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Numerical Analysis of Fluid Flow and Heat Transfer Based on the Cylindrical Coordinate System

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Abstract: In this work we will apply the three-dimensional mathematical modelling of fluid flow and heat transfer inside the furnaces based on the cylindrical coordinate system to describe the behavior of the transport phenomena. This modelling is constructed by using the mass, momentum, and energy conservation laws to achieve the continuity equation, the Navier-Stokes equations, and the energy conservation equation. Due to the moving boundary between the solid and melted materials inside of the furnaces we will impose the Stefan condition to describe the behavior of the free boundary between two phases. We will derive the variational formulation of the system of transport phenomena, then we will discrete the domain to complete the finite element method stages and we will obtain the system of nonlinear equations in 256 equations in 256 unknowns. To get the numerical solution of the large-scale system we will prepare a convenient mathematical work and gain some diagrams where they would be applicable in the design process of the furnaces shapes.

Keywords: Fluid Flow, Heat Transfer, Mathematical Modeling, Stefan Condition, Cylindrical Coordinate

1. Introduction

Mathematical modeling of heat transfer is applied to investigate the environment of the furnaces and transport phenomena inside them, because it has the advantages of low cost and acceptable exactness. Quan-Sheng and You-Lan consider the two-dimensional Stefan problem in (1985) and they used the singularity-separating method to prepare the numerical solution. Ungan and Viskanta in (1986), (1987) investigated modeling of circulation and heat transfer in a glass melting tank in three-dimension. Henry and Stavros in (1996) prepared the convenient work about the mathematical modelling of solidification and melting process. Vuik, Segal, and Vermolen in (2000) studied about the discretization approach for a Stefan problem where they focused on interface reaction at the free boundary.

Pilon, Zhao, and Viskanta in (2002), (2006) in their papers considered three-dimensional flow and researched about the behavior of thermal structures in glass melting furnaces by using the three-dimensional mathematical modeling. Their works were included theoretical and numerical sections where they applied sufficient boundary conditions. Sadov, Shivakumar, Firsov, Lui, and Thulasiram, provided an article about the mathematical model of ice melting on transmission lines in (2007). Choudhary, Venuturumilli, and Matthew in (2010) in their common paper introduce the mathematical modeling of flow and heat transfer phenomena in glass melting where it consists of delivery and forming processes, especially the turbulent conditions has been discussed in the paper for Newtonian and non-Newtonian fluids. Kambourova, and Zheleva had modelized and described temperature distributions ina tank of glass melting furnace in (2002).

The author studied the mathematical modeling of heat transfer and transport phenomena in (2016) in two-dimension with Stefan free boundary based on stream functions. due to the advantages of stream functions mathematical modeling has prepared and by invoking the finite element method the numerical solution of the transport phenomena derived, also we did the same work in three-dimension. In the current work we will apply the mathematical modeling in three-dimension for the especial furnaces with cylindrical shapes, that they are called Garnissage tank. Due to their shapes we will illustrate the mathematical equations of transport phenomena in the cylindrical coordinate system in three-dimension to use its symmetric properties.

The melting process in the Garnissage furnace starts when we impose the heat to the solid material by electrical boosters inside it, but the heat transferring by electrical boosters couldn't melt the whole materials and whenever the central parts are melted the parts far from the boosters is solid. In this case the boundary between the solid and liquid materials are moving during the process, then we will have free boundary problem. The Stefan condition is sufficient tool to describe the behavior of the free boundary and we will handle three-dimensional version of the Stefan condition in the mathematical modeling process.

We state the conservation equations in the cylindrical coordinate, so the Stefan condition, then we convert the system of equations into the weak formulation. For expressing the system in the variational formulation we will use convenient test functions with small support and then we will discrete the domain to follow the finite element method. The approximate values of the variables would be applied instead of the variables, and finally we will get the system of equations that they are combination of linear and nonlinear equations, thus to solve the system numerically the Newton's method would be recommended.

2. Mathematical Modeling

We start the paper by reviewing the mathematical modeling of fluid flow in three-dimension. Cylindrical coordinate is applied for modeling because it provides the convenient environment to express the modeling, hence we can use the symmetric properties to introduce the equations. There are three different parts in the modeling process which we will introduce them.

2.1. Continuity Equation

$$\frac{1}{r}\frac{\partial}{\partial r}(\rho r u_r) + \frac{\partial(\rho u_z)}{\partial z} = 0 \tag{1}$$

2.2. Navier-Stokes Equations

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + u_z\frac{\partial u_r}{\partial z}\right) = -\frac{\partial p}{\partial r} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right) + \frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2}\right) + \rho g_r \tag{2}$$

$$\rho\left(\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + u_z\frac{\partial u_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{\partial^2 u_z}{\partial z^2}\right) + \rho g_z \tag{3}$$

2.3. Energy Equation

$$\frac{\partial \theta}{\partial t} + u_r\frac{\partial \theta}{\partial r} + u_z\frac{\partial \theta}{\partial z} = \frac{k}{\rho c}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta}{\partial r}\right) + \frac{\partial^2 \theta}{\partial z^2}\right) + \frac{1}{\rho c}S \tag{4}$$

Deriving the weak formulation is second stage of the work after the mathematical modeling, then the variational version of the modeling has been obtained by imposing the convenient smooth test function $\eta$with small support and using the integration by parts technique. Like the classical modeling the variational formulation has three sections:

2.4. Continuity Equation (Weak Formulation)

$$\int_\Omega u_r\left(\frac{\eta}{r} - \frac{\partial \eta}{\partial r}\right) - \int_\Omega u_z\frac{\partial \eta}{\partial z} = 0 \tag{5}$$

2.5. Navier-Stokes Equations (Weak Formulation)

$$\int_\Omega u_r\left(\frac{\partial \eta}{\partial t} + \left(\frac{1}{2}u_r + \frac{1}{r}\right)\frac{\partial \eta}{\partial r} + u_z\frac{\partial \eta}{\partial z} + \eta\frac{\partial u_z}{\partial z} - \frac{\partial^2 \eta}{\partial r^2} - \frac{\partial^2 \eta}{\partial z^2}\right) = -\frac{1}{\rho}\int_\Omega p\frac{\partial \eta}{\partial r} - \int_\Omega g_r\eta \tag{6}$$

$$\int_\Omega u_z\left(\frac{\partial \eta}{\partial t} + u_r\frac{\partial \eta}{\partial r} + \eta\frac{\partial u_r}{\partial r} + \frac{1}{2}u_z\frac{\partial \eta}{\partial z} + \frac{\partial^2 \eta}{\partial r\partial z} + \frac{\partial^2 \eta}{\partial z^2}\right) = -\frac{1}{\rho}\int_\Omega p\frac{\partial \eta}{\partial z} - \int_\Omega g_z\eta \tag{7}$$

2.6. Energy Equation (Weak Formulation)

$$\int_\Omega \theta\left(\frac{\partial u_r}{\partial r}\eta + u_r\frac{\partial \eta}{\partial r} + \frac{\partial u_z}{\partial z}\eta + u_z\frac{\partial \eta}{\partial z} + \frac{k}{\rho c}\left(\frac{\partial^2 \eta}{\partial r^2} + \frac{\partial^2 \eta}{\partial z^2}\right)\right) + \int_\Omega \left(\theta + \frac{k\lambda}{\rho c}\right)\frac{\partial \eta}{\partial t} = -\frac{1}{\rho c}S\int_\Omega \eta \tag{8}$$

3. Discretization of Domain

After weak formulation we discrete the domain Ω, and replace the approximate variables u_{r_h}, u_{z_h}, p_h, and θ_h in the variational formulation to obtain the algebraic system of equations. Assume that $V_h \subset H_0^1(\Omega)$, which consists of test functions that they are piecewise continuous with the fix degree, and also suppose that

$$\dim V_h = N(h),$$

$$V_h = span\{\phi_1, \phi_2, \phi_3, \dots, \phi_{N(h)}\},$$

where the basis functions $\phi_i(r, \varphi, z, t), i = 1,2,\dots,N(h)$, have small support. If we choose the finite dimensional subspace V_h, and also the correspond variables in V_h, then we redefine the heat transfer problem as the problem of finding u_{r_h}, u_{z_h}, p_h, and θ_h such that there satisfy in the equations

(5), (6), (7), and (8). Now we define the approximate solutions u_{r_h}, u_{z_h}, p_h, and θ_h in terms of the basis functions $\phi_i(r,\varphi,z,t)$

$$u_{r_h}(r,\varphi,z,t) = \sum_{i=1}^{N(h)} U_{r_i}\phi_i(r,\varphi,z,t), \tag{9}$$

$$u_{z_h}(r,\varphi,z,t) = \sum_{i=1}^{N(h)} U_{z_i}\phi_i(r,\varphi,z,t), \tag{10}$$

$$p_h(r,\varphi,z,t) = \sum_{i=1}^{N(h)} P_i\phi_i(r,\varphi,z,t), \tag{11}$$

$$\theta_h(r,\varphi,z,t) = \sum_{i=1}^{N(h)} \theta_i\phi_i(r,\varphi,z,t), \tag{12}$$

where $U_{r_i}, U_{z_i}, P_i, \theta_i, i = 1,2,3,\ldots,N(h)$, must be determined. We insert the values of $u_{r_h}, u_{z_h}, p_h, \theta_h \in V_h$ instead of u_r, u_z, p, θin the variational formulation of the heat transfer problem. In this case the heat transfer problem is restated as the problem of finding

$$\left(\mathrm{U_{r_1}, U_{r_2}, \ldots, U_{r_{N(h)}}}\right) \in \mathrm{R^{N(h)}},$$

$$\left(\mathrm{U_{z_1}, U_{z_2}, \ldots, U_{z_{N(h)}}}\right) \in \mathrm{R^{N(h)}},$$

$$\left(\theta_1, \theta_2, \ldots, \theta_{\mathrm{N(h)}}\right) \in \mathrm{R^{N(h)}},$$

where they are satisfied in the system of equations

$$\sum_{i=1}^{N(h)}\left(U_{r_i}a_{ik} - U_{z_i}b_{ik}\right) = 0, \tag{13}$$

$$\sum_{i=1}^{N(h)} U_{r_i}A_{ik} + \sum_{i=1}^{N(h)}\sum_{j=1}^{N(h)} U_{r_i}U_{r_j}B_{ijk} + \sum_{i=1}^{N(h)}\sum_{j=1}^{N(h)} U_{r_i}U_{z_j}C_{ijk} + \sum_{i=1}^{N(h)} P_iD_{ik} + E_k = 0, \tag{14}$$

$$\sum_{i=1}^{N(h)} U_{z_i}A'_{ik} + \sum_{i=1}^{N(h)}\sum_{j=1}^{N(h)} U_{z_i}U_{z_j}B'_{ijk} + \sum_{i=1}^{N(h)}\sum_{j=1}^{N(h)} U_{z_i}U_{r_j}C'_{ijk} + \sum_{i=1}^{N(h)} P_iD'_{ik} + E'_k = 0, \tag{15}$$

$$\sum_{i=1}^{N(h)} \theta_iF_{ik} + \sum_{i=1}^{N(h)}\sum_{j=1}^{N(h)} \theta_i\left(U_{r_j} + U_{z_j}\right)G_{jk} + H_k = 0, \tag{16}$$

$$k = 1,2,3,\ldots,N(h),$$

which coefficients of the system are defined as

Table 1. *System Coefficients.*

$a_{ik} = \int_\Omega \phi_i\left(\frac{\phi_k}{r} - \frac{\partial\phi_k}{\partial r}\right)$	$b_{ik} = \int_\Omega \phi_i\frac{\partial\phi_k}{\partial z}$
$A_{ik} = \int_\Omega \phi_i\left(\frac{\partial\phi_k}{\partial t} - \frac{\partial^2\phi_k}{\partial r^2} - \frac{\partial^2\phi_k}{\partial z^2} + \frac{1}{r}\frac{\partial\phi_k}{\partial r}\right)$	$A'_{ik} = \int_\Omega \phi_i\left(\frac{\partial\phi_k}{\partial t} + \frac{\partial^2\phi_k}{\partial r\partial z} + \frac{\partial^2\phi_k}{\partial z^2}\right)$
$B_{ijk} = \frac{1}{2}\int_\Omega \phi_i\phi_j\frac{\partial\phi_k}{\partial r}$	$B'_{ijk} = \frac{1}{2}\int_\Omega \phi_i\phi_j\frac{\partial\phi_k}{\partial z}$
$C_{ijk} = \int_\Omega \phi_i\left(\phi_j\frac{\partial\phi_k}{\partial z} + \phi_k\frac{\partial\phi_j}{\partial z}\right)$	$C'_{ijk} = \int_\Omega \phi_i\left(\phi_j\frac{\partial\phi_k}{\partial r} + \phi_k\frac{\partial\phi_j}{\partial r}\right)$
$D_{ik} = \frac{1}{\rho}\int_\Omega \phi_i\frac{\partial\phi_k}{\partial r}$	$D'_{ik} = \frac{1}{\rho}\int_\Omega \phi_i\frac{\partial\phi_k}{\partial z}$
$E_k = \int_\Omega g_r\phi_k$	$E'_k = \int_\Omega g_z\phi_k$
$F_{ik} = \int_\Omega \phi_i\left(\frac{\partial\phi_k}{\partial t} + \frac{k}{\rho c}\left(\frac{\partial^2\phi_k}{\partial r^2} + \frac{\partial^2\phi_k}{\partial z^2}\right)\right)$	$G_{jk} = \int_\Omega \left(\phi_k\frac{\partial\phi_j}{\partial r} + \phi_j\frac{\partial\phi_k}{\partial r}\right)$
$H_k = \frac{1}{\rho c}\int_\Omega \left(k\lambda\frac{\partial\phi_k}{\partial t} + S\phi_k\right)$	

We need to determine the values of the coefficients to derive the numerical solution of system, then we must define the sufficient domain and discrete it to compute the relevant integrals, and at last we would found all coefficients and finally we will earn the numerical solution of the system for the interpretation of the fluid behavior in the furnaces.

4. Unknown Coefficients

We continue the process by computing the coefficients in the transport system, for this objective we need to divide the domain Ω to the mesh cubes, the Figure 1 shows the ijk −mesh cube where we will construct the test functions on ijk −mesh cube.

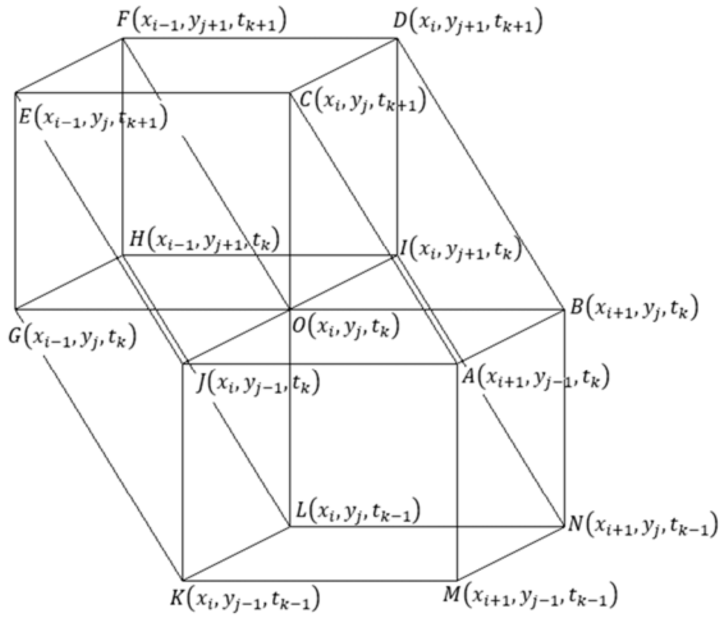

Figure 1. *ijk −Mesh cube.*

To define the test functions we divide the ijk −mesh cube into the 24 tetrahedrons as Figure 2.

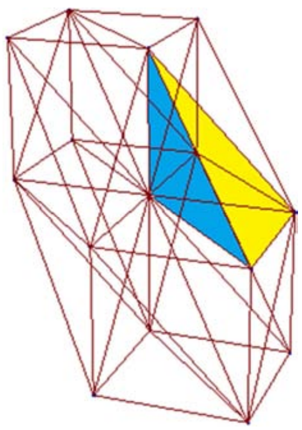

Figure 2. *First tetrahedron.*

Now we are looking for the continuous piecewise linear function $\phi = \phi(x, y, t)$ where it equals to 1 in the origin node ofijk −mesh cube, and 0 in the other nodes.

Suppose that

$$\phi(x, y, t) = a + bx + cy + dt$$

for example to the point (x, y, z)in the first tetrahedron

$$\phi\left(O(x_i, y_j, t_k)\right) = 1$$

$$\phi\left(A(x_{i+1}, y_{j-1}, t_k)\right) = 0$$

$$\phi\left(B(x_{i+1}, y_j, t_k)\right) = 0$$

$$\phi\left(C(x_i, y_j, t_{k+1})\right) = 0$$

then we will have

$$a + x_i b + y_j c + t_k d = 1$$

$$a + x_{i+1} b + y_{j-1} c + t_k d = 0$$

$$a + x_{i+1} b + y_j c + t_k d = 0$$

$$a + x_i b + y_j c + t_{k+1} d = 0$$

We get the solution of the linear system as

$$a = \frac{1}{h}(x_i + t_k) + 1,$$

$$b = d = -\frac{1}{h},$$

$$c = 0,$$

then we derive the test function

$$\phi(x, y, t) = \frac{1}{h}(-x + x_i - t + t_k) + 1$$

We perform the same approach for all of the other tetrahedrons and we demonstrate them as

Parts 1, 2

$$\phi(x, y, t) = \frac{1}{h}(-x + x_i - t + t_k) + 1$$

Parts 3, 4

$$\phi(x, y, t) = \frac{1}{h}(-x + x_i - y + y_j) + 1$$

Parts 5, 6

$$\phi(x, y, t) = \frac{1}{h}(-y + y_j + t - t_k) + 1$$

Parts 7, 8

$$\phi(x,y,t) = \frac{1}{h}(x - x_i + t - t_k) + 1$$

Parts 9, 10

$$\phi(x,y,t) = \frac{1}{h}(x - x_i + y - y_j) + 1$$

Parts 11, 12

$$\phi(x,y,t) = \frac{1}{h}(y - y_j - t + t_k) + 1$$

Parts 13, 14

$$\phi(x,y,t) = \frac{1}{h}(-t + t_k) + 1$$

Parts 15, 16

$$\phi(x,y,t) = \frac{1}{h}(x - x_i) + 1$$

Parts 17, 18

$$\phi(x,y,t) = \frac{1}{h}(-y + y_j) + 1$$

Parts 19, 20

$$\phi(x,y,t) = \frac{1}{h}(t - t_k) + 1$$

Parts 21, 22

$$\phi(x,y,t) = \frac{1}{h}(y - y_j) + 1$$

Parts 23, 24

$$\phi(x,y,t) = \frac{1}{h}(-x + x_i) + 1$$

We are in the position to compute the coefficients of the continuity equation a_{ik} and b_{ik} for the system (13), then we continue the process by determining the a_{ik} for

$$k = i,$$

$$k = i - (N^2 + N - 1),$$

$$k = i + (N^2 + N - 1),$$

and it is trivial that $a_{ik} = 0$ in the other cases. Then we divide the integral as

$$a_{ii} = \int_{\Omega} \phi_i \left(\frac{\phi_i}{r} - \frac{\partial \phi_i}{\partial r}\right)$$

$$= \iiint_{\text{part } 01} \phi_i \left(\frac{\phi_i}{r} - \frac{\partial \phi_i}{\partial r}\right) drdzdt + \iiint_{\text{part } 02} \phi_i \left(\frac{\phi_i}{r} - \frac{\partial \phi_i}{\partial r}\right) drdzdt + \cdots + \iiint_{\text{part } 24} \phi_i \left(\frac{\phi_i}{r} - \frac{\partial \phi_i}{\partial r}\right) drdzdt.$$

We compute the values of integrals in different tetrahedrons separately and derive the final value of a_{ii} as

$$\frac{(x_i + h)(6x_i^2 - 8x_i(x_i + h)^2 + 3(x_i + h)^3)}{6h^2} \ln\left(1 + \frac{h}{x_i}\right) - \frac{x_i^4 - 4x_i h^3 + 3h^4}{6h^2} \ln\left(1 - \frac{h}{x_i}\right) - \frac{x_i(3x_i^2 + h^2)}{9h},$$

and by the similar operation we gain

$$a_{i,i-(N^2+N-1)} = \frac{1}{72h}(12x_i^3 - 6x_i^2 h + 40x_i h^2 - 3h^3) - \frac{1}{6h^2}(x_i^4 + 3x_i^2 h^2 + 2x_i h^3 - 2h^4) \ln\left(1 + \frac{h}{x_i}\right),$$

$$a_{i,i+(N^2+N-1)} = \frac{1}{72h}(12x_i^3 + 6x_i^2 h + 40x_i h^2 + 3h^3) + \frac{1}{6h^2}(x_i^4 + 3x_i^2 h^2 - 2x_i h^3 - 2h^4) \ln\left(1 - \frac{h}{x_i}\right).$$

Also for the coefficient b_{ik} in (13) we get

$$b_{ii} = 0,$$

$$b_{i,i-(N^2+N-1)} = -\frac{h^2}{12},$$

$$b_{i,i+(N^2+N-1)} = \frac{h^2}{12}.$$

Suppose that the domain

$$\Omega = [0.3, 0.9] \times [0.3, 0.9] \times [0.3, 0.9],$$

and $N = 4$, then $h = 0.15$, and

$$x_1 = 0.3, x_2 = 0.45, x_3 = 0.6, x_4 = 0.75,$$

after inserting the values in the system (13) we will reach the coefficients matrix (P, Q) with 64 rows and 128 columns, where

$$P =$$

$$\begin{pmatrix}
0.41 & 0 & \dots & 0 & 0.0009 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0 & 0.62 & 0 & \dots & 0 & 0.0001 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
\vdots & 0 & 0.67 & 0 & \dots & 0 & -0.0003 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0 & \vdots & 0 & 0.41 & 0 & \dots & 0 & -0.0006 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0.003 & 0 & \vdots & 0 & 0.41 & 0 & \dots & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0 & 0.002 & 0 & \vdots & 0 & 0.62 & 0 & \dots & \ddots & 0.0009 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.002 & 0 & \vdots & 0 & 0.67 & 0 & \ddots & 0 & 0.0001 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.002 & 0 & \vdots & 0 & 0.41 & \ddots & \vdots & 0 & -0.0003 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.003 & 0 & \vdots & 0 & \ddots & 0 & \vdots & 0 & -0.0006 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.002 & 0 & \vdots & \ddots & 0.41 & 0 & \vdots & 0 & 0.0009 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.002 & 0 & \ddots & 0 & 0.41 & 0 & \vdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.002 & \ddots & 0 & 0 & 0.62 & 0 & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0.67 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0.003 & 0 & 0 & 0 & 0.41
\end{pmatrix}$$

and

$$Q =$$

$$\begin{pmatrix}
0 & 0 & \dots & 0 & 0.002 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & 0.002 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
\vdots & 0 & 0 & 0 & \dots & 0 & 0.002 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0 & \vdots & 0 & 0 & 0 & \dots & 0 & 0.002 & \ddots & 0 & 0 & 0 & 0 & 0 \\
-0.002 & 0 & \vdots & 0 & 0 & 0 & \dots & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0 & -0.002 & 0 & \vdots & 0 & 0 & 0 & \dots & \ddots & 0.002 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.002 & 0 & \vdots & 0 & 0 & 0 & \ddots & 0 & 0.002 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.002 & 0 & \vdots & 0 & 0 & \ddots & \vdots & 0 & 0.002 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.002 & 0 & \vdots & 0 & \ddots & 0 & \vdots & 0 & 0.002 & 0 \\
0 & 0 & 0 & 0 & 0 & -0.002 & 0 & \vdots & \ddots & 0 & 0 & \vdots & 0 & 0.002 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.002 & 0 & \ddots & 0 & 0 & 0 & \vdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.002 & \ddots & 0 & 0 & 0 & 0 & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & -0.002 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Now we have obtained a linear system of equations that it has 128 variables $U_{r_1}, U_{r_2}, \dots, U_{r_{64}}; U_{z_1}, U_{z_2}, \dots, U_{z_{64}}$ within 64 equations. We will return to the system (13) in the next sections again, thus we focus on the Navier-Stokes system (14) to earn the remainder coefficients.

We continue the process by determining the coefficients of the Navier-Stokes system (14), (15). We start by specifying the coefficient A_{ik}, and as we done before we suppose the ijk −mesh cube that it was divided to 24 tetrahedrons, then after integration process in the whole parts we gain

$$A_{ii} = -\frac{x_i^3+h^3}{3h^2}\ln\left(1+\frac{h}{x_i}\right)+\frac{x_i^3-h^3}{3h^2}\ln\left(1-\frac{h}{x_i}\right)+\frac{6x_i^2+2h^2}{9h}, \tag{17}$$

$$A_{i,i-(N^2+N-1)} = -\frac{1}{18h}(6x_i^2+15x_ih+11h^2)+\frac{(x_i+h)^3}{3h^2}\ln\left(1+\frac{h}{x_i}\right)-\frac{h^2}{12}, \tag{18}$$

$$A_{i,i+(N^2+N-1)} = -\frac{1}{18h}(6x_i^2-15x_ih+11h^2)+\frac{(x_i-h)^3}{3h^2}\ln\left(1-\frac{h}{x_i}\right)+\frac{h^2}{12}. \tag{19}$$

Now we insert the values of A_{ik} from (17), (18), and (19) into the $\sum_{i=1}^{N(h)} U_{r_i} A_{ik}$ from (14) and we earn the coefficient matrix $(A_{ik})_{64\times 64}$ as

$$\begin{pmatrix}
0.008 & 0 & \dots & 0 & 0.004 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0 & 0.003 & 0 & \dots & 0 & 0.002 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
\vdots & 0 & 0.002 & 0 & \dots & 0 & 0.001 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0 & \vdots & 0 & 0.001 & 0 & \dots & 0 & 0.0005 & \ddots & 0 & 0 & 0 & 0 & 0 \\
-0.07 & 0 & \vdots & 0 & 0.008 & 0 & \dots & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0 & -0.33 & 0 & \vdots & 0 & 0.003 & 0 & \dots & \ddots & 0.004 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.78 & 0 & \vdots & 0 & 0.002 & 0 & \ddots & 0 & 0.002 & 0 & 0 & 0 \\
0 & 0 & 0 & -1.43 & 0 & \vdots & 0 & 0.001 & \ddots & \vdots & 0 & 0.001 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.07 & 0 & \vdots & 0 & \ddots & 0 & \vdots & 0 & 0.0005 & 0 \\
0 & 0 & 0 & 0 & 0 & -0.33 & 0 & \vdots & \ddots & 0.001 & 0 & \vdots & 0 & 0.004 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.78 & 0 & \ddots & 0 & 0.008 & 0 & \vdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.43 & \ddots & 0 & 0 & 0.003 & 0 & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0.002 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & -0.07 & 0 & 0 & 0 & 0.001
\end{pmatrix}$$

We continue the process of finding the coefficients of the Navier-Stokes system (14) by focusing on the coefficient B_{ijk} from $\sum_{i=1}^{N(h)}\sum_{j=1}^{N(h)} U_{r_i}U_{r_j}B_{ijk}$, and we achieve

$$B_{iii} = 0$$

$$B_{i,i-(N^2+N-1),i} = -\frac{h^2}{20}$$

$$B_{i,i+(N^2+N-1),i} = \frac{h^2}{20}$$

$$B_{i,i-(N^2+N-1),i-(N^2+N-1)} = \frac{h^2}{20}$$

$$B_{i,i+(N^2+N-1),i+(N^2+N-1)} = -\frac{h^2}{20}$$

$$B_{i,i,i-(N^2+N-1)} = \frac{h^2}{60}$$

$$B_{i,i,i+(N^2+N-1)} = -\frac{h^2}{60}$$

then by applying the values B_{ijk} we will obtain the value of $\sum_{i=1}^{N(h)}\sum_{j=1}^{N(h)} U_{r_i}U_{r_j}B_{ijk}$ as

for $k < 20$

$$\frac{h^2}{10}U_{r_k}U_{r_{k+(N^2+N-1)}} + \frac{h^2}{60}U^2_{r_{k+(N^2+N-1)}}$$

for $20 \le k \le 45$

$$-\frac{h^2}{10}U_{r_k}U_{r_{k-(N^2+N-1)}} + \frac{h^2}{10}U_{r_k}U_{r_{k+(N^2+N-1)}} - \frac{h^2}{60}U^2_{r_{k-(N^2+N-1)}} + \frac{h^2}{60}U^2_{r_{k+(N^2+N-1)}}$$

for $k > 45$

$$-\frac{h^2}{10}U_{r_k}U_{r_{k-(N^2+N-1)}} - \frac{h^2}{60}U^2_{r_{k-(N^2+N-1)}}$$

Also the values of C_{ijk} are as

$$C_{iii} = 0$$

$$C_{i,i-(N^2+N-1),i} = \frac{h^2}{15}$$

$$C_{i,i+(N^2+N-1),i} = -\frac{h^2}{15}$$

$$C_{i,i,i-(N^2+N-1)} = \frac{h^2}{15}$$

$$C_{i,i,i+(N^2+N-1)} = -\frac{h^2}{15}$$

$$C_{i,i-(N^2+N-1),i-(N^2+N-1)} = -\frac{h^2}{5}$$

$$C_{i,i+(N^2+N-1),i+(N^2+N-1)} = \frac{h^2}{5}$$

then we insert the values C_{ijk} to get the $\sum_{i=1}^{N(h)}\sum_{j=1}^{N(h)} U_{r_i}U_{z_j}C_{ijk}$ as

for $k < 20$

$$-\frac{h^2}{15}U_{r_k}U_{z_{k+(N^2+N-1)}} - \frac{h^2}{5}U_{z_k}U_{r_{k+(N^2+N-1)}} + \frac{h^2}{15}U_{r_{k+(N^2+N-1)}}U_{z_{k+(N^2+N-1)}}$$

for $20 \le k \le 45$

$$-\frac{h^2}{15}U_{r_{k-(N^2+N-1)}}U_{z_{k-(N^2+N-1)}} + \frac{h^2}{5}U_{r_{k-(N^2+N-1)}}U_{z_k} + \frac{h^2}{15}U_{r_k}U_{z_{k-(N^2+N-1)}} - \frac{h^2}{15}U_{r_k}U_{z_{k+(N^2+N-1)}} - \frac{h^2}{5}U_{r_{k+(N^2+N-1)}}U_{z_k} + \frac{h^2}{15}U_{r_{k+(N^2+N-1)}}U_{z_{k+(N^2+N-1)}}$$

for $k > 45$

$$\frac{h^2}{15}U_{r_k}U_{z_{k-(N^2+N-1)}} - \frac{h^2}{15}U_{r_{k-(N^2+N-1)}}U_{z_{k-(N^2+N-1)}} + \frac{h^2}{5}U_{r_{k-(N^2+N-1)}}U_{z_k}$$

We terminate this section by computing the coefficients D_{ik} form (14) and we have

$$D_{ii} = 0,$$

$$D_{i,i-(N^2+N-1)} = \frac{h^2}{12},$$

$$D_{i,i+(N^2+N-1)} = -\frac{h^2}{12},$$

then we will get the coefficient matrix $(D_{ik})_{64\times 64}$ from $\sum_{i=1}^{N(h)} P_i D_{ik}$ as

$$\frac{1}{\rho}\begin{pmatrix}
0 & 0 & \dots & 0 & 0.002 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & 0.002 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
\vdots & 0 & 0 & 0 & \dots & 0 & 0.002 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0 & \vdots & 0 & 0 & 0 & \dots & 0 & 0.002 & \ddots & 0 & 0 & 0 & 0 & 0 \\
-0.002 & 0 & \vdots & 0 & 0 & 0 & \dots & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0 & -0.002 & 0 & \vdots & 0 & 0 & 0 & \dots & \ddots & 0.002 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.002 & 0 & \vdots & 0 & 0 & 0 & \ddots & 0 & 0.002 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.002 & 0 & \vdots & 0 & 0 & \ddots & \vdots & 0 & 0.002 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.002 & 0 & \vdots & 0 & \ddots & 0 & \vdots & 0 & 0.002 & 0 \\
0 & 0 & 0 & 0 & 0 & -0.002 & 0 & \vdots & \ddots & 0 & 0 & \vdots & 0 & 0.002 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.002 & 0 & \ddots & 0 & 0 & 0 & \vdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.002 & \ddots & 0 & 0 & 0 & 0 & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & -0.002 & 0 & 0 & 0 & 0
\end{pmatrix}$$

We note that the last coefficient E_i in the system (14) is

$$E_i = g_r h^3.$$

Now we reach to the nonlinear system in 64 variables $U_{r_1}, U_{r_2}, \dots, U_{r_{64}}$ in 64 equations, then we repeat precisely the same process for the second part of the Navier-Stokes equations (15), and again we get the second nonlinear system, that it has 64 variables $U_{z_1}, U_{z_2}, \dots, U_{z_{64}}$within 64 equations.

5. Numerical Solution

As we have shown in the section (4) equation (13) may be exhibited as a linear system in 64 equations in 128 variables, where we got the coefficients matrix (P, Q) before, now we return to the equation (13) and rewrite it in the matrix form as

$$(P, Q)\begin{pmatrix} U_r \\ U_z \end{pmatrix} = 0,$$

$$PU_r + QU_z = 0.$$

Since the matrix P is invertible therefore we can derive the U_r uniquely as

$$U_r = -P^{-1}QU_z \tag{20}$$

Also remember from section (5) that the Navier-Stokes systems (14), and (15) respectively have the styles

$$AU_r + \psi_1(U_r, U_z) + Dp + E = 0, \tag{21}$$

$$A'U_z + \psi_2(U_r, U_z) + D'p + E' = 0, \tag{22}$$

where the matrices A, D, A', and D' are obtained in the section (5) andψ_1, ψ_2 are the nonlinear parts of the systems (14) and (15). Also we know that

$$D = -D',$$

$$E = h^3 g_r 1,$$

$$E' = h^3 g_z 1.$$

After the summation of (21) and (22) we will reach to

$$AU_r + A'U_z + \psi(U_r, U_z) + E + E' = 0, \tag{23}$$

where $\psi(U_r, U_z) = \psi_1(U_r, U_z) + \psi_2(U_r, U_z)$, and we replace the value of U_r from (20) into the (23) to earn

$$(A' - AP^{-1}Q)U_z + \psi(U_z) = -h^3(g_r + g_z)1 \tag{24}$$

The system (24) has 64 equations within 64 variables. Our purpose is to find its numerical solution by invoking the Newton's method, thus we need the initial solution to start the process of iterations, so we assume that the nonlinear part of the system (24) be zero, that is

$$\psi(U_z) = 0,$$

then

$$(A' - AP^{-1}Q)U_z = -h^3(g_r + g_z)1. \tag{25}$$

From the system (25) immediately we compute

$$U_{z_{21}} = U_{z_{22}} = U_{z_{23}} = U_{z_{25}} = U_{z_{26}} = U_{z_{27}} = U_{z_{29}} = U_{z_{30}} = U_{z_{31}} = U_{z_{33}} = U_{z_{34}} = U_{z_{35}} = U_{z_{37}} = U_{z_{38}} = -16.5$$

Newton's algorithm after sufficient iterations leads to the following solutions of the system (24).

Table 2. *Numerical values of U_z.*

$U_{z_1} = -19.964396203271495$	$U_{z_2} = -19.03263160383139$	$U_{z_3} = -21.585584196054814$
$U_{z_4} = -21.15249196304994$	$U_{z_5} = -19.96439620327149$	$U_{z_6} = -21.02739191619664$
$U_{z_7} = -21.585584196045044$	$U_{z_8} = -5.153356893381632$	$U_{z_9} = -15.080509375582018$
$U_{z_{10}} = -26.18565106460826$	$U_{z_{11}} = -18.13195925635669$	$U_{z_{12}} = -5.153356893381636$
$U_{z_{13}} = -15.08035122328177$	$U_{z_{14}} = -26.18565106460826$	$U_{z_{15}} = -18.13195925635669$
$U_{z_{16}} = -5.153356893381636$	$U_{z_{17}} = -15.08050937558201$	$U_{z_{18}} = -26.18565106460826$
$U_{z_{19}} = -18.13138786041306$	$U_{z_{20}} = 0.9044792016387821$	$U_{z_{21}} = 0.9512084266258982$
$U_{z_{22}} = 0.8210418438489437$	$U_{z_{23}} = 0.8364611015836216$	$U_{z_{24}} = 0.9044792016387821$
$U_{z_{25}} = 0.8527276053620682$	$U_{z_{26}} = 0.8210418438493451$	$U_{z_{27}} = 14.525657477973708$
$U_{z_{28}} = 1.2569219242317637$	$U_{z_{29}} = 0.9665538943001553$	$U_{z_{30}} = 1.0909165624593518$
$U_{z_{31}} = 14.525657477973725$	$U_{z_{32}} = 1.256935110922197$	$U_{z_{33}} = 0.9665538943001551$
$U_{z_{34}} = 1.0909165624593518$	$U_{z_{35}} = 14.525657477973725$	$U_{z_{36}} = 1.2569219242317635$
$U_{z_{37}} = 0.9665538943001551$	$U_{z_{38}} = 1.0909428660709701$	$U_{z_{39}} = -16.96838569358458$
$U_{z_{40}} = -15.81750451636952$	$U_{z_{41}} = -19.11862394842311$	$U_{z_{42}} = -18.95208878856998$
$U_{z_{43}} = -16.96838569358458$	$U_{z_{44}} = -18.34556685135166$	$U_{z_{45}} = -19.11862394841126$
$U_{z_{46}} = -2.881538324002666$	$U_{z_{47}} = -8.018433353991593$	$U_{z_{48}} = 12.77607584911194$
$U_{z_{49}} = 3.629362362017885$	$U_{z_{50}} = -2.881538324002671$	$U_{z_{51}} = -8.01837578716835$
$U_{z_{52}} = 12.776075849111944$	$U_{z_{53}} = 3.629362362017885$	$U_{z_{54}} = -2.881538324002670$
$U_{z_{55}} = -8.018433353991593$	$U_{z_{56}} = 12.776075849111944$	$U_{z_{57}} = 3.6285109525148713$
$U_{z_{58}} = 1.365358543778119$	$U_{z_{59}} = 1.0708011223774292$	$U_{z_{60}} = 1.8262529864312615$
$U_{z_{61}} = 1.7875947952490874$	$U_{z_{62}} = 1.3653585437781197$	$U_{z_{63}} = 1.8781813493164021$
$U_{z_{64}} = 1.8262529864328112$		

Now we refer to the relation (10) to simulate the functionu_{z_h}, and the result is exhibited in the Figure 3.

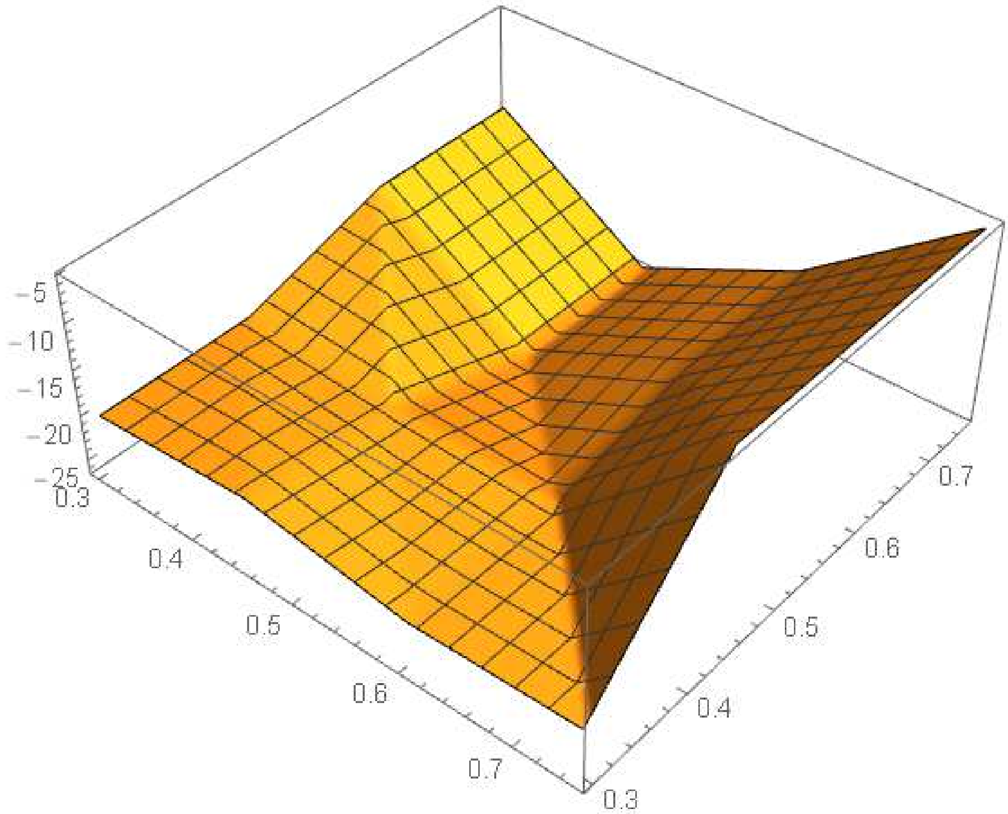

Figure 3. *Simulation of the function* u_{z_h}.

We insert U_z valuesto compute the U_r from (20).

Table 3. *Numerical values of* U_r.

$U_{r_1} = -0.00438008$	$U_{r_2} = -0.00306589$	$U_{r_3} = -0.00245443$
$U_{r_4} = -0.00408989$	$U_{r_5} = -0.00438008$	$U_{r_6} = -0.00274863$
$U_{r_7} = -0.00245443$	$U_{r_8} = -0.0708664$	$U_{r_9} = -0.00605582$
$U_{r_{10}} = -0.0030872$	$U_{r_{11}} = -0.0032878$	$U_{r_{12}} = -0.0708664$
$U_{r_{13}} = -0.0060558$	$U_{r_{14}} = -0.0030872$	$U_{r_{15}} = -0.0032878$
$U_{r_{16}} = -0.0708664$	$U_{r_{17}} = -0.0060558$	$U_{r_{18}} = -0.0030872$
$U_{r_{19}} = -0.0032879$	$U_{r_{20}} = -0.0145846$	$U_{21} = -0.0156674$
$U_{r_{22}} = -0.0079492$	$U_{r_{23}} = -0.0065575$	$U_{r_{24}} = -0.0145846$
$U_{r_{25}} = -0.0130577$	$U_{r_{26}} = -0.0079492$	$U_{r_{27}} = -0.0065490$
$U_{r_{28}} = -0.0343992$	$U_{r_{29}} = -0.1900569$	$U_{r_{30}} = -0.0701881$
$U_{r_{31}} = -0.0065490$	$U_{r_{32}} = -0.0343987$	$U_{r_{33}} = -0.190056$
$U_{r_{34}} = -0.0701881$	$U_{r_{35}} = -0.0065490$	$U_{r_{36}} = -0.0343992$
$U_{r_{37}} = -0.1900564$	$U_{r_{38}} = -0.0701836$	$U_{r_{39}} = -0.0013567$
$U_{r_{40}} = -0.0005378$	$U_{r_{41}} = -0.0046600$	$U_{r_{42}} = -0.0030321$
$U_{r_{43}} = -0.0013567$	$U_{r_{44}} = -0.0049867$	$U_{r_{45}} = -0.0046600$
$U_{r_{46}} = 0.04687816$	$U_{r_{47}} = 0.00385469$	$U_{r_{48}} = 0.00610555$
$U_{r_{49}} = 0.00566393$	$U_{r_{50}} = 0.04687813$	$U_{r_{51}} = 0.00385473$
$U_{r_{52}} = 0.00610555$	$U_{r_{53}} = 0.00566393$	$U_{r_{54}} = 0.0468781$
$U_{r_{55}} = 0.00385469$	$U_{r_{56}} = 0.00610555$	$U_{r_{57}} = 0.00624877$
$U_{r_{58}} = -0.0547324$	$U_{r_{59}} = -0.0472148$	$U_{r_{60}} = -0.0932275$
$U_{r_{61}} = -0.0924344$	$U_{r_{62}} = -0.0547324$	$U_{r_{63}} = -0.054748$
$U_{r_{64}} = -0.0932275$		

Now we insert the values of U_r into the relation (9) to simulate the functionu_{r_h}, and we will show the result in the Figure 4.

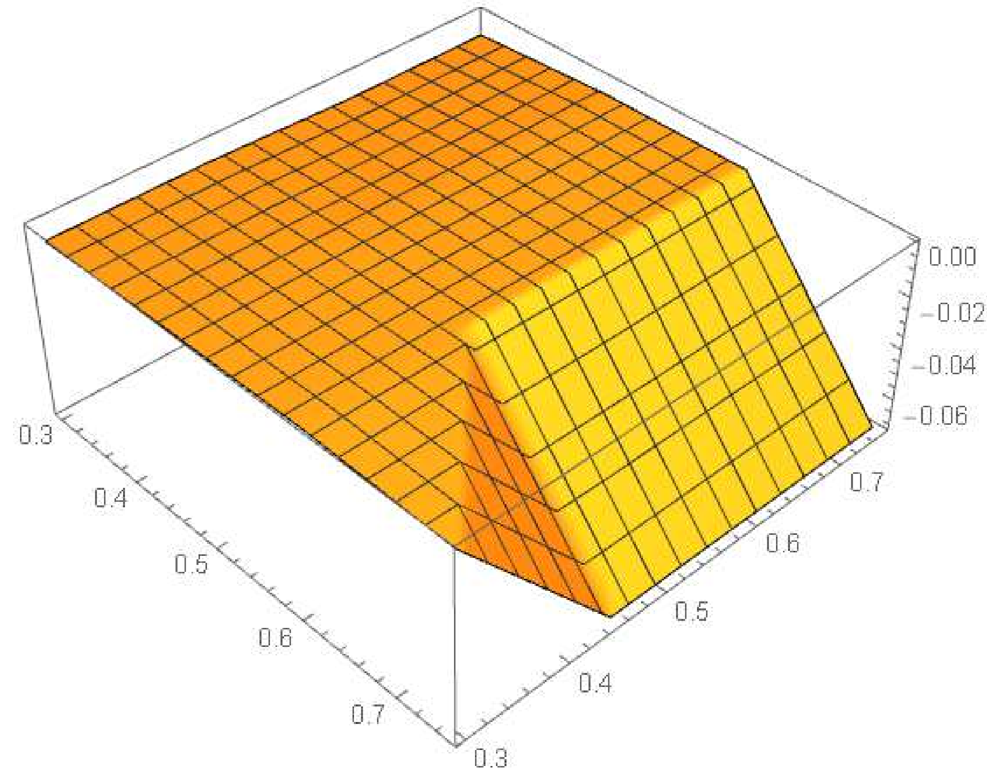

Figure 4. *Simulation of the function* u_{r_h}.

In the section 5 we have computed the values of U_r and U_z, then in the final stage we are ready to determine the coefficients of energy conservation equation. We can apply the computed values of U_r and U_z to earn the linear system of equations in 64 equations in 64 variables. After determining the relevant integrals in 24 tetrahedrons and summation we prepare the necessary coefficients as

$$H_{iii} = 10h,$$

$$H_{i,i-(N^2+N-1),i} = 0,$$

$$H_{i,i-(N^2+N-1),i-(N^2+N-1)} = 0,$$

$$H_{i,i,i-(N^2+N-1)} = \frac{25}{3}h,$$

$$H_{i,i+(N^2+N-1),i} = 0,$$

$$H_{i,i+(N^2+N-1),i+(N^2+N-1)} = 0,$$

$$H_{i,i,i+(N^2+N-1)} = \frac{25}{3}h,$$

$$I_k = -\frac{F}{\rho c}h^3,$$

Also we compute the G_{ijk} by applying the Mathematica codes, and at last we derive θ as

Table 4. *Numerical values of* θ.

$\theta_1 = 267464.5078493500$	$\theta_2 = 248522.6313318577$	$\theta_3 = 307632.2765236602$
$\theta_4 = 307560.1642197024$	$\theta_5 = 279837.7344858227$	$\theta_6 = 321624.6206048220$
$\theta_7 = 427070.98850310955$	$\theta_8 = 196160.3580918937$	$\theta_9 = 3066.7775345885548$
$\theta_{10} = 186800.7827549864$	$\theta_{11} = 121034.4905810277$	$\theta_{12} = 196594.3191942781$
$\theta_{13} = 2152.441237715346$	$\theta_{14} = 188106.5416768024$	$\theta_{15} = 121684.1356482915$
$\theta_{16} = 197507.5771545514$	$\theta_{17} = -74.8655135629144$	$\theta_{18} = 191421.1027365251$
$\theta_{19} = 123429.0998086314$	$\theta_{20} = -13968.2246030608$	$\theta_{21} = -12457.9582485794$

$\theta_{22} = -16553.2395549419$	$\theta_{23} = -15559.4804962354$	$\theta_{24} = -13416.8361963791$
$\theta_{25} = -14444.3019556148$	$\theta_{26} = -12094.5116324607$	$\theta_{27} = 2288889.553754123$
$\theta_{28} = -19412.9119058730$	$\theta_{29} = -26223.6525347567$	$\theta_{30} = -16086.2781566500$
$\theta_{31} = 2294428.379661617$	$\theta_{32} = -19492.9492477656$	$\theta_{33} = -26169.6517726984$
$\theta_{34} = -16049.7064357589$	$\theta_{35} = 2306132.915719399$	$\theta_{36} = -19683.5593562787$
$\theta_{37} = -26029.2197438298$	$\theta_{38} = -15944.7220984291$	$\theta_{39} = 171548.6330508735$
$\theta_{40} = 160669.1329869789$	$\theta_{41} = 196734.1927353769$	$\theta_{42} = 196477.2036791131$
$\theta_{43} = 186387.4932733776$	$\theta_{44} = 220899.00388040626$	$\theta_{45} = 333148.2212944843$
$\theta_{46} = -178843.4387299384$	$\theta_{47} = -198087.481717315$	$\theta_{48} = 85489.45076041944$
$\theta_{49} = 23049.55145269956$	$\theta_{40} = -179359.7530518020$	$\theta_{51} = -200293.621391681$
$\theta_{52} = 82636.400388273181$	$\theta_{53} = 20157.04122994467$	$\theta_{54} = -180465.9496057742$
$\theta_{55} = -205556.800967222$	$\theta_{56} = 75306.140893004793$	$\theta_{57} = 12048.14127260225$
$\theta_{58} = -31523.73111885111$	$\theta_{59} = -31697.6883903928$	$\theta_{60} = -31297.29104957784$
$\theta_{61} = -30510.6513150699$	$\theta_{62} = -30577.26948649386$	$\theta_{63} = -26586.2177068013$
$\theta_{64} = -19116.16815249271$		

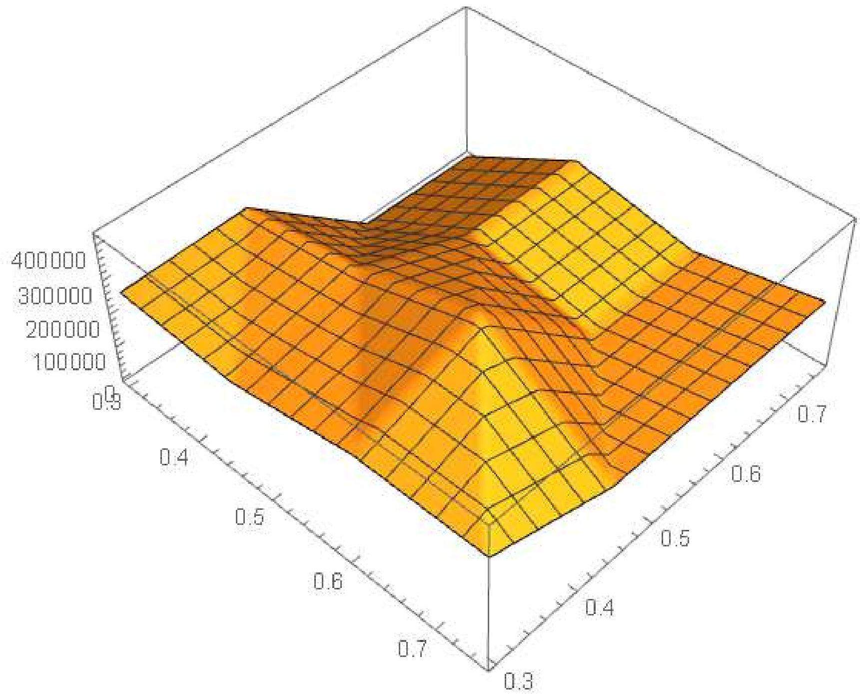

Figure 5. *Simulation of the function θ_h.*

We enter the values of θ into the relation (12) to simulate the function θ_h, and we will show the result in the Figure 5.

Numerical solution of the transport phenomena and its mathematical simulation, where they have been gotten in the current work, are suitable tools to describe the environment of furnaces and fluid flow inside them. Also they help to designers to determine the optimal position of the electrical boosters. In particular designers analyze the mathematical simulation to decide about the optimized style of the furnaces, then we hope this work is useful for them.

6. Summary and Conclusion

In this work we performed the process to get the mathematical modeling of heat transfer in the Garnissage furnace in three dimension in the cylindrical coordinate system. The cylindrical coordinate system has chosen for the modeling process because of its symmetric advantages, then we applied the physical conservation laws, that is the mass, momentum, and energy conservation laws, to achieve the continuity, Navier-Stokes, and heat equation. To modeling the free boundary between the solid and liquid phase we used the three dimensional version of Stefan condition.

When we derived the mathematical modeling of the transport phenomena immediately we started to rearrange the modeling to the weak formulation by handling the sufficient test functions. In this part we divided the domain by cubes and every cubic parts had 24 tetrahedrons. Then we defined the test functions on the tetrahedrons and we got the coefficients after solving the integrals on the mentioned tetrahedrons, and we completed the finite element technic by constructing the system of equations with 128 variables within 64 linear equations and 64 nonlinear equations. Newton's method has been used to achieve the numerical solution of the system and we simulated the numerical solution of the heat transfer system where that was applicable for furnace designers.

Nomenclature

r, φ, z	cylindrical coordinates	u_{r_h}, u_{z_h}	approximate velocity components in cylindrical coordinate
u_r, u_z	velocity components in cylindrical coordinate	p_h	approximate pressure
t	time	θ	temperature
g_r, g_z	gravity acceleration components	λ	latent heat
c	specific heat capacity	μ	dynamic viscosity
k	thermal conductivity	ρ	density
p	pressure	θ_h	approximate temperature
S	heat source	H_0^1	Sobolev space with compact support

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