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Gaussian as Test Functions in Operator Valued Distribution Formulation of QED

Hasimbola Damo Emile Randriamisy[1], Raoelina Andriambololona[1, *], Hanitriarivo Rakotoson[1, 2], Ravo Tokiniaina Ranaivoson[1], Roland Raboanary[2]

[1]Theoretical Physics Department, National Institute for Nuclear Sciences and Technology, Antananarivo, Madagascar

[2]Physics and Applications Department, Faculty of Sciences, University of Antananarivo, Antananarivo, Madagascar

Email address:

hasinadamo@gmail.com (H. D. E. Randriamisy), hasimbola.randriamisy@esti.mg (H. D. E. Randriamisy),
raoelina.andriambololona@gmail.com (R. Andriambololona), raoelinasp@yahoo.fr (R. Andriambololona),
jacquelineraoelina@hotmail.com (R. Andriambololona), tokhiniaina@gmail.com (R. T. Ranaivoson),
tokiniainaravor13@gmail.com (R. T. Ranaivoson), infotsara@gmail.com (H. Rakotoson), r_raboanary@yahoo.fr (R. Raboanary)

[*]Corresponding author

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Abstract: As shown by Epstein and Glaser, the operator valued distribution (OPVD) formalism permits to obtain a non-standard regularization scheme which leads to a divergences-free quantum field theory. The present formulation gives a tool to calculate finite scattering matrix without adding counter-terms. After a short recall about the OPVD formalism in 3+1-dimensions, Gaussian functions and Harmonic Hermite-Gaussian functions are used as test functions. The field resulting from this approach obeys the Klein-Gordon equation. The vacuum fluctuation calculation then gives a result regularized by the Gaussian factor. The example of scalar quantum electrodynamics theory shows that Gaussian functions may be used as test functions in this approach. The result shows that the scale of the theory is described by a factor arising from Heisenberg uncertainties. The Feynman propagators and a study on loop convergence with the example of the tadpole diagram are given. The formulation is extended to Quantum Electrodynamics. Triangle diagrams anomaly are calculated efficiently. In the same approach, using Lagrange formulae to regularize the singular distribution involved in the scattering amplitude gives a good way to avoid infinities. After applying Lagrange formulae, the test function could be reduced to unity since the amplitude is regular. Ward-Takahashi identity is calculated with this method too and this shows that the symmetries of the theory are unbroken.

Keywords: Quantum Electrodynamics, Triangle Anomaly, Ward-Takahashi Identity, Partition of Unity, Operator Valued Distribution, Gaussian Functions, Tadpole Diagram

1. Introduction

Divergences appear when calculating S-matrix in Quantum Theory [1-3]. Many approaches have been considered to deal with this problem.

The standard renormalization techniques based on the dimensional regularization and Cut-Off method has been used [1, 2, 4]. Operator Valued Distribution approach (OPVD) developed by Epstein and Glaser [5] based on the mathematical theory of distribution is considered in this paper.

In this work, the quadri-momentum is denoted by $\bar{p} = p^\mu e_\mu$ for $\mu = 0,1,2,3$ where $\{e_\mu\}$ is a basis in the Minkowski space and the tri-momentum by $\vec{p} = p^i \vec{e_i}$ for $i = 1,2,3$ in which $\{\vec{e_i}\}$ is the 3-dimensional Euclidian basis. The corresponding scalar products are $\bar{p} \cdot \bar{x} = p^\mu x_\mu$ and $\vec{p} \cdot \vec{x} = p^i x_i$.

In the Minkowski space with signature $(1, 3)$, the field is defined by

$$\phi(\bar{x}) = \langle \mathbb{T}_{\bar{x}} \Phi, g \rangle \quad (1)$$

in which

g is a test function

Φ is an OPVD [5-9]: it is an operator on a Fock space and a distribution acting on the set $\{g\}$ of test functions.

$\mathbb{T}_{\bar{x}}$ is the translation operator defined by the relations

$$\begin{cases} \mathbb{T}_{\bar{x}} g(\bar{y}) = g(\bar{y} + \bar{x}) \\ \langle \mathbb{T}_{\bar{x}} \Phi, g \rangle = \langle \Phi, \mathbb{T}_{-\bar{x}} g \rangle \end{cases}$$

The explicit expression of the field is [7]

$$\phi(\vec{x}, t) = \int \frac{d^3\vec{p}}{(\sqrt{2\pi})^3} \frac{1}{\sqrt{2\omega}} [a(\vec{p}) e^{-i\bar{p}.\bar{x}} + b^\dagger(\vec{p}) e^{i\bar{p}.\bar{x}}] \tilde{g}(\omega, \vec{p}) \quad (2)$$

in which

$a(\vec{p})$ and $b^\dagger(\vec{p})$ are respectively particle annihilation and antiparticle creation operators satisfying the commutation relations

$$\begin{cases} [a(\vec{p}), b^\dagger(\vec{q})] = [a(\vec{p}), b(\vec{q})] = 0 \\ [a(\vec{p}), a^\dagger(\vec{q})] = \delta^3(\vec{p} - \vec{q}) \\ [b(\vec{p}), b^\dagger(\vec{q})] = \delta^3(\vec{p} - \vec{q}) \end{cases} \quad (3)$$

$\omega = p_0 = \sqrt{\vec{p}^2 + m^2}$ is the energy of a free particle, m being its mass.

$\tilde{g}$ is the Fourier transform of the test function g

$$\tilde{g}(\bar{p}) = \frac{1}{(\sqrt{2\pi})^4} \int g(\bar{y}) \, e^{-i\bar{p}.\bar{y}} d^4\bar{y} \quad (4)$$

The Klein-Gordon equation

$$(\partial_\mu \partial^\mu - m^2)\phi(\bar{x}) = 0 \quad (5)$$

for $\mu = 0, \dots, 3$ is satisfied. The cases in which the test functions are Gaussian functions namely Harmonic Hermite-Gaussian functions introduced in the references [10-12] are studied. Denoting the latter $g_n(x)$ and their Fourier transforms $\tilde{g}_n(p)$ their expressions are

$$\begin{cases} g_n(\bar{x}) = Q_n(\bar{x}) e^{-\mathcal{B}_{\mu\nu}(x^\mu - X^\mu)(x^\nu - X^\nu) - iP_\mu x^\mu} \\ \tilde{g}_n(\bar{p}) = \mathcal{R}_n(\bar{p}) e^{-\mathcal{A}^{\mu\nu}(p_\mu - P_\mu)(p_\nu - P_\nu) + iX^\mu(p_\mu - P_\mu)} \end{cases} \quad (6)$$

in which $Q_n(\bar{x})$ and $\mathcal{R}_n(\bar{p})$ are polynomials of degree n respectively with variables x^μ and variables $p_\mu, \mu = 0,1,2,3$). X^μ and P_μ are respectively the mean values of the variables x^μ and p_μ

$$\begin{cases} \int |g_n(\bar{x})|^2 d^4\bar{x} = 1 & \int x^\mu |g_n(\bar{x})|^2 d^4\bar{x} = X^\mu \\ \int |\tilde{g}_n(\bar{p})|^2 d^4\bar{p} = 1 & \int p_\mu |\tilde{g}_n(\bar{p})|^2 d^4\bar{p} = P_\mu \end{cases} \quad (7)$$

$\mathcal{A}^{\mu\nu}$ and $\mathcal{B}_{\mu\nu}$ are the components of second order tensors representing the coordinate and momentum dispersion-codispersion operators corresponding to the ground state (n=0) [11-13].

$$\begin{cases} \int (x^\mu - X^\mu)(x^\nu - X^\nu)) |g_0(\bar{x})|^2 d^4\bar{x} = \mathcal{A}^{\mu\nu} \\ \int (p_\mu - P_\mu)(p_\nu - P_\nu)) |\tilde{g}_0(\bar{p})|^2 d^4\bar{p} = \mathcal{B}_{\mu\nu} \\ \mathcal{A}^{\mu\rho} \mathcal{B}_{\rho\nu} = \frac{1}{4} \delta^\mu_\nu \end{cases} \quad (8)$$

The polynomials $Q_n(\bar{x})$ and $\mathcal{R}_n(\bar{p})$ in the relation (6) can be expressed in terms of Hermite polynomials [11].

For the case $\mathcal{A}^{\mu\nu} = \mathcal{B}_{\mu\nu} = 0$ for $\mu \neq \nu$, the uncorrelated Harmonic Hermite Gaussian functions are

$$\begin{cases} Q_n(\bar{x}) = \prod_{\mu=0}^{3} \frac{H_{n^\mu}\left(\frac{x^\mu - X^\mu}{\sqrt{2\mathcal{A}^{\mu\mu}}}\right)}{\sqrt{2^{n^\mu} n^\mu! (\sqrt{2\pi})^{3+1} \sqrt{\mathcal{A}^{\mu\mu}}}} \\ \mathcal{R}_n(\bar{p}) = \prod_{\mu=0}^{3} \frac{H_{n^\mu}\left(\frac{p_\mu - P_\mu}{\sqrt{2\mathcal{B}_{\mu\mu}}}\right)}{\sqrt{2^{n^\mu} n^\mu! (\sqrt{2\pi})^{3+1} \sqrt{\mathcal{B}_{\mu\mu}}}} \end{cases} \quad (9)$$

in which H_{n^μ} is the Hermite polynomials of degree n^μ. For the ground state, $n^\mu = 0 \; for \; \mu = 0,1,2,3$, the relations (6) give

$$\begin{cases} g_0(\bar{x}) = \frac{e^{-\mathcal{B}_{\mu\nu}(x^\mu - X^\mu)(x^\nu - X^\nu) - iP_\mu x^\mu}}{\sqrt{(\sqrt{2\pi})^{3+1} \prod_{\mu=0}^{3}(\sqrt{\mathcal{A}^{\mu\mu}})}} \\ \tilde{g}_0(\bar{p}) = \frac{e^{-\mathcal{A}^{\mu\nu}(p_\mu - P_\mu)(p_\nu - P_\nu) + iX^\mu(p_\mu - P_\mu)}}{\sqrt{(\sqrt{2\pi})^{3+1} \prod_{\mu=0}^{3}(\sqrt{\mathcal{B}_{\mu\mu}})}} \end{cases} \quad (10)$$

2. Gaussian Functions as Test Functions

In this section, the case in which the test function is the Gaussian function $\tilde{g}_0(\bar{p})$ is considered. For sake of simplicity one chooses $X^\mu = 0$. This approach is suggested by the fact that Gaussian functions correspond to the normal probability distribution and appear in the modeling of many phenomena.

2.1. Calculation of Vacuum Fluctuation

The vacuum fluctuation is given by

$$\vartheta(\bar{x}, \bar{y}) = \langle 0 | \phi(\bar{x}) \phi^\dagger(\bar{y}) | 0 \rangle \quad (11)$$

$|0\rangle$ being the vacuum state and

$$\phi^\dagger(\bar{y}) = \int \frac{d^3\vec{p}}{(\sqrt{2\pi})^3} \frac{1}{\sqrt{2\omega}} [a^\dagger(\vec{p}) e^{i\bar{p}.\bar{y}} + b(\vec{p}) e^{-i\bar{p}.\bar{y}}] \tilde{g}_0(\omega, \vec{p}) \quad (12)$$

then

$$\vartheta(\bar{x}, \bar{y}) = \int \frac{d^3\vec{p}}{(\sqrt{2\pi})^3} \frac{d^3\vec{q}}{(\sqrt{2\pi})^3} \frac{1}{2\omega} \langle 0 | a(\vec{p}) a^\dagger(\vec{q}) | 0 \rangle e^{-i\bar{p}.(\bar{x} - \bar{y})} \times \tilde{g}_0(\omega, \vec{p}) \tilde{g}_0(\omega', \vec{q}) \quad (13)$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2\omega} d^3\vec{q} \, \delta^3(\vec{p} - \vec{q}) e^{-i\bar{p}.(\bar{x} - \bar{y})} \times \tilde{g}_0(\omega, \vec{p}) \tilde{g}_0(\omega', \vec{q}) \quad (14)$$

The integration of (14) over $\vec{q}$ gives

$$\vartheta(\bar{x},\bar{y}) = \int \frac{d^3\vec{p}}{2\omega(2\pi)^3} e^{-i\bar{p}.(\bar{x}-\bar{y})} [\tilde{g}_0(\omega,\vec{p})]^2 \tag{15}$$

This result (15) differs from the classical one by the regularizing factor $[\tilde{g}_0(\omega,\vec{p})]^2$.
Using the expression of $\tilde{g}_0$ in the relation (10) and taking the case $p_0 = \omega$ one has explicitly

$$\vartheta(\bar{x},\bar{y}) = \int \frac{d^3\vec{p}}{2\omega(2\pi)^3} e^{-i\bar{p}.(\bar{x}-\bar{y})} \frac{e^{-2\mathcal{A}^{\mu\nu}(p_\mu-P_\mu)(p_\nu-P_\nu)}}{(\sqrt{2\pi})^3 \Pi_{\mu=0}^3(\sqrt{\mathcal{B}_{\mu\mu}})} \tag{16}$$

2.2. Calculation of Feynman Propagator

The Feynman propagator in the configuration space is

$$\Delta_F(\bar{x}-\bar{y}) = -i\langle 0|T\{\phi(\bar{x})\phi^\dagger(\bar{y})\}|0\rangle \tag{17}$$

in which T is the chronological product [1].

$$T\{\phi(\bar{x})\phi^\dagger(\bar{y})\} = \begin{cases} \phi(\bar{x})\phi^\dagger(\bar{y}) \; if \; x^0 \geq y^0 \\ \phi^\dagger(\bar{y})\phi(\bar{x}) \; if \; y^0 \geq x^0 \end{cases} \tag{18}$$

Then

$$\langle 0|T\{\phi(\bar{x})\phi^\dagger(\bar{y})\}|0\rangle = \theta(t)\langle 0|\phi(\bar{x})\phi^\dagger(\bar{y})|0\rangle + \theta(-t)\langle 0|\phi^\dagger(\bar{y})\phi(\bar{x})|0\rangle \tag{19}$$

in which $t = x^0 - y^0$ in natural units $(c = 1)$ and θ is the Heaviside function.

The expressions (1.2) and (2.2) of fields give

$$\langle 0|T\{\phi(\bar{x})\phi^\dagger(\bar{y})\}|0\rangle = \int \frac{d^4\bar{p}}{(2\pi)^4} \frac{i[\tilde{g}_0(p)]^2 e^{-i\bar{p}.(\bar{x}-\bar{y})}}{[(p^0)^2-(\omega_p)^2+i\varepsilon]} \tag{20}$$

$$= \int \frac{d^4\bar{p}}{(2\pi)^4} \frac{i[\tilde{g}_0(p)]^2}{[(p^0)^2-\vec{p}^2-m^2+i\varepsilon]} e^{-i\bar{p}.(\bar{x}-\bar{y})} \tag{21}$$

The Feynman propagator in the configuration space is

$$\Delta_F(\bar{x}-\bar{y}) = \int \frac{d^4\bar{p}}{(2\pi)^4} \frac{e^{-i\bar{p}.(\bar{x}-\bar{y})}}{[\bar{p}^2-m^2+i\varepsilon]} \times \frac{e^{-2\mathcal{A}^{\mu\nu}(p_\mu-P_\mu)(p_\nu-P_\nu)}}{(\sqrt{2\pi})^4 \Pi_{\mu=0}^3(\sqrt{\mathcal{B}_{\mu\mu}})} \tag{22}$$

The Feynman propagator in momentum space is

$$\tilde{\Delta}_F(\bar{p}) = \frac{1}{[\bar{p}^2-m^2+i\varepsilon]} \frac{e^{-2\mathcal{A}^{\mu\nu}(p_\mu-P_\mu)(p_\nu-P_\nu)}}{(\sqrt{2\pi})^4 \Pi_{\mu=0}^3(\sqrt{\mathcal{B}_{\mu\mu}})} \tag{23}$$

3. Harmonic Hermite-Gaussian Functions as Test Functions

The Harmonic Hermite-Gaussian functions defined in the relation (6) are taken as test functions. The vacuum fluctuation and the Feynman propagators (with $X^\mu = 0$ and $p^0 = \omega$) are in this case

$$\langle 0|\phi(\bar{x})\phi^\dagger(\bar{y})|0\rangle = \int \frac{d^3\vec{p}}{2\omega(2\pi)^3} e^{-i\bar{p}.(\bar{x}-\bar{y})} \times \mathcal{R}_n(\omega,\vec{p}) \frac{e^{-2\mathcal{A}^{\mu\nu}(p_\mu-P_\mu)(p_\nu-P_\nu)}}{(\sqrt{2\pi})^4 \Pi_{\mu=0}^3(\sqrt{\mathcal{B}_{\mu\mu}})} \tag{24}$$

$$\Delta_F(\bar{x}-\bar{y}) = \int \frac{d^4\bar{p}}{(2\pi)^4} \frac{1}{[\bar{p}^2-m^2+i\varepsilon]} e^{-i\bar{p}.(\bar{x}-\bar{y})} \times \mathcal{R}_n(\bar{p}) \frac{e^{-2\mathcal{A}^{\mu\nu}(p_\mu-P_\mu)(p_\nu-P_\nu)}}{(\sqrt{2\pi})^4 \Pi_{\mu=0}^3(\sqrt{\mathcal{B}_{\mu\mu}})} \tag{25}$$

$$\tilde{\Delta}_F(\bar{p}) = \frac{1}{[\bar{p}^2-m^2+i\varepsilon]} \mathcal{R}_n(\bar{p}) \frac{e^{-2\mathcal{A}^{\mu\nu}(p_\mu-P_\mu)(p_\nu-P_\nu)}}{(\sqrt{2\pi})^4 \Pi_{\mu=0}^3(\sqrt{\mathcal{B}_{\mu\mu}})} \tag{26}$$

The difference between the expressions (11), (13) and (24), (25), (26) is the presence of the polynomials $\mathcal{R}_n(\bar{p})$ which is reduced to a product of Hermite polynomials in the case of uncorrelated variables.

4. Application to One Loop Scalar Quantum Electrodynamics Diagram

One loop scalar tadpole diagram contributes to the two points correlation function. The associated amplitude is calculated since it is simple and the regularization factor appears.

4.1. Divergence in the Framework of Standard Formulation

The above results may be applied for the calculation of more complicated entities considered in the scalar field theory. For instance for the calculation corresponding to the tadpole diagram which appear in the calculation of photon propagator in conventional scalar QED [1]
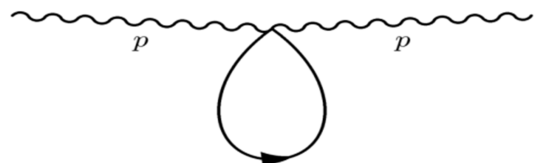

Figure 1. *One loop diagram.*

In the old formulation, the amplitude of this tadpole diagram is given by

$$\mathcal{M}_{\mu\nu} = 2e^2 g_{\mu\nu} \int \frac{d^4\bar{k}}{(2\pi)^4} \frac{i}{\bar{k}^2 - m^2 + i\epsilon} \qquad (27)$$

$$\mathcal{M}_{\mu\nu} = \frac{e^2 g_{\mu\nu}}{8\pi^4} \int_0^{+\infty} \int_0^{\pi} \int_0^{\pi} \int_0^{2\pi} \frac{k_E^3}{k_E^2 + m^2} (\sin\vartheta)^2 (\sin\theta) dk_E d\theta d\vartheta d\varphi = \frac{e^2 g_{\mu\nu}}{4\pi^2} \left[\frac{k_E^2}{2} + \frac{m^2}{2} \ln\left(\frac{k_E^2}{m^2} + 1 \right) \right]_0^{+\infty} \qquad (28)$$

where

$$k_E^2 = k_0^2 + \vec{k}^2$$

The expression (28) shows quadratic and logarithmic divergences. In the standard formulation, one deals with this divergence using regularization and renormalization techniques [1- 2].

4.2. Calculation in the Framework of OPVD Formulation Using Gaussian Functions

Let us now use the Gaussian functions defined in the relation (1.6). The Feynman propagator in momentum space is given by the relation (23). The analog of the integral (27) is then

$$\mathcal{M}_{\mu\nu} = 2e^2 g_{\mu\nu} \int \frac{d^4 k_E}{(2\pi)^4} \frac{1}{k_E^2 + m^2} [\tilde{g}_0(k_E^2)]^2 \qquad (29)$$

As the main interest is the study of convergence, one considers the case

$$\begin{cases} \mathcal{A}^{\mu\nu} = 0 \ if \ \mu \neq \nu \\ \mathcal{A}^{\mu\mu} = \mathcal{A} = \frac{1}{4\mathcal{B}} \\ K_\mu = 0 \end{cases} \qquad (30)$$

The integral (29) becomes

$$\mathcal{M}_{\mu\nu} = 2e^2 g_{\mu\nu} \int \frac{d^4 k_E}{(2\pi)^4} \frac{1}{k_E^2 + m^2} e^{-\frac{k_E^2}{2\mathcal{B}}} \qquad (31)$$

then

$$\mathcal{M}_{\mu\nu} = 4\pi^2 e^2 g_{\mu\nu} \int_0^{+\infty} \frac{k_E^3 e^{-\frac{k_E^2}{2\mathcal{B}}}}{k_E^2 + m^2} dk_E \qquad (32)$$

Performing the variable change $u = \frac{k_E}{\sqrt{2\mathcal{B}}}$, leads to

$$\mathcal{M}_{\mu\nu} = \frac{e^2 g_{\mu\nu} \mathcal{B}}{2\pi^2} \int_0^{+\infty} \frac{u^3 e^{-u^2}}{u^2 + (\frac{m}{\sqrt{\mathcal{B}}})^2} du \qquad (33)$$

Following the interpretation in the references [11-13], the parameter $\sqrt{\mathcal{B}}$ is the common value of the uncertainty (statistical standard deviation) on the values of the energy k_0 and the components k_j $(j = 1,2,3)$ of the spatial momentum of the particle corresponding to the tadpole. In the very large

Introducing quadridimensional spherical coordinate $(k_E, \vartheta, \theta, \varphi)$, the relation (27) becomes energy approximation, $(\frac{m}{\sqrt{\mathcal{B}}})^2 \to 0$, the integral (33) becomes

$$\mathcal{M}_{\mu\nu} = \frac{e^2 \mathcal{B}}{2\pi^2} g_{\mu\nu} \int_0^{+\infty} u e^{-u^2} du = \frac{e^2}{4\pi^2} \mathcal{B} g_{\mu\nu} \qquad (34)$$

As expected, the result is obtained is finite. This result relates explicitly the existence of loop in the photon propagator of scalar QED to the Heisenberg uncertainty relation. This relation is obvious since the Heisenberg principle is already integrated in the representation of quantum phase space formalism.

5. Triangle Axial Anomaly in QED

Let us use Gaussian test function to calculate diagram in QED which involved current conservation. Axial anomaly can be understood by calculating the two triangles graphs
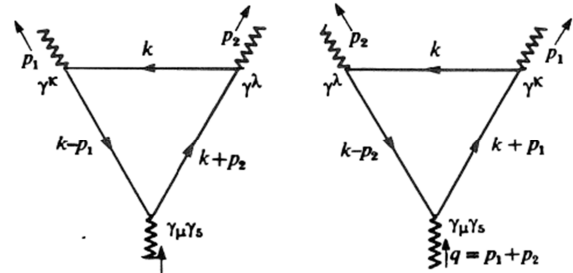

Figure 2. One loop triangle diagrams.

The amplitude of the two graphs contributing to the triangle anomaly can be written as [2]

$$T_{k,\lambda,\mu}(p_1, p_2) = I^1_{k,\lambda,\mu}(p_1, p_2) + I^2_{\lambda,k,\mu}(p_1, p_2) \qquad (35)$$

$I^1_{k,\lambda,\mu}(p_1, p_2)$ represents the amplitude of the first diagram in *Figure 2* and $I^2_{k,\lambda,\mu}(p_1, p_2)$ represents the amplitude of the second diagram in *Figure 2*.

The conservation of the axial currents at the vertices will be shown [2]

$$(p_1 + p_2)^\mu T_{k,\lambda,\mu} = 0 \qquad (36)$$

Using Feynman rules and after some algebra calculation the amplitude is equal to

$$(p_1 + p_2)^\mu (I^1_{k,\lambda,\mu} + I^2_{k,\lambda,\mu}) = 4e^2 \epsilon_{\sigma\lambda\delta k} \int \frac{d^4\bar{k}}{(2\pi)^4} \frac{k^\delta}{\bar{k}^2} \times \{p_1^\sigma \Delta\tilde{g} - p_2^\sigma \Delta\tilde{g}'\}$$

where

$$\begin{cases} \Delta\tilde{g} = \tilde{g}\left(\frac{\bar{k}^2}{2\mathcal{B}}\right)\left[\tilde{g}\left(\frac{(\bar{k}-\bar{p}_2)^2}{2\mathcal{B}}\right) - \tilde{g}\left(\frac{(\bar{k}+\bar{p})^2}{2\mathcal{B}}\right)\right] \\ \Delta\tilde{g}' = \tilde{g}\left(\frac{\bar{k}^2}{2\mathcal{B}}\right)\left[\tilde{g}\left(\frac{(\bar{k}+\bar{p}_1)^2}{2\mathcal{B}}\right) - \tilde{g}\left(\frac{(\bar{k}-\bar{p}_1)^2}{2\mathcal{B}}\right)\right] \end{cases} \quad (37)$$

The result (37) shows that relation (36) is satisfied in the limit where $\bar{p}_1{}^2, \bar{p}_2{}^2 \ll 2\mathcal{B}$. In this limit too, $\tilde{g}$ depends only on the k^2 variable everywhere

$$\begin{cases} \Delta\tilde{g} = \tilde{g}\left(\frac{\bar{k}^2}{2\mathcal{B}}\right)\left[\tilde{g}\left(\frac{\bar{k}^2}{2\mathcal{B}}\right) - \tilde{g}\left(\frac{\bar{k}^2}{2\mathcal{B}}\right)\right] = 0 \\ \Delta\tilde{g}' = \tilde{g}\left(\frac{\bar{k}^2}{2\mathcal{B}}\right)\left[\tilde{g}\left(\frac{\bar{k}^2}{2\mathcal{B}}\right) - \tilde{g}\left(\frac{\bar{k}^2}{2\mathcal{B}}\right)\right] = 0 \end{cases} \quad (38)$$

The conservation of the axial current is then verified by our canonical regularized quantum field.

It may be shown that the conservation of the axial current is kept in general case (see section 6).

In the following, another Gaussian type function is used and the introduction of a partition of unity [5] generalizes the results.

6. Field as OPVD with Gaussian Test Function

The amplitude in the OPVD formalism is

$$A = \int_0^\infty dX\, T(X)g(X) \quad (39)$$

where $g(X)$ is the Gaussian test function. The calculation will be done in one dimension for the sake of simplicity. Four-dimensional extension can be achieved using integration by part.

6.1. Gaussian Test Function as Partition of Unity

One reduces the test function to a partition of unity following the procedure in [5-7] by the convolution of the Gaussian function of the form

$$g(x) = \begin{cases} \mathcal{N}e^{-x^2}\ for\ \|x\| < 1 \\ 0\ elsewhere \end{cases} \quad (40)$$

where $\mathcal{N}$ is a normalization factor. The test function as partition of unity will be denoted f and it can be tend to 1 over the whole domain in a given limit.

6.2. Construction of the Distribution Extension in the UV Limit

In this section, Lagrange formula is used for the case of large momentum. In quantum field theory, the amplitude associated to the Feynman diagrams could be defined as

$$A = \int_0^\infty dX\, \check{T}^>(X)f(X) \quad (41)$$

where $\check{T}^>(X)$ is the extension of the distribution $T(X)$ in the UV limit. This amplitude $\mathcal{A}$ is regularized by the Gaussian test function. $\check{T}^>(X)$ is useful in higher order calculation since it is regular.

6.3. Extension of the Test Function

In order to avoid the pole in the amplitude, the test function is modified using the Taylor remainder [5]. In the UV limit, the test function is equal to its Taylor remainder

$$g^>(X) = -\frac{X}{k!}\int_1^{\eta^2 G_\alpha(\|X\|)} \frac{dt}{t}(1-t)^k \partial_X^{k+1}\left(X^k f^>(Xt)\right) \quad (42)$$

Here, the upper born, $\eta^2 G_\alpha(\|X\|)$, is given by the arbitrary choice of a running support. Following the procedure described in [5], one changes $f^>(X)$ for this value and get

$$A = -\int_0^\infty dX\, T(X)\frac{X}{k!}\int_1^{\eta^2 G_\alpha(\|X\|)} \frac{dt}{t}(1-t)^k \partial_X^{k+1}\left(X^k f^>(Xt)\right)$$

6.4. Extension of the Distribution $T(X)$

Integrating by parts (6.5) and performing the limit $f^>(Xt) \to 1$, which is now possible since the distribution is regularized, the definition of the extension of the distribution in the UV domain [4, 5, 6] is now given by

$$\check{T}^>(X) = \frac{(-X)^k}{k!}\partial_X^{k+1}\left(XT(X)\right)\int_1^{\eta^2}\frac{dt}{t}(1-t)^k \quad (43)$$

This Lagrange formula gives a divergence-free contribution.

7. Ward-Takahashi Identity

In order to keep the gauge symmetry of the theory, one calculates the Ward-Takahashi identity in one loop order. This identity is given by [2]

$$\delta\Gamma^\mu = -\frac{\partial}{\partial(\gamma p)}\Sigma(\gamma p)\Big|_{\gamma p = m} \quad (44)$$

where $\Sigma(p^2)$ is the selfenergy amplitude at one loop order and $\delta\Gamma^\mu$ is the vertex amplitude at one loop order. The Ward-Takahashi identity is a powerful tool used in QED to show that the gauge symmetry of the theory remains after the regularization procedure [2]. This was used to show that some of the conventional regularization procedure breaks the gauge symmetry of the Lagrangian. This is the case of dimensional regularization, hard cut-off procedure and so on. Let us calculate the amplitude involved in the Ward-Takahashi identity. Two diagrams contribute to this identity at one-loop order. The first diagram, *Figure 3*, is the self-energy. In the old fashioned formulation of QED, this diagram contribution is UV-divergent. The second diagram, *Figure 4*, is the vertex.
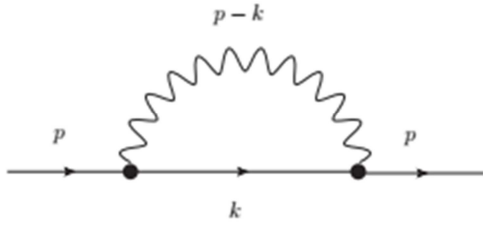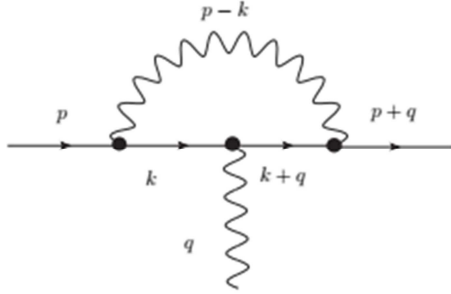
Figure 3. Selfenergy.

Figure 4. One loop Vertex.

One uses the Lagrange formula described above to calculate the two diagrams amplitude.

7.1. One loop Self-energy Diagram

The corresponding amplitude is obtained by using

Feynman rules

$$\Sigma(\gamma p) = \int \frac{d^4\bar{k}}{(2\pi)^4} \frac{(\gamma^\mu k_\mu + m)}{\bar{k}^2 - m^2} \gamma^\alpha \frac{-i}{(\bar{p} - \bar{k})^2 - \mu^2} \gamma_\alpha g(\bar{k}^2) g\left((\bar{p} - \bar{k})^2\right)$$

in which
1) $\bar{k}$ is the quadri-momentum
2) g is a factor emerging from the field formulation as Operator Valued Distribution. A photon mass μ is introduced to avoid IR-divergence [2].
3) γ_α are the Gamma matrices of Dirac

Introducing the Feynman parameterization, the integral becomes

$$\Sigma(\gamma p) = \int_0^1 dx \int \frac{d^4\bar{k}}{(2\pi)^4} \frac{(2x\gamma\bar{p} - 4m)}{\left(\bar{k}^2 - M^2(x,\bar{p}^2)\right)^2} f^2\left(\frac{\bar{p}.\bar{x}}{\Lambda^2}\right) f^2\left(\frac{\bar{k}^2}{\Lambda^2}\right)$$

where

$$M^2(x,\bar{p}^2) = x(1-x)\bar{p}^2 + xm^2 + (1-x)\mu^2$$

and Λ is an arbitrary scale which is related to the dispersion parameter. In order to use the Taylor formula one makes the variable change $X = k^2/\Lambda^2$

Then

$$\Sigma(\gamma p) = \frac{e^2}{16\pi^2} \int_0^1 dx[2x\gamma^\mu p_\mu - 4m] \int_0^{+\infty} \partial_X \frac{X^2}{(X+1)^2} dX \, ln\left(\frac{\eta^2}{M^2(x,\bar{p}^2)}\right) f^2\left(X \frac{M^2(z,\bar{p}^2)}{\Lambda^2}\right) \tag{45}$$

The integral can be split into the following integrals

$$\begin{cases} I_1 = 1 \int_0^\infty dX \frac{1}{(X+1)} f^2\left(X \frac{M^2(x,\bar{p}^2)}{\Lambda^2}\right) \\ I_2 = -3 \int_0^\infty dX \frac{1}{(X+1)^2} f^2\left(X \frac{M^2(x,\bar{p}^2)}{\Lambda^2}\right) \\ I_3 = 2 \int_0^\infty dX \frac{1}{(X+1)^3} f^2\left(X \frac{M^2(x,\bar{p}^2)}{\Lambda^2}\right) \end{cases} \tag{46}$$

The application of the Lagrange formula gives

$$\Sigma(\gamma p) = -\frac{\alpha}{4\pi} \int_0^1 dx[2x\gamma^\mu p_\mu - 4m] \, ln \frac{\eta^2 \Lambda^2}{x(1-x)\bar{p}^2 + xm^2 + (1-x)\mu^2} \tag{47}$$

The self-energy diagram contribution thus obtained is the same as in the dimensional regularization procedure. The Ward-Takahashi identity is related to the derivative of the self-energy.

$$\frac{\partial \Sigma(\gamma p)}{\partial(\gamma p)}\bigg|_{\gamma p = m} = -\frac{\alpha}{4\pi}\left[ln\left(\eta^2 \frac{\Lambda^2}{m^2}\right) - 2ln\left(\frac{\mu^2}{m^2}\right)\right] + o\left(\frac{m^2}{\Lambda^2}, \frac{\mu^2}{\Lambda^2}\right) \tag{48}$$

This result is equivalent to those obtained by conventional calculation up to a constant term which could be introduced in the logarithmic part by some basics properties.

7.2. Vertex Diagram

The last diagram which contributes to the Ward identity in QED at one loop order is the vertex diagram. It involves the coupling between fermion and photon, with a radiative correction [1-2].

The distribution part of the amplitude is equal to

$$\delta\Gamma^\mu = \int \frac{d^4\bar{k}}{(2\pi)^4} \frac{1}{(\bar{k} - \bar{p})^2 + \mu^2} \gamma^\nu \frac{(\gamma^\mu k_\mu + m)}{(\bar{k} + \bar{q})^2 - m^2} \gamma^\mu \frac{(\gamma^\mu k_\mu + \gamma^\mu q_\mu + m)}{\bar{k}^2 - m^2} \gamma_\nu \tag{49}$$

The matrix identities

$$\begin{cases} \gamma^\nu \gamma^\mu \gamma_\nu = -2\gamma^\mu \\ \gamma^\alpha l_\alpha \gamma^\mu \gamma^\beta l_\beta = 2l^\mu l^\nu \gamma_\nu - \gamma^\mu l^\rho l_\rho \end{cases} \tag{50}$$

and the Feynman parametrization lead to the vertex contribution

$$\delta\Gamma^\mu = \frac{\alpha}{2\pi} \int_0^1 dz(1-z) \int_{-\infty}^{+\infty} \frac{d^4\bar{l}}{(2\pi)^4} \frac{\left[-\frac{1}{2}\gamma^\mu \bar{l}^2 + P(z)m^2\gamma^\mu\right]}{[\bar{l}^2 - M^2]^3} f((k + z\bar{p})^2) \tag{51}$$

in which $M^2(z,\bar{p}^2) = (1 - z^2)m^2 + z\mu^2$

After a variable change and introducing a dimensionless variable X, the expression (51) becomes

$$\delta\Gamma^\mu = \frac{\alpha}{2\pi}\int_0^1 dz\,(1-z)\int_0^\infty dX\,\frac{X(X-1)}{[X+1]^3}f^2\left(X\frac{M^2(z,\vec{p}^2)}{\Lambda^2}\right) \quad (52)$$

The method analog to the self-energy calculation may be applied and leads to the expression

$$\delta\Gamma^\mu = -\left.\frac{\partial\Sigma(\gamma p)}{\partial(\gamma p)}\right|_{\gamma p=m} = \frac{\alpha}{4\pi}\left[ln\left(\eta^2\frac{\Lambda^2}{m^2}\right) - 2ln\left(\frac{\mu^2}{m^2}\right)\right] + \sigma\left(\frac{m^2}{\Lambda^2},\frac{\mu^2}{\Lambda^2}\right) \quad (54)$$

This equation proves that introducing Gaussian as test function reduced to partition of unity provide, at one loop order, self-energy and vertex satisfying the Ward-Takahashi identity.

8. Conclusion

It is shown that Gaussian functions may be used as test function in OPVD formulation of a Quantum Field Theory to deal with divergence. QFT is then defined, not in point space-time but smeared in an area described by the Gaussian test function. The quantum scalar fields obey Klein-Gordon equations due to the properties of test functions. One obtains expressions of the vacuum fluctuation and Feynman propagators which contain Gaussian regularization factor. The Loop calculation is performed in the case of one loop tadpole diagram and divergence-free result was obtained.

The Gaussian functions have some mathematical and physical properties which may make this approach interesting. The Heisenberg uncertainty principle is incorporated and it is represented by the factor $\mathcal{B}$. The case in which the mass m $\ll \sqrt{\mathcal{B}}$ is considered and yields finite result. The amplitude is related to the value $\mathcal{B}$ of the dispersion (statistical variance) of the energy and components of spatial momentum. The standard QFT corresponds to a large value of $\mathcal{B}$. The main advantage of this formulation is that it needs no counter-parts to treat divergences in the higher order in perturbation theory. The dimension of space-time is not modified like in the dimensional regularization.

The test function is also reduced in partition of unity. After applying the Taylor formula in the singular distribution, one obtains finite amplitude. Gauge and Poincaré invariance is verified at one loop level and it is the case for the Ward-Takahashi identity too.

$$\delta\Gamma^\mu = \frac{\alpha}{4\pi}\left[ln\left(\frac{\eta^2\Lambda^2}{m^2}\right) - 2ln\left(\frac{\mu^2}{m^2}\right)\right] + \sigma\left(\frac{m^2}{\Lambda^2},\frac{\mu^2}{\Lambda^2}\right) \quad (53)$$

This result is equivalent to those obtained in others regularizations procedure. In our case the Ward-Takahashi identity was verified since

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