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Exact Solution of the Dirac Equation with a Combined Static Electric and Magnetic Field in the Context of Generalized Uncertainty Principle

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Abstract: There is no minimal uncertainty in position measurement in the Heisenberg uncertainty principle is to be considered as the minimum of space resolution, whereas numerous theories of quantum gravity predict the existence of a lower bound to the possible resolution of distances. The minimal length is considered commonly by a modification of the Heisenberg uncertainty principle into the generalized uncertainty principle (GUP). The application of GUP modifies every equation of motion of quantum mechanics and consequently, a new window of research has opened to study quantum mechanical problems under the framework of GUP. In this article, we present an exact solution of the Dirac equation with a combined static electric and magnetic field under the framework of GUP and obtain exact energy spectrums. The spectrums manifest a super-symmetry for the sufficient large magnetic field intensity compared to the electric field intensity. The methodology of the solution is designed for convenient implementation of the key property of the harmonic oscillator, the kinetic and potential energy parts of the Hamiltonian are of equal weight. An obligation for the existence of the solution is found that the magnetic field is stronger than the electric field. Our obtained result is confirmed by rendering energy levels of a relativistic electron in an external normal magnetic field, found in the literature.

Keywords: Dirac Equation, Static Electric Field, Static Magnetic Field, Generalized Uncertainty Principle

1. Introduction

A fundamental length in the Planck length scale has appeared in the perspective of the string theory through the observation that a string cannot probe the distances that are smaller than its length [1-6]. The holographic principle [7], black hole physics [8] and non-commutative geometries [9] all strongly support the appearance of the fundamental length. From the viewpoint of quantum mechanics, this fundamental length has shown up as an extra uncertainty for measurement in a position [10-13]. Therefore, the ordinary canonical commutation relation between the operators of position and momentum turns into a modified form, $[x,p]=i\hbar\left(1+\beta p^2\right)$, where the deformation parameter β has a small positive value. This reformed commutation relation leads to a modification in the Heisenberg uncertainty relation as $\Delta x\Delta p\geq\frac{\hbar}{2}(1+\beta\Delta p^2)$. This adjusted uncertainty product obviously implies the lower bound of uncertainty in a position measurement $\Delta x_{\min}=\hbar\sqrt{\beta}$. This modified uncertainty product is typically known as the generalized uncertainty principle (GUP) [14], or minimal length uncertainty relation. Consequently, in configuration space momentum operator becomes $p=-i\hbar\frac{\partial}{\partial x}\left[1+\frac{\beta}{3}\left(-i\hbar\frac{\partial}{\partial x}\right)^2\right]$ and the position operator persists the same as ordinary operator x [11] due to GUP.

The perception of the GUP provides a suitable framework for the unification of quantum mechanics and general relativity under the dominion of quantum gravity [15]. Besides, this framework of GUP could be of excessive attention in both relativistic and non-relativistic quantum mechanics. An astonishing aspect of the conception is that this minimal length can regularize unexpected divergences in the quantum field theory [16]. The notion of GUP can moreover offer a best latest framework for an effective narrative of complex systems for example several collective excitations in solid or composite particles and quasi-particles, for example, nucleons, nuclei, and molecules [13]. In fact, it is conjectured that this fundamental length may be interpreted as an intrinsic gage depicting the finite size of the measured system and its structure under revision [16]. A minimal length is obviously found to be operative in each quantum system when the non-pointness of the constituent parts of the system like an electron, nucleon, etc. whose positions are determined is accounted for within the framework of quantum measurement [17]. One of the significances of the GUP is that it represents the fascinating UV/IR mixing: when Δp is large (ultraviolet (UV)), Δx is proportional to Δp and hence is also large (infrared (IR)). This kind of relationship is found in non-commutative quantum field theory [18] and in Ads/CFT correspondence [19]. The meaning of UV/IR mixing is physics of short distance can be probed by the physics of long distance.

The framework of the GUP has caused new implications in solving quantum- mechanical problems from the beginning of the present century. A great amount of work has been devoted to study the effect of the GUP on quantum-mechanical problems: The implications of the framework of GUP on the equation of motion for simple harmonic motion have been discussed in [20]. The exact solution of harmonic oscillators Schrödinger equation in D-dimensions has been presented in [21]. Some features of a bouncing particle in Earth's gravitational field under GUP are studied in [22]. The modification to the quadrupole moment of deuteron brought by the generalized uncertainty principle is calculated and an estimation of the minimal length is found in [23]. The approximate energy levels for hydrogen in presence of nonzero minimal length in coordinate space have been gained in [24]. The elastic scattering problem for the Yukawa and the Coulomb potential is studied in [25]. The classical limit of the motion of a particle in a central force potential is investigated in [26]. The modification of quantization of the electromagnetic field for GUP has been analyzed in [27]. The amendment due to GUP to the energy ground state of the deuteron, where Yukawa potential is the binding force between the nucleons, is estimated in [28]. The implications of the GUP on cosmology have been discussed in [29]. The (2 + 1)-dimensional Dirac oscillator under a magnetic field in the presence of a minimal length is investigated in [30]. The free particle Dirac equation through GUP is solved in [31]. The minimal length framework is inserted in the Dirac equation in [32], where an exact spectrum of a one-dimensional Dirac oscillator has been deduced. Dirac oscillator in momentum representation has been solved in [33]. An approximate leading influence of the GUP on the energy spectrum of 1D Dirac equation with a linear scalar potential has been obtained in [34]. The exact scenario of a Dirac particle confined by a mixed vector-scalar linear potential in [35] and a linear scalar potential in [36] has been explored. The wave equation for a relativistic electron in two-dimensional electron gas with a normal external magnetic field has been solved exactly in [37]. The compatibility of noncommutative algebra with GUP for graphene is discussed in [38]. Construction and exploration of the observable consequences of GUP in relativistic quantum field theory are exposed in [39]. A quantum field theoretic toy model built on GUP is formulated in [40].

In this article, we propose an exact solution of the Dirac equation with a combined static electric and magnetic field in the context of the GUP and along in the context of the usual Heisenberg uncertainty principle. The task expresses a relativistic electron that moves in the x-y plane in the presence of a mutual perpendicular electric and magnetic field, where the electric field is in the y-axis and the magnetic field in the z-axis. The uniform electric field is implemented by a scalar potential and the uniform magnetic field is executed by a right selected of the gauge field. The performance of the exact solution depends on the knowledge of the energy spectrum of the harmonic oscillator with GUP. The fundamental property of the harmonic oscillator, kinetic and potential energy parts of the Hamiltonian are of equal weight is used. The solution of the Dirac equation with a combined static electric and magnetic field within the framework of GUP shown here is of curiosity since it gives a model of a two-dimensional electron gas system in presence of a mutual perpendicular static electric and magnetic field and has various applications in advanced solid-state physics. The present work is of much interest since it offers a suitable framework for the exploration of phenomenological aspects of quantum gravity proposals.

The arrangement of the article is as follows: In section 2, we formulate Dirac equation for the electron in presence of a combined static electric and magnetic field by an appropriate choice of the scalar and vector potential giving rise to an electric and magnetic field respectively and two coupled differential equations are obtained. We then solve the differential equations in the context of the ordinary Heisenberg algebra in sub section 2.1 and of a modified algebra for the GUP in sub section 2.2, where the energy spectrums are obtained exactly. The ending section 3 gives a brief conclusion.

2. Exact Solution of the Dirac Equation with a Combined Static Electric and Magnetic Field

In the presence of an external electromagnetic field, the Dirac equation for an electron of mass m, charge $-e(e>0)$ moving with linear momentum $\vec{p}$ is [41]

$$i\hbar\frac{\partial\psi(\vec{x},t)}{\partial t}=\left[c\vec{\alpha}\cdot\left(\frac{\hbar}{i}\vec{\nabla}+\frac{e}{c}\vec{A}\right)-e\phi+\beta mc^2\right]\psi(\vec{x},t),\quad(1)$$

where, $\vec{\alpha}$ and β are Dirac matrices, ϕ scalar potential, $\vec{A}$ the vector potential, c the speed of light and $\psi(\vec{x},t)$ is a four-component spinor. In the case of stationary states, we can write

$$\psi(\vec{x},t)=\psi(\vec{x})e^{-i\frac{Et}{\hbar}}.\quad(2)$$

Using this in equation (1), we obtain

$$\left[c\vec{\alpha}\cdot\left(\vec{p}+\frac{e}{c}\vec{A}\right)-e\phi+\beta mc^2\right]\psi(\vec{x})=E\psi(\vec{x}).\quad(3)$$

We consider that a relativistic electron is moving in x-y plane with a uniform velocity. A uniform electric field is applied in y-axis and a uniform magnetic field normal to the x-y plane that is along z-axis. The scalar and vector potentials are

$$\phi=-\varepsilon y\quad(4)$$

and

$$A_x=-B_0y,\ A_Y=A_Z=0,\quad(5)$$

where ε and B_0 are electric and the magnetic field strength respectively.

For the electron, we can write the equation (3) as

$$\left(c\alpha_xp_x+c\alpha_yp_y+e\alpha_xA_x+e\varepsilon y+\beta mc^2\right)\psi(x,y)=E\psi(x,y),\quad(6)$$

A standard representation of Dirac matrices for two spatial dimensions is

$$\alpha_x=\sigma_x=\begin{pmatrix}0&1\\1&0\end{pmatrix},$$

$$\alpha_y=\sigma_y=\begin{pmatrix}0&-i\\i&0\end{pmatrix}\quad(7)$$

and

$$\beta=\sigma_z=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad(8)$$

where $\sigma_i\left(i=x,y,z\right)$ are the 2×2 Pauli matrices and a two component form for ψ is

$$\psi=\begin{pmatrix}\psi_1\\\psi_2\end{pmatrix},\quad(9)$$

which is a spinor mixing spin-up and down-components with positive and negative energy. Using the representation of Dirac matrices and the spinor, we obtain the following coupled equations:

$$\left(cp_x-icp_y-eB_0y\right)\psi_2(x,y)=\left(E-e\varepsilon y-mc^2\right)\psi_1(x,y),\quad(10)$$

and

$$\left(cp_x+icp_y-eB_0y\right)\psi_1(x,y)=\left(E-e\varepsilon y+mc^2\right)\psi_2(x,y).\quad(11)$$

Putting the value of ψ_2 from equation (11) in equation (10) and putting the value of ψ_1 from equation (10) in equation (11), the following two coupled equations can be obtained:

$$\left[c^2p_x^2-2ceB_0p_xy+e^2B_0^2y^2+c^2p_y^2-iceB_0[y,p_y]\right]\psi_1(x,y)=\left[\left(E-e\varepsilon y\right)^2-m^2c^4\right]\psi_1(x,y)\quad(12)$$

and

$$\left[c^2p_x^2-2ceB_0p_xy+e^2B_0^2y^2+c^2p_y^2+iceB_0[y,p_y]\right]\psi_2(x,y)=\left[\left(E-e\varepsilon y\right)^2-m^2c^4\right]\psi_2(x,y).\quad(13)$$

In deriving stages of the equations (12) and (13), we have assumed that $[p_x,p_y]=[y,p_x]=0$.

2.1. Perspective of the Ordinary Heisenberg Uncertainty Principle

Now we focus to the problem of solving equations (12) and (13) in the framework of ordinary Heisenberg algebra. Using the Heisenberg algebra $[y,p_y]=i\hbar,\ p_x=-i\hbar\frac{\partial}{\partial x},\ p_y=-i\hbar\frac{\partial}{\partial y}$ and considering the solutions as

$$\psi_1(x,y)=\frac{1}{\sqrt{2\pi}}e^{\frac{i}{\hbar}xp_x}\psi_1(y)\quad(14)$$

and

$$\psi_2(x,y)=\frac{1}{\sqrt{2\pi}}e^{\frac{i}{\hbar}xp_x}\psi_2(y),\quad(15)$$

equation (12) and (13) can be cast into

$$\left[p_y^2+\frac{e^2\left(B_0^2-\varepsilon^2\right)}{c^2}\left\{y-\frac{cB_0p_x-\varepsilon E}{e\left(B_0^2-\varepsilon^2\right)}\right\}^2\right]\psi_1(y)=\left[\frac{E^2}{c^2}-m^2c^2-\frac{eB_0\hbar}{c}-p_x^2+\frac{\left(cB_0p_x-\varepsilon E\right)^2}{c^2\left(B_0^2-\varepsilon^2\right)}\right]\psi_1(y) \tag{16}$$

and

$$\left[p_y^2+\frac{e^2\left(B_0^2-\varepsilon^2\right)}{c^2}\left\{y-\frac{cB_0p_x-\varepsilon E}{e\left(B_0^2-\varepsilon^2\right)}\right\}^2\right]\psi_2(y)=\left[\frac{E^2}{c^2}-m^2c^2+\frac{eB_0\hbar}{c}-p_x^2+\frac{\left(cB_0p_x-\varepsilon E\right)^2}{c^2\left(B_0^2-\varepsilon^2\right)}\right]\psi_2(y). \tag{17}$$

Let us write $y'=y-\frac{cB_0p_x-\varepsilon E}{e\left(B_0^2-\varepsilon^2\right)}$. Under the variable conversion $p_y=-i\hbar\frac{\partial}{\partial y}=-i\hbar\frac{\partial}{\partial y'}=p_y'$. We divide both sides of equations (16) and (17) by *2m*, then we have

$$\left[\frac{p_y'^2}{2m}+\frac{1}{2}m\omega^2y'^2\right]\psi_1(y')=\left[\frac{E^2}{2mc^2}-\frac{mc^2}{2}-\frac{eB_0\hbar}{2cm}-\frac{p_x^2}{2m}+\frac{\left(cB_0p_x-\varepsilon E\right)^2}{2mc^2\left(B_0^2-\varepsilon^2\right)}\right]\psi_1(y'), \tag{18}$$

and

$$\left[\frac{p_y'^2}{2m}+\frac{1}{2}m\omega^2y'^2\right]\psi_2(y')=\left[\frac{E^2}{2mc^2}-\frac{mc^2}{2}+\frac{eB_0\hbar}{2cm}-\frac{p_x^2}{2m}+\frac{\left(cB_0p_x-\varepsilon E\right)^2}{2mc^2\left(B_0^2-\varepsilon^2\right)}\right]\psi_2(y'), \tag{19}$$

where $\omega\left(=\frac{e\sqrt{B_0^2-\varepsilon^2}}{mc}\right)$ is an angular frequency in e.s.u. Both equations, (18) and (19), are the simple harmonic oscillators Schrödinger equations having frequency ω. Hence for equation (18), we have

$$\left[\frac{E^2}{2mc^2}-\frac{mc^2}{2}-\frac{eB_0\hbar}{2cm}-\frac{p_x^2}{2m}+\frac{\left(cB_0p_x-\varepsilon E\right)^2}{2mc^2\left(B_0^2-\varepsilon^2\right)}\right]=(2n+1)\frac{\hbar\omega}{2}. \tag{20}$$

This equation gives the following quadratic equation for energy

$$B_0^2E^2-2cB_0p_x\varepsilon E+ceB_0\hbar\varepsilon^2-ceB_0^3\hbar-(2n+1)ce\hbar\left(B_0^2-\varepsilon^2\right)^{\frac{3}{2}}-B_0^2m^2c^4+\varepsilon^2m^2c^4+c^2\varepsilon^2p_x^2=0. \tag{21}$$

From this quadratic equation, the spectrum of ψ_1 is given by

$$E_n=cp_x\frac{\varepsilon}{B_0}\pm\sqrt{\left[(2n+1)\left\{1-\left(\frac{\varepsilon}{B_0}\right)^2\right\}^{\frac{3}{2}}+1\right]\hbar eB_0c+m^2c^4\left\{1-\left(\frac{\varepsilon}{B_0}\right)^2\right\}-\hbar eB_0c\left(\frac{\varepsilon}{B_0}\right)^2}, \tag{22}$$

n=0,1,2....

For equation (19), we have

$$\left[\frac{E^2}{2mc^2}-\frac{mc^2}{2}+\frac{eB_0\hbar}{2cm}-\frac{p_x^2}{2m}+\frac{\left(cB_0p_x-\varepsilon E\right)^2}{2mc^2\left(B_0^2-\varepsilon^2\right)}\right]=(2n'+1)\frac{\hbar\omega}{2}. \tag{23}$$

This equation gives the following quadratic equation for energy

$$B_0^2E^2-2cB_0p_x\varepsilon E-ceB_0\hbar\varepsilon^2+ceB_0^3\hbar-(2n'+1)ce\hbar\left(B_0^2-\varepsilon^2\right)^{\frac{3}{2}}-B_0^2m^2c^4+\varepsilon^2m^2c^4+c^2\varepsilon^2p_x^2=0. \tag{24}$$

From this quadratic equation, the spectrum of ψ_2 is given by

$$E_{n'}=cp_x\frac{\varepsilon}{B_0}\pm\sqrt{\left[(2n'+1)\left\{1-\left(\frac{\varepsilon}{B_0}\right)^2\right\}^{\frac{3}{2}}-1\right]\hbar eB_0c+m^2c^4\left\{1-\left(\frac{\varepsilon}{B_0}\right)^2\right\}+\hbar eB_0c\left(\frac{\varepsilon}{B_0}\right)^2}, \tag{25}$$

n′=0,1,2....

The exact form of $\psi_1(x,y)$ and $\psi_2(x,y)$ are

$$\psi_1(x,y)=\frac{1}{\sqrt{2\pi}}e^{\frac{i}{\hbar}xp_x}\psi_{1n}\left(y-\frac{cB_0p_x-\varepsilon E}{e\left(B_0^2-\varepsilon^2\right)}\right), \tag{26}$$

and

$$\psi_2(x,y)=\frac{1}{\sqrt{2\pi}}e^{\frac{i}{\hbar}xp_x}\psi_{2n'}\left(y-\frac{cB_0p_x-\varepsilon E}{e\left(B_0^2-\varepsilon^2\right)}\right), \tag{27}$$

respectively, where $\psi_{1n}(y')$ and $\psi_{2n'}(y')$ are the wave functions of simple harmonic oscillator.

2.2. Perspective of the Generalized Uncertainty Principle

In this section, we will focus to the problem of solving Equations (12) and (13) under the perspective of the generalized Heisenberg algebra. Using the generalized commutation relation, $[y,p_y]=i\hbar\left(1+\beta p_y^2\right)$ in equation (12) and (13), we obtain respectively

$$[\,c^2p_x^2-2ceB_0p_xy+e^2B_0^2y^2+c^2p_y^2+ceB_0\hbar\beta p_y^2]\;\psi_1(x,y)\;=\;(\mathrm{E}^2-2e\varepsilon Ey+e^2\varepsilon^2y^2-m^2c^4-ceB_0\hbar)\;\psi_1(x,y) \tag{28}$$

and

$$[c^2p_x^2-2ceB_0p_xy+e^2B_0^2y^2+c^2p_y^2-ceB_0\hbar\beta p_y^2]\;\psi_2(x,y)\;=\;(\mathrm{E}^2-2e\varepsilon Ey+e^2\varepsilon^2y^2-m^2c^4+ceB_0\hbar)\;\psi_2(x,y)\,. \tag{29}$$

Let us assume the solutions as

$$\psi_1(x,y)=\frac{1}{\sqrt{2\pi}}e^{\frac{i}{\hbar}xp_x}\psi_1(y), \tag{30}$$

and

$$\psi_2(x,y)=\frac{1}{\sqrt{2\pi}}e^{\frac{i}{\hbar}xp_x}\psi_2(y)\;, \tag{31}$$

Using $p_x=-i\hbar\frac{\partial}{\partial x}\left[1+\frac{\beta}{3}\left(-i\hbar\frac{\partial}{\partial x}\right)^2\right]$ and $p_y=-i\hbar\frac{\partial}{\partial y}\left[1+\frac{\beta}{3}\left(-i\hbar\frac{\partial}{\partial y}\right)^2\right]$ and performing some algebraic operations, one easily can find that

$$p_x\psi_1(x,y)=\left(p_x+\frac{\beta}{3}p_x^3\right)\frac{1}{\sqrt{2\pi}}e^{\frac{i}{\hbar}xp_x}\psi_1(y), \tag{32}$$

$$p_x^2\psi_1(x,y)=\left(p_x+\frac{\beta}{3}p_x^3\right)^2\frac{1}{\sqrt{2\pi}}e^{\frac{i}{\hbar}xp_x}\psi_1(y), \tag{33}$$

$$p_y^2\psi_1(x,y)=\frac{1}{\sqrt{2\pi}}e^{\frac{i}{\hbar}xp_x}p_y^2\psi_1(y), \tag{34}$$

$$p_x\psi_2(x,y)=\left(p_x+\frac{\beta}{3}p_x^3\right)\frac{1}{\sqrt{2\pi}}e^{\frac{i}{\hbar}xp_x}\psi_2(y), \tag{35}$$

$$p_x^2\psi_2(x,y)=\left(p_x+\frac{\beta}{3}p_x^3\right)^2\frac{1}{\sqrt{2\pi}}e^{\frac{i}{\hbar}xp_x}\psi_2(y), \tag{36}$$

and

$$p_y^2\psi_2(x,y)=\frac{1}{\sqrt{2\pi}}e^{\frac{i}{\hbar}xp_x}p_y^2\psi_2(y). \tag{37}$$

Using these, equation (28) and (29) can be cast into

$$\begin{gathered}\left[\left(1+\beta\hbar\frac{eB_0}{c}\right)p_y^2+\frac{e^2\left(B_0^2-\varepsilon^2\right)}{c^2}\left\{y-\frac{cB_0\left(p_x+\frac{\beta}{3}p_x^3\right)-\varepsilon E}{e\left(B_0^2-\varepsilon^2\right)}\right\}^2\right]\psi_1(y)\\=\left[\frac{E^2}{c^2}-m^2c^2-\frac{eB_0\hbar}{c}-\left(p_x+\frac{\beta}{3}p_x^3\right)^2+\frac{\left\{cB_0\left(p_x+\frac{\beta}{3}p_x^3\right)-\varepsilon E\right\}^2}{c^2\left(B_0^2-\varepsilon^2\right)}\right]\psi_1(y)\end{gathered} \tag{38}$$

and

$$\begin{gathered}\left[\left(1-\beta\hbar\frac{eB_0}{c}\right)p_y^2+\frac{e^2\left(B_0^2-\varepsilon^2\right)}{c^2}\left\{y-\frac{cB_0\left(p_x+\frac{\beta}{3}p_x^3\right)-\varepsilon E}{e\left(B_0^2-\varepsilon^2\right)}\right\}^2\right]\psi_2(y)\\=\left[\frac{E^2}{c^2}-m^2c^2+\frac{eB_0\hbar}{c}-\left(p_x+\frac{\beta}{3}p_x^3\right)^2+\frac{\left\{cB_0\left(p_x+\frac{\beta}{3}p_x^3\right)-\varepsilon E\right\}^2}{c^2\left(B_0^2-\varepsilon^2\right)}\right]\psi_2(y)\ .\end{gathered} \tag{39}$$

Let $y' = y - \frac{cB_0\left(p_x + \frac{\beta}{3}p_x^3\right) - \varepsilon E}{e\left(B_0^2 - \varepsilon^2\right)}$. Under the variable conversion

$$p_y = -i\hbar\frac{\partial}{\partial y}\left[1 + \frac{\beta}{3}\left(-i\hbar\frac{\partial}{\partial y}\right)^2\right] = -i\hbar\frac{\partial}{\partial y'}\left[1 + \frac{\beta}{3}\left(-i\hbar\frac{\partial}{\partial y'}\right)^2\right] = p'_y.$$

After dividing both sides of equations (38) and (39) by *2m*, the equations are turned into

$$\left[\left(1 + \beta\hbar\frac{eB_0}{c}\right)\frac{p'^2_y}{2m} + \frac{1}{2}m\left(\frac{e\sqrt{B_0^2 - \varepsilon^2}}{mc}\right)^2 y'^2\right]\psi_1(y') = \left[\frac{E^2}{2mc^2} - \frac{mc^2}{2} - \frac{eB_0\hbar}{2cm} - \frac{\left(p_x + \frac{\beta}{3}p_x^3\right)^2}{2m} + \frac{\left\{cB_0\left(p_x + \frac{\beta}{3}p_x^3\right) - \varepsilon E\right\}^2}{2mc^2\left(B_0^2 - \varepsilon^2\right)}\right]\psi_1(y') \tag{40}$$

and

$$\left[\left(1 - \beta\hbar\frac{eB_0}{c}\right)\frac{p'^2_y}{2m} + \frac{1}{2}m\left(\frac{e\sqrt{B_0^2 - \varepsilon^2}}{mc}\right)^2 y'^2\right]\psi_2(y') = \left[\frac{E^2}{2mc^2} - \frac{mc^2}{2} + \frac{eB_0\hbar}{2cm} - \frac{\left(p_x + \frac{\beta}{3}p_x^3\right)^2}{2m} + \frac{\left\{cB_0\left(p_x + \frac{\beta}{3}p_x^3\right) - \varepsilon E\right\}^2}{2mc^2\left(B_0^2 - \varepsilon^2\right)}\right]\psi_2(y'). \tag{41}$$

Let us set $\frac{e\sqrt{B_0^2-\varepsilon^2}}{mc} = \omega$ (an angular frequency in e.s.u) which imposes a requirement for the existence of the solution that the magnetic field intensity must be greater than the electric field intensity,

$$\frac{E^2}{2mc^2} - \frac{mc^2}{2} - \frac{eB_0\hbar}{2cm} - \frac{\left(p_x + \frac{\beta}{3}p_x^3\right)^2}{2m} + \frac{\left\{cB_0\left(p_x + \frac{\beta}{3}p_x^3\right) - \varepsilon E\right\}^2}{2mc^2\left(B_0^2 - \varepsilon^2\right)} = E' \tag{42}$$

and

$$\frac{E^2}{2mc^2} - \frac{mc^2}{2} + \frac{eB_0\hbar}{2cm} - \frac{\left(p_x + \frac{\beta}{3}p_x^3\right)^2}{2m} + \frac{\left\{cB_0\left(p_x + \frac{\beta}{3}p_x^3\right) - \varepsilon E\right\}^2}{2mc^2\left(B_0^2 - \varepsilon^2\right)} = E''. \tag{43}$$

Then equations (40) and (41) can be turned into the form as

$$\left[\left(1 + \beta\hbar\frac{eB_0}{c}\right)\frac{p'^2_y}{2m} + \frac{1}{2}m\omega^2 y'^2\right]\psi_1(y') = E'\psi_1(y') \tag{44}$$

and

$$\left[\left(1 - \beta\hbar\frac{eB_0}{c}\right)\frac{p'^2_y}{2m} + \frac{1}{2}m\omega^2 y'^2\right]\psi_2(y') = E''\psi_2(y'). \tag{45}$$

It is noticeable that $1\pm\beta\hbar\frac{eB_0}{c}$ are dimensionless factors and both factors occur as a multiplication of kinetic energy operator $\left(\frac{p_y'^2}{2m}\right)$, one is in equation (44) and other in equation (45). Obviously, equation (44) and (45) both are deformed Harmonic Oscillator apart from the dimensionless factors. Therefore, we can make use of the aspect of harmonic oscillator, the kinetic and potential energy parts of the Hamiltonian are of equal weight, for obtaining energy spectrums. The property also retains classically. To find the energy levels of ψ_1 and ψ_2, the aspect can be applied in the following method [36]: Divide the harmonic oscillator energy spectra due to GUP into two halves and multiply the one half by $1+\beta\hbar\frac{eB_0}{c}$ for equation (44) and $1-\beta\hbar\frac{eB_0}{c}$ for equation (45). The harmonic oscillator energy levels due to GUP are [21]

$$E_n=\left(n+\frac{1}{2}\right)\hbar\omega+\left(n^2+n+\frac{1}{2}\right)\frac{\beta m\hbar^2\omega^2}{2},\quad n=0,1,2.... \tag{46}$$

Using the treatment as stated, the following equation is found for equation (44):

$$E_n'=(2n+1)\frac{\hbar\omega}{2}+(2n+1)\frac{\beta eB_0\hbar^2\omega}{4c}+\left(2n^2+2n+1\right)\frac{\beta m\hbar^2\omega^2}{4}+\left(2n^2+2n+1\right)\frac{\beta^2 meB_0\hbar^3\omega^2}{8c}. \tag{47}$$

Up to 1st order in β, E_n' is given by

$$E_n'=(2n+1)\frac{\hbar\omega}{2}+(2n+1)\frac{\beta eB_0\hbar^2\omega}{4c}+\left(2n^2+2n+1\right)\frac{\beta m\hbar^2\omega^2}{4}. \tag{48}$$

Replacing E_n' and ω by their values, we get the following quadratic equation of energy

$$\begin{aligned}&2B_0^2E^2-4cB_0\left(p_x+\frac{\beta}{3}p_x^3\right)\varepsilon E+2ceB_0\hbar\varepsilon^2-2ceB_0^3\hbar-2(2n+1)ce\hbar\left(B_0^2-\varepsilon^2\right)^{\frac{3}{2}}-\left(2n^2+2n+1\right)\beta e^2\hbar^2\left(B_0^2-\varepsilon^2\right)^2\\&-(2n+1)\beta e^2B_0\hbar^2\left(B_0^2-\varepsilon^2\right)^{\frac{3}{2}}-2B_0^2m^2c^4+2\varepsilon^2m^2c^4+2c^2\varepsilon^2\left(p_x+\frac{\beta}{3}p_x^3\right)=0.\end{aligned} \tag{49}$$

From this quadratic equation, the spectrum of ψ_1 is given by

$$\begin{aligned}E_n=c\left(p_x+\frac{\beta}{3}p_x^3\right)\frac{\varepsilon}{B_0}\pm\Bigg[&\left\{(2n+1)\left(1-\frac{\varepsilon^2}{B_0^2}\right)^{\frac{3}{2}}+1\right\}\hbar eB_0c+m^2c^4\left(1-\frac{\varepsilon^2}{B_0^2}\right)-\hbar eB_0c\left(\frac{\varepsilon}{B_0}\right)^2\\&+\left(n^2+n+\frac{1}{2}\right)\beta\hbar^2e^2B_0^2\left(1-\frac{\varepsilon^2}{B_0^2}\right)^2+\left(n+\frac{1}{2}\right)\beta\hbar^2e^2B_0^2\left(1-\frac{\varepsilon^2}{B_0^2}\right)^{\frac{3}{2}}\Bigg]^{\frac{1}{2}},\end{aligned}\quad n=0,1,2,.... \tag{50}$$

Again using the treatment as stated, the following equation is found for equation (45):

$$E_n''=(2n'+1)\frac{\hbar\omega}{2}-(2n'+1)\frac{\beta eB_0\hbar^2\omega}{4c}+\left(2n'^2+2n'+1\right)\frac{\beta m\hbar^2\omega^2}{4}-\left(2n'^2+2n'+1\right)\frac{\beta^2 meB_0\hbar^3\omega^2}{8c}. \tag{51}$$

Up to 1st order in β, E_n'' is given by

$$E_n''=(2n'+1)\frac{\hbar\omega}{2}-(2n'+1)\frac{\beta eB_0\hbar^2\omega}{4c}+\left(2n'^2+2n'+1\right)\frac{\beta m\hbar^2\omega^2}{4}. \tag{52}$$

Replacing E_n'' and ω by their values, we get the following quadratic equation of energy

$$2B_0^2E^2-4cB_0\left(p_x+\frac{\beta}{3}p_x^3\right)\varepsilon E-2ceB_0\hbar\varepsilon^2+2ceB_0^3\hbar-2(2n+1)ce\hbar\left(B_0^2-\varepsilon^2\right)^{\frac{3}{2}}-\left(2n^2+2n+1\right)\beta e^2\hbar^2\left(B_0^2-\varepsilon^2\right)^2$$
$$+(2n+1)\beta e^2B_0\hbar^2\left(B_0^2-\varepsilon^2\right)^{\frac{3}{2}}-2B_0^2m^2c^4+2\varepsilon^2m^2c^4+2c^2\varepsilon^2\left(p_x+\frac{\beta}{3}p_x^3\right)=0. \quad (53)$$

The quadratic equation (53) gives the spectrum of ψ_2 as

$$E_{n'}=c\left(p_x+\frac{\beta}{3}p_x^3\right)\frac{\varepsilon}{B_0}\pm\left[\left\{(2n'+1)\left(1-\frac{\varepsilon^2}{B_0^2}\right)^{\frac{3}{2}}-1\right\}\hbar eB_0c+m^2c^4\left(1-\frac{\varepsilon^2}{B_0^2}\right)+\hbar eB_0c\left(\frac{\varepsilon}{B_0}\right)^2\right.$$
$$\left.+\left(n'^2+n'+\frac{1}{2}\right)\beta\hbar^2e^2B_0^2\left(1-\frac{\varepsilon^2}{B_0^2}\right)^2-\left(n'+\frac{1}{2}\right)\beta\hbar^2e^2B_0^2\left(1-\frac{\varepsilon^2}{B_0^2}\right)^{\frac{3}{2}}\right]^{\frac{1}{2}},\quad n'=0,1,2\ldots \quad (54)$$

The equations (50) and (54) give the exact energy spectrums of the Dirac equation leaded by GUP with a combined static electric and magnetic field. The spectrums (equation (50) and (54)) manifest a broken super-symmetry. We see that a remarkable significance of the minimal observable length is the attendance of a square of quantum number in the energy spectrums, since it is a feature of a particle in confined potential. If the GUP parameter $\beta=0$ is set in the equations (50) and (54) then they reproduce the equations (22) and (25) respectively.

In absence of electric field, that is, when only magnetic field is presence, the expression of energy spectrum of ψ_1 cuts to

$$E_n=\pm\sqrt{2(n+1)\hbar eB_0c+m^2c^4+(n+1)^2\beta\hbar^2e^2B_0^2},\quad n=0,1,2\ldots \quad (55)$$

and the expression of energy spectrum of ψ_2 cuts to

$$E_{n'}=\pm\sqrt{2n'\hbar eB_0c+m^2c^4+n'^2\beta\hbar^2e^2B_0^2},\quad n'=0,1,2\ldots \quad (56)$$

which fully coincide with the expression of the energy spectrums of a relativistic electron in an external uniform magnetic field under the perception of the nonzero minimal length [37].

When magnetic field intensity is large enough compared to the electric field intensity i.e., $B_0 \gg \varepsilon$, we can neglect the terms of higher order of $\frac{\varepsilon}{B_0}$ containing in both equations (50) and (54). In this case, therefore, the spectrum of ψ_1 is given by

$$E_n=c\left(p_x+\frac{\beta}{3}p_x^3\right)\frac{\varepsilon}{B_0}\pm\sqrt{2(n+1)\hbar eB_0c+m^2c^4+(n+1)^2\beta\hbar^2e^2B_0^2},\quad n=0,1,2\ldots \quad (57)$$

and the spectrum of ψ_2 is given by

$$E_{n'}=c\left(p_x+\frac{\beta}{3}p_x^3\right)\frac{\varepsilon}{B_0}\pm\sqrt{2n'\hbar eB_0c+m^2c^4+n'^2\beta\hbar^2e^2B_0^2},\quad n'=0,1,2\ldots \quad (58)$$

Now, the E_n given by equation (57) is exactly the same as $E_{n'}$ given by equation (58) except that $E_{n'}$ holds an additional level, the ground state with $n'=0$. Therefore, ψ_1 & ψ_2 form a super-symmetric partner, where magnetic field intensity is sufficiently large compared to the electric field intensity.

3. Conclusion

In this article, we have formulated the problem of the Dirac equation for an electron in presence of a combined stationary electric and magnetic field by an appropriate setting of a scalar potential for the electric field and a gauge field for the magnetic field, leading to two coupled differential equations. These equations have been solved through analogy with quantum harmonic oscillator under the ordinary Heisenberg uncertainty relation and under the minimal length uncertainty relation separately, although the chief attention of this article is to do solve the coupled equations within the GUP framework. The coupled equations can be cast into the form of the harmonic oscillator under GUP with an additional dimensionless factor that occurs as a multiplication of kinetic energy operator. Implementing the property of harmonic oscillator, the kinetic and potential energy fragments of Hamiltonian are equal in weight, we have gained the energy spectrums of the coupled equations under GUP. The energy levels can be reduced correctly to the ordinary result by setting the deformation parameter is zero.

The accuracy of the obtained result is confirmed by rendering the energy levels of the relativistic electron in presence of normal magnetic field by putting the electric field intensity $\varepsilon = 0$. A requirement for the existence of the solution is found that the magnetic field intensity must be greater than the electric field intensity. Unlike the traditional case without GUP, the exact energy spectrums of the electron show n^2-dependency as the spectrums of the particles in confining potentials that is a remarkable significance which is a result due to the existence of a lower bound of uncertainty in position. The spectrums reveal a manifestation of broken super-symmetry. When magnetic field intensity is large enough compared to the electric field intensity, the spectrums reveal a manifestation of super-symmetry.

In brief, we have explored a solution of the (2+1) dimensional Dirac equation with a combined stationary electric and magnetic field under the influence of the Generalized Uncertainty Principle and obtained the exact energy levels. This article thus describes a relativistic electron in a combined external stationary electric and magnetic field under the GUP scenario.

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