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Low-Energy Effective Theories of the 1/2 - Filled Hubbard Model in the Continuum Limit

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Abstract: It has been observed that when the gauge fields are present on the link, fermion propagation is possible in the weak coupling limit due to the dominance of the hopping term, which corresponds to the colour gauge interaction in the lattice QCD formulation. The production of low energy skyrmionic excitation at the fermionic site destroys the underlying antiferromagnetic order. In the continuum limit, the kinetic term in the lattice QCD corresponds to the rearrangement of the fermionic constituents through their propagation within the confined domain of the bound stateconfigurations of the interacting system which gives rise to a running coupling constant leading to asymptotic freedom. When one can assign a colour to a particular quantum number of a fermionic component in a limited state, it shows that QCD may be thought of as a generalised non-Abelian gauge field theory since these degrees of freedom play a part in the restricted area of the system and examines the continuous limit of the Hubbard-like model and the weak coupling limit that results from the abolition of the antiferromagnetic order and fermion propagation. This is equivalent to the non-Abelian color gauge field interaction. It is noted that the generalised spin fluctuation may be linked to the colour gauge field. This formalism's discovery of pseudoscalar Goldstone bosons associated with chiral symmetry breaking is in line with (3+1)D continuum QCD.

Keywords: Skyrmionic Excitation, QCD, Antiferromagnetic Order, D-theory, Non-Abelian Gauge Field

1. Introduction

In this note, it will emphasize that the generalised Hubbard-like model [1] that describes a correlated system is a replica of the lattice QCD's weak coupling model. The analogy between the ground state of lattice QCD and the generalised antiferromagnet in the strong coupling limit is explained by the fact that the Hubbard-like model reduces to an antiferromagnet in the strong coupling limit with half filling. Then using the D-theory it is shown that the continuum limit of the antiferromagnetic spin system is equivalent to the principal chiral model of QCD which incorporates dimensional reduction. Again the weak coupling limit of lattice QCD where the fermion kinetic term allows fermion propagation finds its relevance in the hopping term of the Hubbard-like model. To this goal, the Hubbard-like model's Hamiltonian has been defined here in terms of generalised fermions with flavour and colour degrees of freedom. Since the colour degrees of freedom are constrained, it stands to reason that the system will be relevant for a fermion bound state. As a result, the bound state configurations of the interacting system will be the only place where fermion propagation can occur. It should be noted that the constituent fermions in the bound state are repelled hard in this situation by the on-site repulsion term in the standard Hubbard model. In the weak coupling limit the underlying antiferromagnetic order in the lattice QCD is destroyed by the creation of low energy skyrmionic excitation at the fermionic position in the lattice which drives the fermion propagation from one site to the other [2]. This shows how the colour gauge field interacts with the link where the gauge field is located. The colour change that corresponds to the colour gauge transformation occurs as fermions move from one site to another. The rearrangement of fermionic elements through their propagation within the constrained domain of the bound state configurations of the interacting system corresponds to the kinetic term in the lattice QCD in the continuous limit. It is discovered that this produces a running coupling constant that results in asymptotic freedom. The interaction with non-Abelian colour gauge fields is analogous to this. The spin fluctuation of the fermionic elements may be related to the colour gauge fields. This suggests that QCD may be considered to bea generalized non-Abelian gauge field theory when one can assign color to a specific quantum number of a fermionic constituent in a bound state such that these degrees of freedom have their role in the confined region of the system. The corresponding non-Abelian gauge field theory has the gauge group $SU(N_C)$, N_C being the number of colors. In accordance with the accepted observation, the current formalism also implies that the Goldstone bosons associated with the chiral symmetry breaking in QCD are pseudoscalars. In this section, we'll look at the Hubbard-like model's continuum limit as well as the weak coupling limit that occurs when the antiferromagnetic order is abolished and fermion propagation occurs.

2. Theoretical Background

A. Weak Coupling Theory and a Model Similar to Hubbard Model

In the strong coupling limit and at half-filling, now have argued that the Hubbard-like model reduces to the antiferromagnetic system, which is equivalent to the main chiral model of QCD. However, when the antiferromagnetic order parameter is eliminated, the weak coupling theory deviates from this. Fermions will be able to jump between sites as a result of this. In the weak coupling limit of lattice QCD, the fermion kinetic energy term will be quite important. By creating a topological defect, suchas the low energy skyrmionic excitation at each point of the fermionic site, which acts as amagnetic field for the carrier, it is possible to destroy the underlying antiferromagnetic order parameter [2].

To study lattice QCD, now first consider that the fermionic field $\psi^a_{\alpha,i}$ resides on the site i andtransforms under the color gauge group $SU(N_C)$

$$\psi^a_{\alpha,i} \to U_{ab}\psi^b_{\alpha,i} \tag{1}$$

The fermions also transform under the global flavor gauge group $SU(N_F)$.

$$\psi^a_{\alpha,i} \to g_{\alpha\beta}\psi^\beta_{\alpha,i}\,,\, g \in SU(N_F) \tag{2}$$

The degrees of freedom of lattice QCD are the gauge fields U_{ab} which are unitary operators and the associated color electric fields E_{ab} both of which live on the links $\langle ij\rangle$ of the lattice and transform as

$$U_{ij} \to V_i U_{ij} V_j^\dagger\,,\, E_{ij} \to V_i E_{ij} V_j^\dagger \tag{3}$$

The reflection conditions are

$$U_{ij} = U^\dagger_{ji} \tag{4}$$

$$E_{ji} = -U^\dagger_{ij} E_{ij} U_{ij} \tag{5}$$

The QCD Hamiltonian on the lattice is given by

$$\mathrm{H} = \sum_{<ij>}[\psi^{a\dagger}_{a,i} u^{ab}_{ij}\psi^b_{a,j} + h.c. + \frac{e^2}{2}\sum_{A=1}^{N_C^2}(E^A_{ij})^2] + \frac{1}{2e^2}\sum Tr[\Pi U + \Pi U^\dagger] \tag{6}$$

Here now have assumed that the gauge group is $U(N_C)$ [3-5]. The first sum over links $\langle ij\rangle$ is the fermionic kinetic energy and total electric energy respectively and the second over plaquettes □ is the magnetic energy. The kinetic energy term must have phase which produces an effective $U(1)$ magnetic flux per plaquette [6-7]. The electric fields E_{ij} are Lie algebra valued operators and can be expanded in terms of the generators of $U(N_C)$

$$E_{ij} = \sum_{A=0}^{N_C^2-1} E^A_{ij} T^A \tag{7}$$

Here T^0 is the unit matrix which corresponds to the generator of $U(1)$ and T^A are the generators of $SU(N_C)$. The electric energy in the Hamiltonian is the sum over the gauge group Laplacians which act on the color gauge degrees of freedom associated with each link. The magnetic energy corresponds to the Wilson energy function for a gauge field.

The lattice formulation of the fermion kinetic energy implies staggered fermions [8]. These staggered fermions have a relativistic continuum limit when their density is $\frac{1}{2}$ of the maximum that is allowed by Fermi statistics which is $\frac{N_C N_F}{2}$ in the present case. On the lattice, staggered fermions do not have any continuum chiral symmetry. They have a discrete chiral symme- try which is associated with the translation by one site. This forbids explicit fermion mass term [8, 9]. A fermion mass term is a staggered density operator

$$\bar{\psi}_i \tau^A \psi_i \approx (-1)^{\sum n_i} \sum_{\alpha,\beta,\gamma} \psi^a_{\alpha,i}\tau^A_{\alpha\beta}\psi^\beta_{\alpha,i} = \sum_{\alpha\beta}\tau^A_{\alpha\beta}\mu_{\alpha\beta} \tag{8}$$

Thus the antiferromagnetic order parameterand the order parameter for chiral symmetry breaking with a flavor vector condensate are identical. In fact in the strong coupling limit $e^2 \to \alpha$ the problem of finding the ground state of lattice QCD is identical to that of solving the generalized antiferromagnet where Neel order plays the role of chiral symmetry breaking. The strong coupling limit effectively supresses fermion propagation as the fermion kinetic term in the Hamiltonian becomes subdominant.

In the continuum model of the antiferomagnetic spin system the effective action may be taken in Euclidean time dimension as

$$S=-M^2\int d^4xTr\partial_\mu g\partial_\mu g^{-1} \tag{9}$$

with M being a constant having the dimension of mass.

Now note that the action (9) corresponds to the nonlinear sigma model describing a skyrmion. In fact we can add a topological term to this so that now write

$$S=-M^2\int d^4x\partial_\mu g^{-1}\partial_\mu g+N\Gamma \tag{10}$$

where Γ is the Wess-Zumino term

$$\Gamma=\frac{1}{240\pi^2}\int d^5x\varepsilon_{\mu\nu\lambda\sigma\rho}Tr[g^{-1}\partial_\mu gg^{-1}\partial_\nu gg^{-1}\partial_\lambda gg^{-1}\partial_\sigma gg^{-1}\partial_\rho g] \tag{11}$$

the physical space-time being the boundary of a five dimensional manifold. The coefficient of Γ given by N is an integer so that the relevant skyrmion has a consistent statistics.

At each point of the fermionic site, now take into account these low energy skyrmionic excitations. The antiferromagnetic order, which is only valid under the Hubbard-like model's strong coupling and half-filling limits, will be destroyed as a result. This will enable fermion propagation between two sites. This is equivalent to the colour gauge field interaction in lattice QCD when the gauge fields are on the link. The colour change that is connected to the colour gauge transformation occurs together with fermion propagation. It should be noted that fermion doubling on a lattice is an issue we face. However, the inability of chiral anomalies on a lattice to exist is what leads to the doubling dilemma [10]. However as the chiral anomaly is responsible for the generation of mass as well as Pontryagin index which is associated with the charge [11], the nonexistence of chiral anomaly on a lattice suggests that we cannot have well defined local charge as well local mass of a fermion on a lattice [10]. As a result, the fermions on a lattice will have both staggered mass and charge. It is possible to think of the correlation between an antiferromagnet's staggered mass density and staggered magnetization as an equivalent of lattice QCD with an antiferromagnet, which is only true in the strong coupling limit.

It has been noted that the hard core repulsion, which represents the on-site repulsion, will be subdominant when hopping is considerable in the Hubbard-like model for vast distances (low energy area). This suggests that fermions will transfer between sites. This circumstance suggests that fermions will hopp within the lattice sites with change in colours that occur through an interaction with the colour gauge fields residing on the link when we view it in terms of lattice QCD.

Now define a generalized spin operator in the algebra of $SU(N_F)$, N_F being the number of flavors.

$$S^A=\psi_\alpha^{a\dagger}\tau_{\alpha\beta}^A\psi_a^\beta \tag{12}$$

Where $\psi_\alpha^{a\dagger}$ and ψ_a^β are the creation and destruction operators of fermion oscillators which satisfy the algebra.

$$\left\{\psi_{\alpha,i}^a,\psi_{\beta,j}^{b\dagger}\right\}=\delta_{ab}\delta_{\alpha\beta}\delta_{ij} \tag{13}$$

Now denote the indices α, β= 1, 2,....., N_F, the flavor index, a, b = 1, 2,......., N_C color index and i, j are spatial positions.

The color gauge field may be associated with spin fluctuation when the generalized spin is defined as (12). The low energy skyrmionic excitation at each site of the fermion will drive this fermion propagation by destroying the underlying antiferromagnetic order. Again at short distance (high energy region) the on-site repulsion will dominate and this will suppress fermionic propagation and thus will drive the system towards a noninteracting regime leading to asymptotic freedom. Now may mention here that the Goldstone bosons associated with chiral symmetry breaking in this formalism will be pseudoscalars. This results from the fact that the separation of the flavour symmetry group into left-handed and right-handed groups $SU(N_F)_L\otimes SU(N_F)_R$ suggests that flavours will only manifest in the left-handed or right-handed universe and the mass condensate in the entire space will carry no flavour. Space-inversion violation brought on by the chiral symmetry breakdown will result in pseudoscalar bosons. This is in line with the outcome of continuum QCD in (3 + 1) D.

B. Fermion Propagation in the Continuum Limit

It has been pointed out that in lattice QCD, in the weak coupling limit, a topological defect, such as a skyrmionic excitation, is induced at each position of the fermion. This defect is essential for the destruction of the underlying antiferromagnetic order and permits fermionic propagation from one site to the next. The colour gauge interaction, which is the kinetic term in the lattice QCD Hamiltonian, carries out this function. In the continuum limit, now can see how the skyrmion that corresponds to each position of a fermionic ingredient in a particle's bound state configuration moves from one position to the next inside the constrained region of the interacting system's bound state configurations. This corresponds to the color gauge field interaction with the fermionic constituent which allows the change in color of the constituent fermion within the confined region of the configuration. As mentioned earlier, this color gauge field can be associated with spin fluctuation when the generalized spin operator is defined by eqn. (12). This skyrmionic propagation may be visualized as the rearrangement of the fermionic constituents within the bound state configurations of the interacting system through planar and nonplanar diagrams which satisfy $s-t$ and $t-u$ duality respectively, s, t, u having their usual meanings. These two diagrams have phase factors $\exp[i2\pi\gamma]$ and 1 respectively where γ is a suitable parameter. The rearrangement amplitudes are given by [12-15].

$$T_1(s,t)\sim[(p_a+p_b)^2]^{-\gamma}[(p_c+p_d)^2]^{-\gamma}\sim s^{-2\gamma} \tag{14}$$

Here the factor $[(p_i + p_j)^2]^{-\gamma}$ corresponds to the rearrangement of one constituent from an interacting particle with momentum p_i to that with p_j. T_1 and T_2 correspond to planar and nonplanar diagrams respectively. It may be noted that there is a correspondence between $T(s,t)$ and the Regge amplitude T_Regge with strong degenerate trajectories $\alpha(t)$ and residue $\beta(t)$ in the forward regions if we take $-2\gamma+1=\alpha(t)$ [12]. This is evident from the following expressions.

$$T_1(s,t)\big|_{t=0} + T_2(s,t)\big|_{t=0} \sim [1 \mp \exp(i2\pi\gamma)]s^{-2\gamma}$$

And

$$T_Regge = \frac{\beta(t)[1 \pm \exp(-i\pi\alpha(t)]}{\sin \pi\alpha(t)} s^{\alpha(t)-1} \tag{15}$$

It may be mentioned that in any interaction $AB \rightarrow CD$ for n number of constituents rearranged have

$$T(s,t) \sim \prod_n [(p_i + p_j)^2]^{-\gamma} \sim s^{-n\gamma} \tag{16}$$

Now note that the net effect of this rearrangement amplitude may be accomodated in an effective coupling constant given by

$$g(s) = g.s^{-n\gamma} \tag{17}$$

which represents a running coupling constant ensuring asymptotic freedom.

It has been noted that non-Abelian gauge theory's distinctive characteristic is its asymptotic freedom. In light of this, now may speculate that the above-mentioned fermionic propagation may be connected to interactions with non-Abelian gauge fields. As was previously indicated, when a spin operator is produced by generalised fermions with flavour and colour degrees of freedom as provided by equation, non-Abelian colour gauge fields result from spin fluctuations (1). The quantization of a Dirac spinor can be accomplished when we introduce an internal variable that appears as a direction vector representing spin degrees of freedom [17, 18], according to Nelson's stochastic quantization approach [16]. It has been shown that the internal variable gives rise to $SU(2)$ gauge degrees of freedom so that the spin may be depicted as an $SU(2)$ gauge bundle. In this picture a massive fermion appears as a skyrmion. Extrapolating this result may view that when a spin is constructed from generalized fermions having flavor and color indices the spin may be depicted as $SU(N_F) \otimes SU(N_F)$ gauge bundle where $SU(N_F)$ is the global flavor gauge group and $SU(N_C)$ is the color gauge group. Such a spin exhibits only flavour quantum numbers in the static situation because the confinement causes the colour degrees of freedom to be fixed. Spin fluctuations cause the concealed colour degrees of freedom to become visible. Spin fluctuations do, in fact, cause the circumstances under which colour gauge transformations occur. It is possible to assume that the interaction with these non-Abelian gauge fields is what causes a fermionic constituent to propagate from one site to another with a change in colour in the confined domain of the bound state configurations of the interacting system.

3. Conclusion

It should be noted that the colour quantum number is a second quantum number (in addition to flavour) that is only relevant in the restricted domain of a bound state configuration that represents a physical particle. This concept gives colour degrees of freedom a broader definition than the simple quantum numbers that are naively assigned to the fermionic components of a bound state that represents a physical particle. These degrees of freedom may be connected to a particular characteristic of a fermionic ingredient that may result from the bound state's dynamical aspect. In a previous paper [19], it was demonstrated that flavour quantum numbers and the internal symmetry of hadrons may both result from a particular configuration scheme in which the fermionic constituents move in an anisotropic space (a fictitious magnetic field) within the confines of the composite state. In this composite system a fermion appears to move in the field of a magnetic monopole whenit can have orbital angular momentu $l = 1/2$ and the background magnetic field associatedwith the carrier implies that this effectively corresponds to a skyrmion. Evidently the total angular momentum of such a constituent can take the value $J=1$ with its three J_z values+1, 0, *and* − 1. Now can assign these three quantum numbers as color degrees of freedom and the rearrangement of the constituents within the confined region of the relevant systems will be accompanied by a change in these values. The spin fluctuation may be taken to give rise to a color gauge field $SU(N_C)$ with $N_C = 3$.

4. Discussion

The rearrangement of the constituents within the confined domain may be taken to be equivalent to the color gauge field interaction with the constituents having the gauge group $SU(3)$. This analysis suggests that QCD may be regarded as a generalised non-Abelian gauge field theory when can give colour gauge degrees of freedom to any particular characteristic of a constituent (either static or dynamical) in a bound state so that these characteristics are only significant in the confined area of the system. As is wellknown QCD has been generalized from the physical value $N_C = 3$ to any arbitrary value and its inverse $\frac{1}{N_C}$ is treated as an effective expansion parameter [20]. Witten [21] has shown that QCD is equivalent to an effective theoryof weakly interacting mesons. By weakly interacting mesons, it is intended that baryons appear as the soliton solution of the effective meson theory and an effective four meson vertex scale is $\frac{1}{N_C}$ Skyrmions taken to be low energy degrees of freedom of QCD [22–24]. According to our study, there may be common physical circumstances that permit arbitrary assignment of N_C .

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