which the discrete solutions are obtained (that is, under which closed exterior forms are obtained from evolutionary form), are realized as a transition from nonintegrable accompanying manifold (on which the evolutionary form is defined) to the integrable structures with a closed forms. Mathematically, it is a transition from one frame of reference to another nonequivalent frame of reference (from accompanying frame of reference to a locally-inertial on obtained integrable structures).

3. Process of Originating Vorticity and Turbulence

Below it will be shown that inexact solutions describe a non-equilibrium state of a gas-dynamic system, whereas the generalized solutions describe a locally-equilibrium state of a gas-dynamic system. Process of originating vorticity and turbulence is connected with a transition of a gas-dynamic system from non-equilibrium state into a locally-equilibrium one and is described by a transition of inexact solutions to discrete generalized solutions.

3.1. Physical Meaning of Solutions to the Euler and Navier-Stokes Equations

Physical meaning of inexact solutions. Non-equilibrium state of a gas-dynamic system

Inexact solutions have a physical meaning. They describe a nonequilibrium state of gas-dynamic system. This follows from the evolutionary relation. Evolutionary relation (8) has an unique physical meaning because this relation includes a differential of entropy s, which is a state functional. The entropy entered into the evolutionary relation is the functional, which characterizes the state of gas-dynamic system. [Here, it should be called attention to the fact that the entropy, which enters into the evolutionary relation for a gas-dynamic system, depends on space-time coordinates rather then on thermodynamical variables like the entropy entered into the thermodynamical relation (1). The state of gas-dynamic system is characterized by the entropy, which depends on space-time variables. And the entropy that depends on thermodynamical variables characterizes a state of thermodynamic system (a gas particles). In the gas-dynamic system the entropy depended on thermodynamical variables characterizes only the state of a gas rather then the state of gas-dynamic system itself.]

If from relation (8) the differential of entropy could be obtained, this would point to the fact that entropy is a state function. And this would mean that the state of a gas-dynamic system is an equilibrium one.

But, since relation (8) is a nonidentical relation, from that one cannot obtain the differential of entropy and find the state function. This means that the gas-dynamic system is in a non-equilibrium state.

One can see that *the solutions of the Euler and Navier-Stokes equations, which are not functions, describe a nonequilibrium state of gas-dynamic system*.

The nonequilibrium means that in a gas-dynamic system an internal force acts. It is evident that the internal force is described by the commutator of skew-symmetric form ω, on which the inexact solutions of the Euler and Navier-Stokes equations depend. (If the evolutionary form commutator be zero, the evolutionary relation would be identical, and this would point out to the equilibrium state, i.e. the absence of internal forces.) Everything that gives a contribution into the commutator of the evolutionary form ω leads to emergence of internal force that causes the non-equilibrium state of a gas-dynamic system.

From the analysis of the expression A_μ in formulas (3) and (7) one can see that the terms, which are related to the multiple connectedness of the flow domain, the nonpotentiality of the external forces and the nonstationarity of the flow contribute into the commutator (see, formula (7)). In the case of a viscous, non-heat-conducting gas described by the Navier-Stokes equations, the terms related to the transport processes will contribute into the commutator (see, formula (3)). (In a general case the term related to physical-chemical processes will make a contribution into the commutator.)

All these factors lead to emergence of internal forces, that is, to nonequilibrium, and to development of various types of instability. And yet for every type of instability one can find an appropriate term giving contribution into the evolutionary form commutator, which is responsible for this type of instability. Thus, there is an unambiguous connection between the type of instability and the terms that contribute into the evolutionary form commutator in the evolutionary relation. (It can be noted that, for the case of ideal gas described by the Euler equations, Lagrange derived a condition of the eddy-free stable flow. This condition is as follows: the domain must be simple connected one, forces must be potential and the flow must be stationary. One can see that under fulfillment of these conditions, there are no terms that contribute into the commutator).

Here it can be noted that the nonidentity of the evolutionary relation is connected with a noncommutativity of conservation laws. And this points out to the fact that the noncommutativity of conservation laws is a cause of nonequilibrium state of a gas-dynamic system.

Evolutionary relation also describes a variation of non-equilibrium state. This is due to another peculiarity of nonidentical evolutionary relation, namely, this relation is a selfvarying relation.

The process of the evolutionary relation selfvariation describes the process of selfvariation of the gas-dynamic system state. This process proceeds under the internal force action and is described by inexact solutions.

Physical meaning of generalized solutions (discrete functions). Locally-equilibrium state of a gas-dynamic system

As it has been shown above, under degenerate transformation the identical relation is obtained from nonidentical one.

From identical relation one can obtain the differential of entropy *ds* and find entropy *s* as a function of space-time coordinates. It is precisely the entropy that will be a gas-dynamic function of state. The availability of gas-dynamic function of state would point out to equilibrium state of a gas-dynamic system. However, since the identical relation is satisfied only under additional conditions, such a state of gas-dynamic system will be a locally-equilibrium one.

The generalized solutions, which are discrete functions, describe such locally-equilibrium state of gas-dynamic system. Below it will be shown to what discrete function correspond.

3.2. Transition of Gas-Dynamic System from Nonequilibrium State to Locally-Equilibrium State Origination of Vorticity and Turbulence

One can see that the transition from inexact solutions to exact (generalized) solutions is assigned to the transition of gas-dynamic system from non-equilibrium state to locally-equilibrium state.

Since the non-equilibrium state has been induced by an availability of internal force and in the case of locally-equilibrium state there is no internal force (in local domain of gas-dynamic system), it is evident that under transition of gas-dynamic system from non-equilibrium state into locally-equilibrium state the nonmeasurable quantity, which acts as internal force, converts into a measurable quantity. This manifests itself in the form of arising a certain observable measurable formation. Waves, vortices, turbulent pulsations and so on are examples of such formations.

Exact generalized solutions to the Euler and Navier-Stokes equations, which are discrete functions, describe such observable formations.

One can see that only with the help of double solutions such processes like the emergency of vorticity and the development of turbulence can be described. (If there were existed only smooth solution, the description of the turbulence origination would be impossible.) [It should be noted that closed dual forms and closed inexact exterior forms, which are realized under degenerate transformations, made up a differential-geometric structure, i.e. a pseudostructure (integrable structure) with conservative quantity (closed exterior form describes a conservative quantity because its differential equals zero). Realization of such differential-geometric structure (under degenerate transformation) points out to emergence of physical structure. The characteristics, the singular points, the envelopes of characteristics, and other structures with conserved quantities are examples of such physical structures. The origination of physical structure reveals as a new measurable and observable formation (waves, vertices, turbulent pulsations) that spontaneously arises in a gas-dynamic systems.]

Thus, one can see that the transition from inexact solutions to exact (generalized) solutions is assigned to the transition of gas-dynamic system from a non-equilibrium state to a locally-equilibrium state, which is accompanied by the emergence of observable formations. Such observable formations are described by generalized solutions of the Euler and Navier-Stoles equations. In this case the discontinuities of a function, which corresponds to generalized solutions, or their derivatives are defined by a quantity that is described by the commutator of unclosed form ω and acts as an internal force. Such a quantity defines the intensity of formations arisen (if the commutator be equal to zero, the intensity of formation would be equal to zero, i.e. the formation could't arise).

The process of arising observable formations discloses a mechanism of emergence waves, vortices, turbulent pulsations and such phenomena as an emergence of vorticity and turbulence.

[Here it should be emphasized that the conservation laws for energy, linear momentum, and mass, which are noncommutative ones, play a controlling role in these processes.]

3.3. Some Examples of Instability Development and Appearing Observable Formations.

Let as analyze which types of instability and what gas-dynamic formations can originate under given external action.

1) *Flow of ideal (inviscous, heat nonconductive) gas around bodies, described by the Euler equations. Action of nonpotential forces.*

The instability develops because of the multiple connectedness of the flow domain and a nonpotentiality of the body forces. The contribution into the commutator comes from the second and third terms in formula (7). Since the gas is ideal one and $\partial s / \partial \xi^1 = A_1 = 0$, that is, there is no contribution into each fluid particle, an instability of convective type develops. For $U > a$ (U is the velocity of the gas particle, a is the speed of sound) a set of equations of the balance conservation laws belongs to the hyperbolic type and hence the transition to the locally equilibrium state is possible on the characteristics and on the envelopes of characteristics as well, and weak shocks and shock waves are the structures of the system. If $U < a$, when the equations are of elliptic type, such a transition is possible only at singular points. The formations emerged due to a convection are of the vortex type. Under long acting the large-scale structures can be produced.

One can see that in gas-dynamic system, even in the case of ideal gas, it can originate the physical structures and relevant formations that lead to emergence of vorticity.

2) *Flow of a viscous, heat-conducting gas around bodies, described by the Navier-Stokes equations. Boundary layer.*

In the case of a viscous gas the term A_1 (see, formula (3)) connected with transport phenomena will contribute into the evolutionary form commutator. The instability originates due to the multiple connectedness of the domain and the transport phenomena (an effect of viscosity and thermal conductivity). Contributions into the commutator produce the second term in formula (3) and the second and third terms in formula (7). The transition to the locally equilibrium state is allowed at singular points. because in this case $\partial s / \partial \xi^1 = A_1 \neq 0$, that is, the external exposure acts onto the gas particle separately, the development of instability and the transitions to the locally equilibrium state are allowed only in an individual fluid particle. Hence, the formations emerged behave as pulsations. These are turbulent pulsations.

[Studying the instability on the basis of the analysis of entropy behavior was carried out in the works by Prigogine and co-authors [5]. In that works entropy was considered as the thermodynamic function of state (though its behavior along the trajectory was analyzed). By means of such state function one can trace the development (in gas fluxes) of the thermodynamic instability only. To investigate the gas dynamic instability it is necessary to consider entropy as a gas dynamic state function, i.e. as a function of the space-time coordinates. Whereas for studying the thermodynamic instability one has to analyze the commutator constructed by the mixed derivatives of entropy with respect to the thermodynamic variables, for studying the gas-dynamic instability it is necessary to analyze the commutators constructed by the mixed derivatives of entropy with respect to the space-time coordinates.]

3.4. On the Problem of Numerical Solving the Euler and Navier-Stokes Equations

Problems of numerical solving the Euler and Navier-Stokes equations relate to the fact that these solutions are defined on distinct spatial objects. The solutions of one type are defined on initial coordinate space whereas the solutions of another type are defined on integrable structures. Since the solutions are defined on distinct spatial objects, they cannot be obtained by a continuous numerical simulation of derivatives. So, it turns out to be impossible to describe the origination of vorticity and turbulence by this way.

The solutions of first type can be obtained only by numerical modeling the equations on the original nonintegrable manifold (it is impossible to find such a solution by analytical method).

The solutions of the second type (generalized solution) cannot be obtained by modeling the equations on initial coordinate space, since they are obtained on integrable structures.

To obtain the generalized solutions by numerical simulation, one must use second systems of reference (on integrable structure). The generalized solutions can be obtained by analytical methods if the integrability conditions are imposed on the equations. The methods of characteristics, symmetries, eigen-functions and others are examples of such methods.

Therefore, to describe the origination of vorticity and turbulence by numerical simulation, one must use two systems of reference.

That is, a description of evolutionary processes is possible only either by numerical methods, but with two frames of reference, or by using simultaneously numerical and analytical methods.

4. Conclusion

The application of the mathematical apparatus of skew-symmetric differential forms enables to disclose peculiarities of the solutions to the Euler and Navier-Stokes equations, their mathematical and physical meaning.

From the Euler and Navier-Stokes equations one obtains the evolutionary relation, from which it follows that, firstly, the Euler and Navier-Stokes equations have double solutions (this enables to describe discrete transitions) and, secondly, this property discloses a physical meaning of these solutions (the ability to describe a state of gas-dynamic system). Such properties of the solutions to the Euler and Navier-Stokes equations disclose a mechanism of the processes of emerging the vorticity and turbulence [6-10].

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