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The Solvability of a New Boundary Value Problem with Derivatives on the Boundary Conditions for Forward-Backward Semi Linear Systems Mixed of Keldysh Type in Multivariate Dimension

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Abstract: Abstract. In present paper we investigate solvability of a new boundary value problem with derivatives on the boundary conditions for semi-linear systems of mixed hyperbolic-elliptic of Keldysh type equations in multivariate dimension with the changing time direction. Considered problem and system equations are new and belong to modern level of partial differential equations, moreover contain partition degenerating elliptic, degenerating hyperbolic, mixed and composite type differential equations. Applying methods of functional analysis, topological methods, "ε -regularizing» and continuation by the parameter at the same time with aid of a prior estimates, under assumptions conditions on coefficients of equations of system, the existence and uniqueness of generalized and regular solutions of a boundary value problem are established in a weighted Sobolev's space. In this work one of main idea, the identity of strong and weak solution is established.

Keywords: Changing Time Direction, Weighted Sobolev's Space, Equation of Mixed Type, Strong, Weak and Regular Solution, Forward-Backward Equations, System Equations of Mixed Hyperbolic-Elliptic Keldysh Type

1. Introduction

Note that, non-classical equations arises in applications in the field of hydro-gas dynamics, aerodynamics, plasma and some of modeling of physical process (for example, in [3], [5], [6], [7], [11], [19], etc. and the references given therein). Many authors investigated nonlinear and semi-linear mixed type equations (for example, in [2], [7], [13], [16], etc. the references given therein). In the work [12] considered the Direchlet problem for elliptic-hyperbolic equation of Keldysh type, and in [6] existence smooth solution for a Keldysh type equation is proved. Note that a parabolic equation and mixed equation with changing time direction also has physical applications. The boundary value problems with such sewing conditions appear when modeling, for example, process of interaction between two reciprocal flows with mutual permeating, or when designing certain heat exchangers. Frankly speaking forward-backward equations (equations of changing time direction) arise in supersonic dynamics, boundary layer theory and plasma. Therefore the boundary value problems for equations of mixed hyperbolic-elliptic type with changing time direction and equation of parabolic type with changing time direction (forward-backward equations)(e.g. [14], [18], and the references given therein) resents attention as important object for all investigators. As it noted in the work [20] that for interesting, the non-classical model is defined as the model of mathematical physics, which is represented in the form of the equation or systems of partial differential equations that does not fit into one of the classical types as elliptic, parabolic, or hyperbolic. In particular, non-classical models are described by equations of mixed type (for example, the Tricomi equation), degenerate equations (for example, the Keldysh equation or the equations of Sobolev type (e.g., the Barenblatt-Zsolt-Kachina equation), the equation of the mixed type with the changing time direction and forward-backward equations. As it shown in the work [20], the theory of boundary value problems for degenerate equations and equations of mixed-type, as it shown in the work (e.g. , [5], [8]) the well-posedness and the class of its correctness essentially depend on the coefficient of the first order derivative (younger member) of equations . The solvability a some of new kind of boundary value problems for linear system equations of mixed type with the changing time direction had been studied details in [19],[20], [21]. Frankly speaking great difficulties come into being in the investigation of systems of degenerate elliptic and hyperbolic equations. Note that solvability of different boundary value problems for nonlinear and semi-linear system equations of hyperbolic-elliptic type including property of changing time direction and in case of multivariate dimension has not been extensively investigated. Now in this paper we will study such important problem.

2. Well-Posed Boundary Value Problem and Notation, Preliminaries

Let G be a bounded domain in the Euclidean space R^n of the point $x=(x_1,...,x_n)$, including a part of hyper plane $x_n=0$ and with sufficiently smooth boundary $\partial G\in C^2$, $G^+=G\bigcap\{x_n>0\}$, $G^-=G\bigcap\{x_n<0\}$. The boundary of G^+ consists of a part of hyper -plane $x_n=0$ for $x_n>0$ and smooth surface ∂G^+ . Analogically, the boundary G^- consists of a part of hyper -plane $x_n=0$ for $x_n<0$ and smooth surface ∂G^- .Assume that $D=G\times(-T,T)$, $T>0$; $S=\partial G\times(-T,T)$, where $\Gamma=\partial D$ is a boundary of domain D . In the domain D consider the system of equations:

$$\left.\begin{aligned}
&L_1(u,\upsilon)=k_1^{(1)}(t)u_{tt}+k_2(x)\Delta_x u+\sum_{i=1}^n a_{i1}^{(1)}(x,t)u_{x_i}+\sum_{i=1}^n a_{i2}^{(1)}(x,t)\upsilon_{x_i}+\\
&+b_{11}(x,t)u_t+b_{12}(x,t)\upsilon_t+c_{11}(x,t)u+c_{12}(x,t)\upsilon+c_1(x)|u|^{\rho_1}u=f_1(x,t,u,\upsilon)\\
&L_2(u,\upsilon)=k_1^{(2)}(t)\upsilon_{tt}-\Delta_x\upsilon+\sum_{i=1}^n a_{i1}^{(2)}(x,t)u_{x_i}+\sum_{i=1}^n a_{i2}^{(2)}(x,t)\upsilon_{x_i}+\\
&+b_{21}(x,t)u_t+b_{22}(x,t)\upsilon_t+c_{21}(x,t)u+c_{22}(x,t)\upsilon-|\upsilon|^{\rho_2}\upsilon=f_2(x,t,u,\upsilon)
\end{aligned}\right\}\qquad(2.1)$$

Where the Δ_x is Laplace operator

$$\Delta_x=\frac{\partial^2}{\partial x_1^2}+...+\frac{\partial^2}{\partial x_n^2}$$

Everywhere we will assume that the coefficients of the systems of equations (2.1) are sufficiently smooth and the conditions $tk_1^{(i)}(t)>0$ for $t\neq0,\ t\in(-T,T)$, $i=1,2$, $x_nc_1(x)>0,\ x_nk_2(x)<0$, $x_n\neq0,\ x=(x_1,...,x_n)\in G\in R^n$ are satisfied. As well as is known that quadratic form of equations of system (2.1) changes, then this system contain partition degenerating elliptic, degenerating hyperbolic, mixed and composite type differential equations at the same time including changing direction time of variable in the domain D .

Assume the ntations

$$\Gamma_{-T}^+=\{(x,t)\in\Gamma:x_n>0,\ t=-T\},\ \Gamma_{-T}^-=\{(x,t)\in\Gamma:x_n<0,\ t=-T\},\ \Gamma_T^+=\{(x,t)\in\Gamma:x_n>0,\ t=T\},$$
$$\Gamma_T^-=\{(x,t)\in\Gamma:x_n<0,\ t=T\}\ S^+=\partial G^+\times[-T,T],\ S^-=\partial G^-\times[-T,T],\ D^+=D\bigcap\{x_n>0\},\ D^-=D\bigcap\{x_n<0\}$$

The boundary value problem: Find the solution of system equations (1.1) in the domain $\overline{D}$, satisfying the conditions:

$$u|_\Gamma=0,\ u_t|_{\overline{\Gamma_{-T}^-}}=0,\ u_t|_{\overline{\Gamma_T^-}}=0,\qquad(2.2)$$

$$\upsilon|_\Gamma=0,\ \upsilon_t|_{\overline{\Gamma_{-T}^-}}=0,\ \upsilon_t|_{\overline{\Gamma_T^+}}=0\qquad(2.3)$$

By the symbol C_L we denote a class of twice continuously differentiable functions in the closed domain D, satisfying the boundary conditions (2.2), (2.3), by $H_{1,L}(D)$, $H_{2,L}(D)$ in Sobolev's space with weighted spaces obtained by the class C_L which is closed by the norm:

$$\|u\|_{H_{1,L}(D)}^2=\int_D(u_t^2+|k_2(x)|\sum_{i=1}^n u_{x_i}^2+u^2)dD,$$

$$\|u\|_{H_{2,L}(D)}^2=\int_D(u_{tt}^2+k_2^2(x)\sum_{i=1}^n u_{x_ix_i}^2+|k_2(x)|\sum_{i=1}^n u_{x_it}^2+|k_2(x)|\sum_{i=1}^n u_{x_i}^2+u_t^2+u^2)dD,$$

respectively.

Introduce, the space $W_2^k(D)$ Sobolev's with the norm(e.g.[1],[15]):

$$\|u\|_{W_2^k(D)}^2=\|u\|_{K,D}^2=\int_D\sum_{|\alpha|\le k}|D^\alpha u|^2\,dxdt,$$

$$|\alpha|=\alpha_0+...+\alpha_n,\ D^\alpha=D_0^{\alpha_0}...D_n^{\alpha_n},\ D_0=\frac{\partial}{\partial t},\ D_i=\frac{\partial}{\partial x_i}.$$

Since $k_2(x) \neq 0$ for $x_n \neq 0$, by the Sobolev's embedding Theorem (in [1], [15]) the functions from the spaces $H_{2,L}(D)$ will satisfy the boundary conditions (2.2), (2.3).

3. The Existence Generalized Solution of Problem (2.1)- (2.3)

Before describing our theorem on existence, we take as decaying systems equations for (2.1) in the following form:

$$L_3(u) = k_1^{(1)}(t)u_{tt} + k_2(x)\Delta u + \sum_{i=1}^{n} a_{i1}^{(1)} u_{x_i} + b_{11}u_t + c_{11}u + c_1(x)|u|^{\rho_1} u = f_1(x,t,u,\upsilon), \tag{3.1}$$

$$L_4(\upsilon) = k_1^{(2)}(t)\upsilon_{tt} - \Delta\upsilon + \sum_{i=1}^{n} a_{i2}^{(2)} \upsilon_{x_i} + b_{22}\upsilon_t + c_{22}\upsilon - |\upsilon|^{\rho_2} \upsilon = f_2(x,t,u,\upsilon). \tag{3.2}$$

Now we able pose to formulate a definition of the generalized solution for the system equations under consideration.

Definition 3.1. The functions $u(x,t)$ and $\upsilon(x,t$) are generalized solutions of the problems (3.1), (2.2) and (3.2), (2.3), if for any functions $u(x,t) \in H_1(D) \cap L_{\rho_1+2,|c_1|}(D)$, $\upsilon(x,t) \in W_2^1(D) \cap L_{\rho_2+2}(D)$ respectively, and for $\phi(x,t) \in W$ the following identities holds:

$$\begin{aligned} B_1(u,\phi) = &-\int_D k_1^{(1)}(t)u_t\phi_t dD + \int_D \left(b_{11} - k_{1t}^{(1)}(t)\right)u_t\phi_t dD - \int_D \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(k_2(x)u\right)\phi_{x_i} dD \\ &+\int_D \left(c_{11} - \sum_{i=1}^{n}(a_{i1x_i}^{(1)} - k_{2x_ix_i})\right)u\phi_{x_i} dD - \int_D \left(\sum_{i=1}^{n}(a_{i1x_i}^{(1)} - 2k_{2x_i})\right)u\phi_{x_i} dD \\ &+\int_D c_1(x)|u|^{\rho_1} u\phi dD = \int_D f_1\phi dD \end{aligned} \tag{3.3}$$

$$\begin{aligned} B_2(\upsilon,\phi) = &-\int_D k_1^{(2)}(t)\upsilon_t\phi_t dD + \int_D \left(b_{22} - k_{1t}^{(2)}(t)\right)\upsilon_t\phi dD - \int_D \sum_{i=1}^{n} \upsilon_{x_i}\phi_{x_i} dD \\ &+\int_D \left(c_{22} - \sum_{i=1}^{n} a_{i2x_i}^{(2)}\right)\upsilon\phi dD - \int_D \sum_{i=1}^{n} a_{i2}^{(2)}\upsilon\phi_{x_i} dD - \int_D |\upsilon|^{\rho_2} \upsilon\phi dD = \int_D f_2\phi dD, \end{aligned} \tag{3.4}$$

where

$$W = \left\{\phi : \phi \in C^2(\bar{D}),\ \ \phi|_\Gamma = 0,\ \ \phi_t|_{\overline{\Gamma_T^+}} = \phi_t|_{\overline{\Gamma_{-T}^-}} = 0\right\}. \tag{3.5}$$

respectively, holds true the following theorem.

Theorem 3.1 (*existence of generalized solution of problem (*3.1), (2.2) and (3.2), (2.3)).Suppose that

(i) $b_{11}(x,t) - \frac{1}{2}k_{1t}^{(1)}(t) \leq -\delta < 0$, $b_{22}(x,t) - \frac{1}{2}k_{1t}^{(2)}(t) \leq -\delta < 0 \quad \forall(x,t) \in D.$

(ii) $x_n c_{11} > 0$ for $x_n \neq 0, c_{11t} \geq 0, \quad x = (x_1,...,x_n) \in G,$ $t \in [-T,T]$ (Or $-(c_{11}\alpha_{1t} - c_{11t}\alpha_1) \geq 0$)

(iii) $-c_{22t}\alpha_1 - c_{22}\alpha_{1t} \geq 0,\ \forall(x,t) \in D$,

(iv) $\sum_{i=1}^{n}\left(a_{i1}^{(1)} - k_{2x_i}\right)^2 \leq M|k_2(x)|;\ \ k_{2x_n}^2 \leq M|k_2(x)|$;

(v) $\rho_1 > -1,\ \ -1 < \rho_2 < \dfrac{2}{n-2}$ (vi) $c_{11} - \sum_{i=1}^{n}(a_{i1x_i}^{(1)} - k_{2x_ix_i}) > 0$, $\forall(x,t) \in D$; (vii) $\sum_{i=1}^{n} a_{i2}^{(2)}(x,t) + b_{22}(x,t) \geq 0, (x,t) \in D.$

Assume that the functions $f_1(x,t,u,\upsilon), f_2(x,t,u,\upsilon) \in L_2(D)$ are continuous respect to u, υ and $\|f_1(x,t,u,\upsilon)\|_{L_2} \leq C_1^* + C_2^*\int_D |u|^{\rho_1^*} dD, \rho_1^* < \rho_1 + 2$,

$\|f_2(x,t,u,\upsilon)\|_{L_2} \leq C_3^* + C_4^*\int_D |\upsilon|^{\rho_2^*} dD, \rho_2^* < \rho_2 + 2$ (where C_1^*, C_2^*, C_3^*, C_4^* are constants), then, there exists a generalized solution of the problem (3.1),(2.2) and (3.2),(2.3) $u(x,t) \in H_1(D) \cap L_{\rho_1+2,|c_1|}(D)$, $\upsilon(x,t) \in W_2^1(D) \cap L_{\rho_2+2}(D)$ respectively,

Proof. For prove of theorem existence we will apply the method of Faedo-Galerkin chosen a complete system of orthonormal bases $\{\phi^i(x,t)\}$ in space $L_2(D)$ and $\phi^i(x,t) \in C^2(D)$ which satisfy the conditions (3.5). According to (e.g., [10]) the functions $\phi^i(x,t)$ must satisfy the ordinary differential equation

$$\phi^i(x,t) = (-tx_n - M_i)\psi_t^i(x,t) \ \text{ in } D \tag{3.6}$$

and the solution of (3.6) which satisfies the following conditions:

$$\psi_t^i(x,-T) = 0 \text{ for } x_n < 0;\ \psi_t^i(x,T) = 0 \text{ for } x_n > 0,\ x = (x_1,...,x_n) \in \Omega\ \psi^i(x,t)\big|_\Gamma = 0 \tag{3.7}$$

Furthermore, we will seek approximate solutions in the forms:

$$u^m(x,t) = \sum_{i=1}^{n} c_{im}\psi^i(x,t),\ \upsilon^m(x,t) = \sum_{i=1}^{n} c_{im}^{'}\psi^i(x,t),$$

where the constants c_{im} and $c_{im}^{'}$ will be determined from the nonlinearly algebraic equations

$$B_1\left(u^m,\phi^i\right)_{L_2(D)}=\left(f_1^*,\phi^i\right)_{L_2(D)} \quad (3.8)$$

$$B_2\left(\upsilon^m,\phi^i\right)_{L_2(D)}=\left(f_2^*,\phi^i\right)_{L_2(D)} \quad (3.9)$$

It is easy to see that the integral identity has meaning. Here we use the ideas from [10, chapter 1, 4], to prove the existence of generalized solutions to the problems (3.1),(2.2) and (3.2) ,(2.3). We exploit the Faedo-Galerkin method, a priori estimates and compactness arguments. Our study is motivated by proof of solvability identity of (3.8),(3.9) follows from the conditions (i) – (vii) and from Lemma 1.3 (e.g., [10, chapter 1, p. 25]) and taking into account estimates on the approximate solution which will be obtained in future. Therefore, we may now state that to obtain a prior estimation for the approximate solution of problems (3.1), (2.2) and (3.2),(2.3) , and multiply the identity of (3.8) (or (3.9) by c_{im}, (or $c_{im}^{'}$) and summing over of i from 1 to m , respectively, then we get

$$m_1^*\left\|u^m\right\|_{H_1(D)}^2+\frac{1}{\rho_1+2}\left\|\left|c_1\right|^{\frac{1}{\rho_1+2}}u^m\right\|_{L_{\rho_1+2}\left(D^+\right)}^{\rho_1+2}+\frac{1}{\rho_1+2}\left\|\left|c_1\right|^{\frac{1}{\rho_1+2}}u^m\right\|_{L_{\rho_1+2}\left(D^-\right)}^{\rho_1+2}\le\int\limits_{D^+}(\alpha_1f_1)^2dD^++\int\limits_{D^-}(\alpha_1f_1)^2dD^-. \quad (3.10)$$

$$m_2^*\left\|\upsilon^m\right\|_{W_2^1(D)}^2+\frac{1}{\rho_2+2}\left\|\upsilon^m\right\|_{L_{\rho_2+2}\left(D^+\right)}^{\rho_2+2}+\frac{1}{\rho_2+2}\left\|\upsilon^m\right\|_{L_{\rho_2+2}\left(D^-\right)}^{\rho_2+2}\le\int\limits_{D^+}(\alpha_1f_2)^2dD^++\int\limits_{D^-}(\alpha_1f_2)^2dD^- \quad (3.11)$$

where $\alpha_1=\left(-tx_n-M_1\right)$ and constant m_1^*,m_2^* are independent of functions $u^m\left(x,t\right)$, $\upsilon^m\left(x,t\right)$ and m. Consequently, there exists two subsequences (we denote it again by $u^m\left(x,t\right)$ $\upsilon^m\left(x,t\right)$) and the functions $u(x,t)\in H_1\left(D\right)\cap L_{\rho_1+2,|c_1|}\left(D\right)$ and $\upsilon(x,t)\in W_2^1\left(D\right)\cap L_{\rho_2+2}(D)$ such that

$u^m\to u$ weakly in $H_1\left(D\right)$, $\upsilon^m\to\upsilon$ weakly in $W_2^1(D)$ (3.12)

$u^m\to u$ weakly in $L_{\rho_1+2}(D)$, $\upsilon^m\to\upsilon$ weakly in $L_{\rho_2+2}(D)$ (3.13)

It follows from (3.12), one can pass to the limit in the linear terms left side in (3.8),(3.9). Now, we need to show that, in (3.8), (3.9) terms of nonlinearity can be omitted to pass a limit. For this aim we take $w_{x_n}^m=\sqrt{\left|k_2\right|}u_{x_i}^m$ and it is easy to see that $w^m\left(x,t\right)\in W_2^1\left(D\right)$. As usually, in this situation the imbedding theorems play very an important role, but in the present case we deal with weighted spaces [14] roughly speaking in this case directly standard passing to limit for degenerating weighted function in (3.8) and (3.9) is very difficult. From the representation of functions w^m, we get $w_{x_n}^m=\sqrt{\left|k_2\right|}u_{x_i}^m+\frac{1}{2}\frac{k_{2x_i}}{\sqrt{\left|k_2\right|}}u^m,\ i=1,2,...,n,\ \ w_t^m=\sqrt{\left|k_2\right|}u_t^m$. Hence, $\left\|w^m\right\|_{W_2^1(D)}\le M_2$ for any m , M_2 is constant. According to Sobolev's embedding theorem(e.g., [1], [15]) there exists the functions $w,\upsilon\in W_2^1\left(D\right)$ and subsequences (which we again denote by $u^m\left(x,t\right)$, $\upsilon^m\left(x,t\right)$) such that $w^m\to w$ strongly almost everywhere (a.e.) in $L_2\left(D\right),\upsilon^m\to\upsilon$ strongly almost everywhere (a.e.) in $L_2\left(D\right)$.Consequently, $\sqrt{\left|k_2\right|}u^m\to w$ a.e. in D for any m and $\upsilon^m\to\upsilon$ a.e. in D. Since $k_2\left(x\right)$ vanishes only on the hyper-plane $x_n=0$, then $w^m\to\frac{w}{\sqrt{\left|k_2\right|}}$ a.e. Then we conclude that $u^m\to u$ a.e. in D and $\left|c_1\right|^{\frac{1}{\rho_1}}\operatorname{sgn}c_1\left|u^m\right|^{\rho_1}u^m\to\left|c_1\right|^{\frac{1}{\rho_1}}\operatorname{sgn}c_1\left|u\right|^{\rho_1}u$ a.e. in D.

Hence, taking into account (3.10), (3.11) and by virtue of the compactness of imbedding $W_2^1\left(D\right)$ in $L_2\left(D\right)$, and moreover by Lemma 1.3 (e.g., [10, chapter1, p. 25]) guarantees

$$\left|\upsilon^m\right|^{\rho_{21}}\upsilon^m\to\left|\upsilon\right|^{\rho_1}\upsilon \text{ weakly in } L_{\frac{\rho_2+2}{\rho_2+1}}\left(D\right).$$

Hence, $\left\{c_1\left|u^m\right|^{\rho_1}u^m\right\}\to c_1\left|u\right|^{\rho_1}u$ weakly in $L_{\frac{\rho_1+2}{\rho_1+1}}\left(D\right)$.

Thus, we able to pass to the limit the nonlinear terms of left in (3.8) (3.9) By virtue of the continuity of $f_1(x,t,u,\upsilon)$ and $f_2(x,t,u,\upsilon)$ functions respect to components, we have

$$f_1(x,t,u^m,\upsilon^m)\to f_1(x,t,u,\upsilon)a.e.,$$
$$f_2(x,t,u^m,\upsilon^m)\to f_2(x,t,u,\upsilon)a.e..$$

It follows from the same Lemma 1.3 (e.g., [10, chapter1, p. 25]) that

$$f_1(x,t,u^m,\upsilon^m)\to f_1(x,t,u,\upsilon)\text{ weakly in } L_{\frac{\rho_1+2}{\rho_1+1}}(D),$$

$$f_2(x,t,u^m,\upsilon^m)\to f_2(x,t,u,\upsilon)\text{ weakly in } L_{\frac{\rho_2+2}{\rho_2+1}}(D).$$

Thus, we able pose to pass to the limit in terms of nonlinearities of the right in (3.8) and (3.9).This completes the proof of Theorem 3.1, if we prove the solvability of systems (3.8) and (3.9). We put

$$c=(c_1,...,c_m),A(c)=(A^1(c),...,A^m(c)),$$

$$A^m(c)=-\sum_{i=1}^m c_{im}\int_D L_1\phi^i\psi^i dD-\sum_{i=1}^m c_{im}\int_D\left|\sum_{i=1}^m c_{im}\phi^i\right|^{\rho_1}\phi^i\psi^i dD-\int_D f_1\left(x,t,c_{im}\phi^i,c'_{im}\phi^j\right)\psi^i dD.$$

$$c'=(c'_1,...,c'_m),A(c')=(A^1(c'),...,A^m(c')),$$

$$A^m(c')=-\sum_{i=1}^m c'_{im}\int_D L_2\phi^i\psi^i dD-\sum_{i=1}^m c'_{im}\int_D\left|\sum_{i=1}^m c'_{im}\phi^i\right|^{\rho_1}\phi^i\psi^i dD-\int_D f_2^*\left(x,t,c_{im}\phi^i,c'_{im}\phi^j\right)\psi^i dD.$$

To establish solvability of (8.8) and (8.9) with respect c, c' we employ the Lemma4.3 (e.g.,[10, chapter1, p. 66],which is also known as Sharp Angle Lemma),it suffices to prove that $A^n(c)$, $A^n(c')$ are continuous functions and $(A(c),c)\geq p_0|c|^2-p_1,\ p_0>0,\ p_1\geq 0$, $(A(c'),c')\geq p^*_0|c'|^2-p^*_1,\ p^*_0>0,\ p^*_1\geq 0$ The term connected with $B(\lambda)=|\lambda|^{\rho_1}\lambda$, after integration by parts, analogous to the one carried out in the proof of the lemma, gives nonnegative quantity.

4. The Uniqueness of Solution of Problems (3.1), (2.2) and (3.2), (2.3)

Theorem 4.1 Suppose that the conditions (i) – (vii) of Theorem 3.1 are fulfilled and moreover, we assume that $c_1(x)>0$ for $x_n<-2\varepsilon$; $c_1(x)\equiv 0$ for $-2\varepsilon<x_n<2\varepsilon$ and if the norms $\|f_1(x,t,u,\upsilon)\|_{L_2}\leq C_1^*+C_2^*\int_D|u|^{\rho_1^*}dD,\rho_1^*<\rho_1+2,$ $\|f_2(x,t,u,\upsilon)\|_{L_2}\leq C_3^*+C_4^*\int_D|\upsilon|^{\rho_2^*}dD,$ $\rho_2^*<\rho_2+2$ $\|f_2\|_{L_2(D)}<+\infty,$ $\|f_1\|_{L_2(D)}<+\infty,$ sufficiently small, then $\rho_2^*<\rho_2+2$ there exists a unique generalized solution problems (3.1),(2.2) ((3.2),(2.3)) $u(x,t)\in H_1(D)\cap L_{\rho_1+2,|c_1|}(D)$ ($\upsilon(x,t)\in W_2^1(D)\cap L_{\rho_2+2}(D)$ respectively.

Proof. By the inequality of (3.10) we can write

$$P=\int_{D^+}(\alpha_1 f_1)^2dD^++\int_{D^-}(\alpha_1 f_1)^2dD^-\geq\int_{D^+}\left\{\delta_1u_t^2+\varepsilon(\alpha_{1t}-M_1)\sum_{i=1}^n u_{x_i}^2+u^2\right\}dD^+$$
$$+\int_{D^-}\left\{\delta_2u_t^2+\varepsilon(\alpha_{1t}-M_1)\sum_{i=1}^n u_{x_i}^2+u^2\right\}dD^--\frac{1}{\rho_1+2}\int_{D^+}c_1(x)|u|^{\rho+2}\alpha_t dD^+-\frac{1}{\rho_1+2}\int_{D^-}c_1(x)|u|^{\rho+2}\alpha_t dD^-$$

where, $\delta_1=\min\limits_{D^+}\left(2b_{11}-k_{1t}^{(1)}(t)-\alpha_{1t}k_1^{(1)}(t)-2\right),\ \delta_2=\min\limits_{D^-}\left(2b_{11}-k_{1t}^{(1)}(t)-\alpha_{1t}k_1^{(1)}(t)-2\right)..$

Let $u_1(x,t),u_2(x,t)$ be two solutions of problem (3.1),(3.2) from the space $H_1(D)\cap L_{\rho_1+2,|c_1|}(D)$ set $u\equiv u_1-u_2$, then we get

$$P\geq\int_{D_\varepsilon^+}\left[\delta_1u_t^2+\varepsilon(-\alpha_{1t}-M)\sum_{i=1}^n u_{x_i}^2\right]dD^++\int_{D_\varepsilon^-}\left[\delta_2u_t^2+\varepsilon(\alpha_{1t}-M)\sum_{i=1}^n u_{x_i}^2\right]dD^-$$

$$J\leq\delta_3(\rho_1+1)\|u_t\|_{L_2(D_{2\varepsilon}^+)}\|u\|_{L_{\rho_1}(D_{2\varepsilon}^+)}\|g\|_{L_q(D_{2\varepsilon}^+)}+\delta_3(\rho_1+1)\|u_t\|_{L_2(D_{2\varepsilon}^-)}\|u\|_{L_{\rho_1}(D_{2\varepsilon}^-)}\|g\|_{L_q(D_{2\varepsilon}^-)}.$$

where $\frac{1}{2}+\frac{1}{\rho_1}+\frac{1}{q}=1,\ g=|u_1+\theta u_2|^{\rho_1}(u_1+\theta u_2),\ 0<\theta<1$ and $\delta_3>0$ is constant. Hence, using multiplicatively inequality from the work (e.g., [9, chapter1, p.28]), and by the Sobolev's embedding Theorem (e.g., [1],[15]) , we obtain

$$\int_{D_{2\varepsilon}^+}\left[\delta_1u_t^2+\varepsilon(-\alpha_{1t}-M)\sum_{i=1}^n u_{x_i}^2\right]e^{\lambda t}dD^++\int_{D_{2\varepsilon}^-}\left[\delta_2u_t^2+\varepsilon(\alpha_{1t}-M)\sum_{i=1}^n u_{x_i}^2\right]dD^-\leq$$
$$\leq\beta\delta_3(\rho_1+1)\left[\|u\|^2_{W_2^1(D_{2\varepsilon}^+)}\|g\|_{L_6(D_{2\varepsilon}^+)}+e^{\mu t}\|u\|^2_{W_2^1(D_{2\varepsilon}^-)}\|g\|_{L_6(D_{2\varepsilon}^-)}\right].$$

Now, let's to estimate the function $g=|u_1+\theta u_2|^{\rho_1}(u_1+\theta u_2)$, $0<\theta<1$ in space $L_6(\overline{D})$ and we get

$$\min\{\min[\delta_1,\varepsilon(-\alpha_{1t}-M)],\min[\delta_2,\varepsilon(\alpha_{1t}-M)]\}\cdot\left[\|u\|^2_{W_2^1(D_{2\varepsilon}^+)}+\|u\|^2_{W_2^1(D_{2\varepsilon}^-)}\right]\le$$

$$\le\delta_3(\rho_1+1)\left[\frac{2(n-1)}{n-2}\right]^{\frac{n}{6}}[3(\rho_1+1)]^{\frac{3\rho_1+2}{3}}\cdot P^{\frac{\rho_1+1}{2}}\times$$

$$\times\max\left\{\left[\frac{1}{\min[\delta_1,\varepsilon(-\alpha_{1t}-M)]}\right]^{\frac{\rho_1+1}{2}},\left[\frac{1}{\min[\delta_2,\varepsilon(\alpha_{1t}-M)]}\right]^{\frac{\rho_1+1}{2}}\right\}\cdot\left[\|u\|^2_{W_2^1(D_{2\varepsilon}^+)}+\|u\|^2_{W_2^1(D_{2\varepsilon}^-)}\right].$$

Hence, if the inequality holds true:

$$\min\{\min[\delta_1,\varepsilon(-\alpha_{1t}-M)],\min[\delta_2,\varepsilon(\alpha_{1t}-M)]\}>\delta_3(\rho_1+1)\left[\frac{2(n-1)}{n-2}\right]^{\frac{n}{6}}\times$$

$$\times[3(\rho_1+1)]^{\frac{3\rho_1+2}{3}}\cdot P^{\frac{\rho_1+1}{2}}\cdot\max\left\{\left[\frac{1}{\min[\delta_1,\varepsilon(-\alpha_{1t}-M)]}\right]^{\frac{\rho_1+1}{2}},\left[\frac{1}{\min[\delta_2,\varepsilon(\alpha_{1t}-M)]}\right]^{\frac{\rho_1+1}{2}}\right\}$$

then, we obtain contrary proposition. This implies that, $u\equiv 0$ and $u_1\equiv u_2$. If the conditions of the Theorem 3.2 are satisfied, then there exists a unique solution in the space $W_2^1(D_{2\varepsilon}^+\cup D_{2\varepsilon}^-)$ (or in $H_1(D_{2\varepsilon}^+\cup D_{2\varepsilon}^-)$). In the case where $x_n=-2\varepsilon$ and $x_n=2\varepsilon$ we have trace inequality

$$\|u\|_{H_1(D_\varepsilon)}\le m\|f_1^*\|_{L_2(D)},$$

where, the constant m>0, which is obtained in Theorem 4.1.By virtue of the conditions of restriction on $c_1(x)$, we have $c_1(x)|u|^{\rho_1}u\equiv 0$ in the domain $\tilde{D}=D\setminus(D_{2\varepsilon}^-\cup D_{2\varepsilon}^+)$. Thus, there exists a unique generalized solution $u(x,t)$ of problems (3.1), (2.2) in the space $H_1(D)\cap L_{\rho_1+2,|c_1|}(D)$. Let $\upsilon_1(x,t),\upsilon_2(x,t)$ be two solutions of problem (3.2), (2.3) in the space $\upsilon(x,t)\in W_2^1(D)\cap L_{\rho_2+2}(D)$ and set $\upsilon\equiv\upsilon_1-\upsilon_2$ then by according standard Approaches we have

$$\frac{1}{2}\int_G[k_1^2(t)\upsilon_t^2+(\delta_1^*)|\nabla\upsilon|^2]dx\Big|^{t=t}+(\delta-\varepsilon_1)\int_{D_t}\upsilon_t^2dD-\frac{1}{4\varepsilon_1}\int_{D_t}f_2^2dD\le\int_{D_t}\left(|\upsilon_1|^{\rho_2}\upsilon_1-|\upsilon_2|^{\rho_2}\upsilon_2\right)\upsilon_tdD \quad (4.1)$$

$$\int_{D_t}\left(|\upsilon_1|^{\rho_2}\upsilon_1-|\upsilon_2|^{\rho_2}\upsilon_2\right)\upsilon_tdD\equiv\int_{-T}^{t}J(\tau)d\tau.$$

By Holder's inequality we have $J(t)\le C^*(\||\upsilon_1|^{\rho_2}\|_{L_2(G)}+\||\upsilon_2|^{\rho_2}\|_{L_n(G)})\|\upsilon(x,t)\|_{L_r(G)}\|\upsilon_t(x,t)\|_{L_2(G)}$. Since $\||\upsilon_1|^{\rho_2}\|_{L_2(G)}+\||\upsilon_2|^{\rho_2}\|_{L_n(G)}\le c^{**}$, for $\rho_2 n\le r$, then we get $J(t)\le c^*\|\nabla\upsilon(x,t)\|_{L_2(G)}\|\upsilon_t(x,t)\|_{L_2(G)}$. Hence , taking into account inequality (4.1),it follows $\|\nabla\upsilon(x,t)\|^2_{L_2(G)}+\int_{D_t}\upsilon_t^2dD\le C\int_{-T}^{t}\|\nabla\upsilon\|^2_{L_2(G)}d\tau.$ Finally, by Gromwell's lemma, we conclude that $\upsilon\equiv\upsilon_1-\upsilon_2=0,\upsilon_1=\upsilon_2$.

Remark4.1. If instead of smallest of $|a_{i2}^{(2)}(x,t)|$ be satisfied the condition $|a_{i2}^{(2)}(x,t)|^2\le M|k_2(x)|$ then from the Theorem 3.1 it follows that for any function $f_2(x,t,u,\upsilon)\in L_2(D)$, there exists unique generalized solution of problem (3.2), (2.3) from the space $\upsilon(x,t)\in H_1(D)\cap L_{\rho_2+2}(D)$.

5. Strong (Regular) Solution of Problems (3.1), (2.2) and (3.2), (2.3)

Theorem 5.1.Suppose that the following conditions are fulfilled:

(i) $2b_{22}(x,t)-\frac{1}{2}k_{1t}^{(2)}(t)\le-\delta<0 \quad \forall(x,t)\in D;$

(ii) $-1<\rho_2<\frac{2}{n-2}$, (iii) $-c_{22t}\alpha_1-c_{22}\alpha_{1t}\ge 0;$

(iv) $\sum_{i=1}^{n}a_{i2}^{(2)}(x,t)+b_{22}(x,t)\ge 0,(x,t)\in D.$ Then for any function $f_2(x,t,u,\upsilon),f_{2t}(x,t,u,\upsilon)\in L_2(D)$ and

$\|f_2(x,t,u,\upsilon)\|_{L_2} \le C_3^* + C_4^* \int_D |\upsilon|^{\rho_2^*} dD, \quad \rho_2^* < \rho_2 + 2$ (where C_3^*, C_4^* are constants) there exists unique generalized solution of problem (3.2), (2.3), from the space $\upsilon(x,t) \in W_2^2(D) \cap L_{\rho_2+2}(D)$.

Proof. Since $k_{1t}^{(2)}(t)$ has sign fixed defined in domain D, then (following [5] and [9, chapter 1, p. 22] or may repeat again steps proof of Therems3.1 and 4.1) we conclude that conditions of the theorem due to provided the coerciveness of the following operator

$$L_2\upsilon = k_1^{(2)}(t)\upsilon_{tt} - \Delta\upsilon + \sum_{i=1}^n a_{i2}^{(2)}\upsilon_{x_i} + b_{22}\upsilon_t + |\upsilon|^{\rho_2}\upsilon + c_{22}\upsilon$$

$$\Delta\upsilon = -f_2 + k_1^{(2)}(t)\upsilon_{tt} + \sum_{i=1}^n a_{i2}^{(2)}\upsilon_{x_i} + b_{22}\upsilon_t + |\upsilon|^{\rho_2}\upsilon + c_{22}\upsilon \equiv F.$$

Other side, accordance to Sobolev's embedding Theorem (e.g., [1], [15]) and by results of Theorems 3.1 and 4.1 we have $|\upsilon|^{\rho_2}\upsilon \in L_2(D)$, $F \in L_2(D)$. Then for $\rho_2 < \frac{2}{n-2}$ there exists unique generalized solution of (3.2),(2.3) in $W_2^1(D)$. If $\upsilon(x,t) \in W_2^1(D)$, then $|\upsilon|^{\rho_2}\upsilon \in L_2(D)$, and consequently, we get $f_2 + |\upsilon|^{\rho_2}\upsilon \in L_2(D)$. Therefore, any solution (3.2), (2.3) from the space $W_2^1(D)$, will be an element of space $W_2^2(D)$ (e.g. [9, chapter 4, p. 216], [10.chapter 1, p. 27-33]). Hence, we can conclude that under assumptions of Theorems 3.1 and 4.1 there exists a unique generalized solution of (3.2), (2.3) $\upsilon(x,t) \in W_2^2(D) \cap L_{\rho_2+2}(D)$.

Definition5.1.The functions $u(x,t) \in H_{2,L}(D^+) \cap L_{\rho_1+2}(D)$ $(u(x,t) \in H_{2,L}(D^-) \cap L_{\rho_1+2}(D))$, $\upsilon(x,t) \in W_2^2(D) \cap L_{\rho_2+2}(D)$ is said to be a regular solution of problem (3.1), (2.2) ((3.2), (2.3)) if it is generalized solution which satisfy almost everywhere equations (3.1) ((3.2)) in domain D^+ (D^-).

Lemma 5.1. Assume that the conditions of Theorems 3.1 and 4.1 are fulfilled and sign of function $k_{1t}^{(2)}(t)$ arbitrarily. Then regular solutions of problems (3.1), (2.3) and (3.2), (2.3) are unique. The Lemma 5.1 can be proved similarly way to the Theorems 3.1.Let consider in the domain D^+, "$\varepsilon-$ regularized" equation of mixed type:

$$\left.\begin{aligned} L_{1\varepsilon}(u_\varepsilon) &= k_1^{(1)}(t)u_{tt\varepsilon} + (k_2(x) - \varepsilon)\Delta u_\varepsilon + \sum_{i=1}^n a_{i1}^{(1)}u_{\varepsilon x_i} + b_{11}u_{\varepsilon t} + c_{11}u_{\alpha} + c_1(x)|u_\varepsilon|^{\rho_1}u_\varepsilon = f_1(x,t,u_\varepsilon,\upsilon_\varepsilon) \\ L_{2\varepsilon}(\upsilon_\varepsilon) &= (k_1^{(2)}(t) - \varepsilon)\upsilon_{\varepsilon t} - \Delta\upsilon_\varepsilon + \sum_{i=1}^n a_{i2}^{(2)}\upsilon\varepsilon_{x_i} + b_{22}\upsilon_{\varepsilon t} + c_{22}\upsilon_\varepsilon - |\upsilon_\varepsilon|^{\rho_2}\upsilon_\varepsilon = f_2(x,t,u_\varepsilon,\upsilon_\varepsilon) \end{aligned}\right\} \tag{5.1}$$

and we state for its the boundary value problem:

$$u_\varepsilon\big|_{x_n=0} = 0,\ u_\varepsilon\big|_{S^+} = 0,\ u_\varepsilon\big|_{\overline{\Gamma_T^+}} = 0,\ u_\varepsilon\big|_{\overline{\Gamma_{-T}^+}} = 0\ u_{\varepsilon t}\big|_{\overline{\Gamma_T^+}} = 0 \tag{5.2}$$

$$\upsilon_{\varepsilon t}\big|_{\overline{\Gamma_{-T}^+}} = 0,\ \upsilon_\varepsilon\big|_{S^+} = 0,\ \upsilon_\varepsilon\big|_{\overline{\Gamma_T^+}} = 0,\ \upsilon_\varepsilon\big|_{\overline{\Gamma_{-T}^+}} = 0\ \upsilon_{\varepsilon t}\big|_{\overline{\Gamma_T^+}} = 0.$$

Analogically, we will consider the following boundary value problem:

$$\left.\begin{aligned} L_{1\varepsilon}(u_\varepsilon) &= k_1^{(1)}(t)u_{\varepsilon tt} + (k_2 + \varepsilon)\Delta u_\varepsilon + b_{11}u_{\varepsilon t} + \sum_{i=1}^n a_{i1}^{(1)}u_{\varepsilon x_i} + c_{11}u_\varepsilon + c_1(x)|u_\varepsilon|^{\rho_1}u_\varepsilon = f_1(x,t,u_\varepsilon,\upsilon_\varepsilon) \\ L_{2\varepsilon}(\upsilon_\varepsilon) &= (k_1^{(2)}(t) + \varepsilon)\upsilon_{\varepsilon tt} - \Delta\upsilon_\varepsilon + \sum_{i=1}^n a_{i2}^{(2)}\upsilon\varepsilon_{x_i} + b_{22}\upsilon_{\varepsilon t} + c_{22}\upsilon_\varepsilon - |\upsilon_\varepsilon|^{\rho_2}\upsilon_\varepsilon = f_2(x,t,u_\varepsilon,\upsilon_\varepsilon) \end{aligned}\right\} \tag{5.3}$$

$$u_\varepsilon\big|_{x_n=0} = 0,\ u_\varepsilon\big|_{S^-} = 0,\ u_\varepsilon\big|_{\overline{\Gamma_{-T}^-}} = 0,\ u_\varepsilon\big|_{\overline{\Gamma_T^-}} = 0\ u_{t\varepsilon}\big|_{\overline{\Gamma_T^-}} = 0 \tag{5.4}$$

$$\upsilon_\varepsilon\big|_{\overline{\Gamma_{-T}^-}} = 0,\ \upsilon_\varepsilon\big|_{S^-} = 0,\ \upsilon_{\varepsilon t}\big|_{\overline{\Gamma_{-T}^-}} = 0,\ \upsilon_\varepsilon\big|_{\overline{\Gamma_T^-}} = 0\ \upsilon_{t\varepsilon}\big|_{\overline{\Gamma_T^-}} = 0.$$

Proceeding from the results of the papers [5], [20], we can affirm in our case the following proposition.

Remark 5.1. If the conditions of Theorems3.1, 4.2 and $2b_{11}(x,t) - |k_{1t}^{(1)}| \le -\delta < 0$ $2b_{22}(x,t) - |k_{1t}^{(1)}| \le -\delta < 0$ $(x,t) \in D$ are satisfied, then for any right-hand sides of (3.1),(3.2), $f_1(x,t,u,\upsilon), f_{1t}(x,t,u,\upsilon) \in L_2(D)$, $f_2(x,t,u,\upsilon), f_2(x,t,u,\upsilon) \in L_2(D)$ there exists a unique solution of boundary value problems(5.1),(5.2) and (5.3),(5.4) from the space $u_\varepsilon(x,t) \in W_2^2(D^+)$ $(u_\varepsilon(x,t) \in W_2^2(D^-))$ and $\upsilon(x,t) \in W_2^2(D)$ this solutions admissible estimates

$$\|f_1\|_{L_2(D^+)}^2 + \|f_{1t}\|_{L_2(D^+)}^2 \ge m_3^*\|u_\varepsilon\|_{W_2^2(D^+)\cap L_{\rho_1+2}(D)}^2,$$

$$\|f_1\|_{L_2(D^-)}^2 + \|f_{1t}\|_{L_2(D^{-+})}^2 \ge m_4^*\|u_\varepsilon\|_{W_2^2(D^-)\cap L_{\rho_1+2}(D)}^2$$

$$\|f_2\|_{L_2(D)}^2 + \|f_{2t}\|_{L_2(D)}^2 \ge m_5^*\|u_\varepsilon\|_{W_2^2(D)\cap L_{\rho_2+2}(D)}^2 \tag{5.5}$$

where, the constants m_3^*, m_4^*, m_5^* are independent from the function $\varepsilon, u_\varepsilon(x,t), \upsilon_\varepsilon(x,t)$.

Proof. This Remark can be proved by similarly to Theorems 3.1, 4.1 and 5.1.

Theorem 5.2. (On the solvability of problems (3.1), (2.2) and (3.2),(2.3) in D^+). Suppose that the conditions of

Theorem 3.1, 4.1 and $\left|k_{2x_i}k_{2x_j}\right| \le M_1|k_2|(x),\ f_1(x,t,u,\upsilon),$ $f_{1t}(x,t,u,\upsilon), f_2(x,t,u,\upsilon), f_{2t}(x,t,u,\upsilon) \in L_2(D),$ $2b_{22}(x,t) - \left|k_{1t}^{(2)}\right| \le -\delta < 0$, $2b_{11}(x,t) - \left|k_{1t}^{(1)}(t)\right| \le -\delta < 0,$ for $(x,t) \in D^+$, $i,j = 1,2,...,n$ are satisfied, then there exists a unique regular solution of problems (3.1), (2.2) and (3.2),(2.3)from the space $u(x,t) \in H_{2,L}(D^+) \cap L_{\rho_1+2}(D),$ $\upsilon(x,t) \in W_2^2(D^+) \cap L_{\rho_2+2}(D).$

Theorem.5.3 (On the solvability of problems (3.1), (2.2)and (3.2),(2.3))in D^-) Assume that the conditions of Theorem3.1,4.1 and $\left|k_{2x_i}k_{2x_j}\right| \le M_1|k_2|(x),\ f_1(x,t,u,\upsilon),$ $f_{1t}(x,t,u,\upsilon), f_2(x,t,u,\upsilon), f_{2t}(x,t,u,\upsilon) \in L_2(D),$ $2b_{22}(x,t) - \left|k_{1t}^{(2)}\right| \le -\delta < 0$, $2b_{11}(x,t) - \left|k_{1t}^{(1)}(t)\right| \le -\delta < 0,$ for $(x,t) \in D^-$, $i,j = 1,2,...,n$ are satisfied, then there exists a unique regular solution of problems (3.1), (2.2) and (3.2),(2.3)from the spaces $u(x,t) \in H_{2,L}(D^-) \cap L_{\rho_1+2}(D),$ $\upsilon(x,t) \in W_2^2(D^-) \cap L_{\rho_2+2}(D).$

Proof. The Theorems 5.2 and 5.3 are proved exactly and similarly way to the Theorems 4.1and 4.2.

In this case we need to obtain second a prior estimate for nonlinear terms. For this purpose, applying Holder's inequality we have

$$(\rho_1+1)\int_{D_t} |c_1(x)||u_\varepsilon|^{\rho_1} u_{\varepsilon t}u_{\varepsilon tt}dD \le (\rho_1+1)\left\||u_\varepsilon|^{\rho_1}\right\|_{L_n(D)} \|u_{\varepsilon t}\|_{L_r(D)} \|u_{\varepsilon tt}\|_{L_2(D)} \tag{5.6}$$

Where, $\frac{1}{n}+\frac{1}{r}+\frac{1}{2}=1$. Since $n\rho \le r$, then at the same time analogous we have

$$\left\||\upsilon_\varepsilon|^{\rho_2}\right\|_{L_n(D)} \le \|\upsilon_\varepsilon\|_{L_r(D)}^{\rho_2}, \left\||u_\varepsilon|^{\rho_1}\right\|_{L_n(D)} \le \|u_\varepsilon\|_{n(D}^{\rho_1}.$$

According to Sobolev's embedding theorem (e.g. [1], [15])we get: $W_2^1(G) \subset L_r(G), r = \frac{2n}{n-2}$, $n \ge 3,$ $\left\||u_\varepsilon|^{\rho_1}\right\|_{L_n(G)} \le C_0^*\|u_\varepsilon\|_{W_2^1(G)}^{\rho_1} \le C^*, \left\||\upsilon_\varepsilon|^{\rho_{21}}\right\|_{L_n(G)} \le C_2^*\|\upsilon_\varepsilon\|_{W_2^1(G)}^{\rho_{21}} \le C_2^*.$

Hence, taking account into inequality of (5.6) obtains that, for any $t \in (-T,T)$ the inequality is valid:

$$(\rho_1+1)\int_{D_t} |c_1(x)||u_\varepsilon|^{\rho_1} u_{\varepsilon t}u_{\varepsilon tt}dD \le C^*\|\nabla u_{\varepsilon t}\|_{L_r(G)} \|u_{\varepsilon tt}\|_{L_2(G)}$$

$$(\rho_2+1)\int_{D} \upsilon_\varepsilon\upsilon_{\varepsilon tt}dD \le C^*\|\nabla \upsilon_{\varepsilon tt}\|_{L_r(G)} \|\upsilon_{\varepsilon tt}\|_{L_2(G)}.$$

Hence, using Gromwell's lemma, we get $\|u_{\varepsilon tt}\|_{H_1(D)} \le M_5$, $\|u_{\varepsilon tt}\|_{L_2(D)} \le M_5$ $\left\|\sqrt{\left|K_1^{(1)}\right|}u_{\varepsilon tt}\right\|_{L_2(D)} \le M_5$ $\|\upsilon_{\varepsilon tt}\|_{W_2^1(D)} \le M_6$, $\left\|\sqrt{\left|K_1^{(2)}\right|}\upsilon_{\varepsilon tt}\right\|_{L_2(D)} \le M_6$, $\|\upsilon_{\varepsilon tt}\|_{L_2(D)} \le M_6$. (where M_6, M_3 are constants)

Definition 5.3 (following by [17],[20]) The functions $u(x,t) \in H_{1,L}(D^+) \cap L_{\rho_1+2}(D)$ $\left(u(x,t) \in H_{1,L}(D^-) \cap L_{\rho_1+2}(D)\right)$ and $\upsilon(x,t) \in W_2^1(D) \cap L_{\rho_2+2}(D).$ functions is said to be a strong solution of boundary value problem (3.1), (2.2) and (3.2), (2.3), if there exists a sequences of functions $\{u_{\varepsilon n}\} \in C_L'(D^+)$ $\left(\{u_{\varepsilon n}\} \in C_L'(D^-)\right)$, $\{\upsilon_{\varepsilon n}\} \in C_L'(D)$ (where the spaces $C_L'(D^+),\ C_L'(D^-)\ C_L'(D)$ are infinitely differentiable functions which satisfies boundary conditions of (2.8),(2.9) ,(2.4) , respectively) and such that equalities

$$\lim_{n\to\infty}\left\|L_1(u_n) - f_1(x,t,u,\upsilon)\right\|_{L_2(D^+)} = \lim_{n\to\infty}\|u_n - u\|_{H_{1,L}(D^+)} = 0$$

in the domain D^- as well if instead of the domain taken D^+ and

$$\lim_{n\to\infty}\left\|L_2(\upsilon_n) - f_2(x,t,u,\upsilon)\right\|_{L_2(D)} = \lim_{n\to\infty}\|\upsilon_n - \upsilon\|_{H_{1,L}(D)} = 0$$

in the domain Dare fulfilled.

The following theorem on the existence of strong solution holds.

Theorem5.4 Strong (regular) solution of problems (3.1), (2.2) and (3.2), (2.3))

Suppose that the conditions of Theorem 3.1, 4.1 and $\left|k_{2x_i}k_{2x_j}\right| \le M_1\left|k_2(x)\right|,\quad i,j = 1,...,n \qquad 2b_{22}(x,t) - \left|k_{1t}^{(1)}\right| \le -\delta < 0,$ $2b_{11} - \left|k_1^{(1)}(t)\right| \le -\delta < 0,\quad (x,t) \in D$ are satisfied.

Then for any functions $f_1(x,t,u,\upsilon), f_{1t}(x,t,u,\upsilon) \in L_2(D^+)$ $(f_1(x,t,u,\upsilon), f_{1t}(x,t,u,\upsilon) \in L_2(D^-))$, $f_2(x,t,u,\upsilon), f_{2t}(x,t,u,\upsilon) \in L_2(D)$ there exists a unique strong solution of boundary value problem (5.1), (5.2) from the space $u(x,t) \in H_{2,L}(D^+) \cap L_{\rho_1+2}(D),$ (for problem (5,3), (5.4) from $u(x,t) \in H_{2,L}(D^-) \cap L_{\rho_1+2}(D),$) and $\upsilon(x,t) \in W_2^2(D^+) \cap L_{\rho_2+2}(D).$

Proof. From these Theorem 3.1, Theorem 4.1, Theorem4.2 and Theorem 5.1 there exists ($u^+(x,t)\ \upsilon(x,t)$) solution of problem ((5.1), (5.2)), ($u^-(x,t)$, $\upsilon(x,t)$) solution of problem ((5.3), (5.4)) in the domains D^+ and D^- , respectively, and belonging respectively to the spaces $H_{2,L}(D^+) \cap W_2^2(D)$ and $H_{2,L}(D^-) \cap W_2^2$. Then by the construction of such spaces there exists sequences $\{u_n\} \in C_L'(D^+)$ $\left(\{u_n\} \in C_L'(D^-)\right)$ $\{\upsilon_{\varepsilon n}\} \in C_L'(D)$ such that

$$\lim_{n\to\infty}\|u_n^+ - u^+\|_{H_{2,L}(D^+)} = \lim_{n\to\infty}\|u_n^- - u^-\|_{H_{2,L}(D^-)} = 0\ \lim_{n\to\infty}\|\upsilon_n - \upsilon\|_{H_{2,L}(D^+)} = 0 .$$

From the obvious inequality

$$\|u_n^+\|_{H_{2,L}(D^+)} \geq m\left\|L_1\left(u_n^+\right)\right\|_{L_2\left(D^+\right)},\ \|u_n^-\|_{H_{2,L}(D^-)} \geq m\left\|L_1\left(u_n^-\right)\right\|_{L_2\left(D^-\right)}$$
$$\|\upsilon_n\|_{H_{2,L}(D)} \geq m\left\|L_2(\upsilon)\right\|$$

it follows that $\{L_1(u_n^+)\} \to f_1^+$ in $L_2(D^+)$, for $n \to \infty$. $\{L_1(u_n^-)\} \to f_1^-$ in $L_2(D^-)$, $\{L_2(\upsilon_n)\} \to f_2$ in $L_2(D)$, for $n \to \infty$. Thus, suppose that $f_{1t}^+ \in L_2(D^+)$, $f_{1t}^- \in L_2(D^-)$, then regular solutions υ, u^+ and u^- are strong solution. We are constructing the sequences of functions $f_{1n}^+ \in W_2^1(D^+)$, $f_{1n}^- \in W_2^1(D^-)$ such that $\{f_{1n}^+\} \to f_1^+$ in $L_2(D^+)$, $\{f_{1n}^-\} \to f_1^-$ $\{f_{2n}\} \to f_2$ in $L_2(D^-)$, $\{f_{2n}\} \to f_2$ in $L_2(D)$ for $n \to \infty$. Then for the functions f_1^+ and f_1^-, f_2 there exists strong solution problem of ((5.1), (5.2)) and ((5.3), (5.4))from the spaces $H_{2,L}(D^+) \cap W_2^2(D), \cap L_{\rho_1,1}(D)$ $H_{2,L}(D^-) \cap W_2^2(D) \cap L_{\rho_2}(D)$ respectively. Hence, we can include that $u_n^+ \to u^+$ in $H_{1,L}(D^+)$, $u_n^- \to u^-$ in $H_{1,L}(D^-)$, $\upsilon_n \to \upsilon$ for $n \to \infty$ and these functions are strong of problem ((5.1), (5.2)) and((5.3), (5.4)) respectively.

6. The Solvability of Problem ((2.1)-(2.3))

Theom6.1. (Gluing solutions in the spaces) Suppose that the functions u^+, u^- from the spaces $u^+ \in H_{i,L}(D^+)$, $u^- \in H_{i,L}(D^-)$, $i = 1,2.$ Then the constructed function

$$u(x,t) = \begin{cases} u^+(x,t),\ (x,t) \in D^+, \\ u^-(x,t),\ (x,t) \in D^- \end{cases} \tag{6.1}$$

will also be from the class $u(x,t) \in H_{i,L}(D)$, $i = 1,2$.

Proof. The Theorem6.1 proved exactly and similarly way to the Remark6.1 (e .g. [20]).

Thus, we have the proof of the following theorem accordance essentially a combination of the proof of Theorems 3.1, 4.1, 5.1, 5.2, 5.3, 5.4, Lemma5.1 and Theorem 6.1.

Theorem6.2. (On the solvability of problem (3.1), (2.3) and (3.2),(2.4) in D) Let the conditions of Theorems3.1, 4.1, 5.1, 5.2, 5.3, 5.4, and Theorem 6.1 are satisfied.

Then for any functions $f_1(x,t,u,\upsilon), f_{1t}(x,t,u,\upsilon) \in L_2(D)$ and $f_2(x,t,u,\upsilon), f_{2t}(x,t,u,\upsilon) \in L_2(D)$ there exists a unique generalized solution of problem (3.1), (2.2) and (3.2),(2.3) from the space $H_{2,L}(D) \cap L_{\rho_1+2}(D)$ and $\upsilon(x,t) \in W_2^2(D) \cap L_{\rho_2+2}(D)$.

Proof. Since on the base of Theorem 4.1, Theorem 4.2 and Theorem 5.1 there exists a unique solution $u^+(x,t)$, $u^-(x,t)$ of problems ((5.1), (5.2)) and ((5.3), (5.4))from the space $H_{2,L}(D^+) \cap L_{\rho_1+2}(D)$ and $H_{2,L}(D^-) \cap L_{\rho_1+2}(D)$ respectively. Then function $u(x,t)$ which is constructed by formula (6.1) will also be from the class $u(x,t) \in H_{2,L}(D)$ and at the same time is generalized solution of equation (5), moreover, the functions $u^+(x,t)$ and $u^-(x,t)$ are strong generalized solution of problems((3.1), (2.2)) and ((3.2), (2.3)).Consequently, it means that the strong and weak solutions of corresponding problems are identity (see, e.g., [17],). It follows that the problem ((3.1), (2.2)) and ((3.2), (2.3)) are solvability. The uniqueness of problem ((3.1), (2.2)) and ((3.2), (2.3)) follows by means of inequality of Theorem 3.1. That is proof of Theorem 6.2. Analogically, the existence strong solution of problem ((3.1), (2.2)) and ((3.2), (2.3)) from the space $H_{1,L}(D)$ can be proved. Now we must prove solvability of problem (2.1) ,(2.2) ,(2.3). Let

$$M_1^*\overline{u} = k\overline{u}_{tt} + \sum_{i=1}^n A_i\overline{u}_{x_i} + B\overline{u}_t + D^*\overline{u},$$
$$N_1\overline{u} = \sum_{i=1}^n P_i\overline{u}_{x_i} + Q\overline{u}_t + R\overline{u}.$$

Where

$$K = \begin{pmatrix} k_1^{(1)} & 0 \\ 0 & k_1^{(2)} \end{pmatrix},\ A_i = \begin{pmatrix} a_{i1}^{(1)} & 0 \\ 0 & a_{i2}^{(2)} \end{pmatrix},\ B = \begin{pmatrix} b_{11} & 0 \\ 0 & b_{22} \end{pmatrix},$$
$$C = \begin{pmatrix} k_2\Delta + c_{11} & 0 \\ 0 & \Delta + c_{22} \end{pmatrix}\ P_i = \begin{pmatrix} 0 & a_{i2}^{(1)} \\ a_{i1}^{(2)} & 0 \end{pmatrix},\ Q = \begin{pmatrix} 0 & b_{12} \\ b_{21} & 0 \end{pmatrix},$$
$$R = \begin{pmatrix} 0 & c_{12} \\ c_{21} & 0 \end{pmatrix},\ \overline{u} = \begin{pmatrix} u \\ \upsilon \end{pmatrix},\ f^* = \begin{pmatrix} f_1(x,t,u,\upsilon) \\ f_2(x,t,u,\upsilon) \end{pmatrix}$$
$$D^* = \begin{pmatrix} k_2(x)\Delta + c_{11} + c_1|u|^{\rho_1} & 0 \\ 0 & \Delta + c_{22} - |\upsilon|^{\rho_2} \end{pmatrix},$$
$$D_\varepsilon^+ = D \cap \{x_n > 0\},\ D_\varepsilon^- = D \cap \{x_n < -\varepsilon\}.$$
$$\tilde{D} = D_{2\varepsilon}^+ \cup D_{2\varepsilon}^-,\ \tilde{D}_\varepsilon = D_\varepsilon^+ \cup D_\varepsilon^-.$$

Then the system equations (2.1) can be written in the form

$$L^*\overline{u} = M_1^*\overline{u} + N_1\overline{u} = f^*. \tag{6.2}$$

Theorem6.2. Assume that the conditions $f_2(x,-T,u,\upsilon) = 0$, $f_1(x,t,u,\upsilon), f_{1t}(x,t,u,\upsilon), f_2(x,t,u,\upsilon), f_{2t}(x,t,u,\upsilon) \in L_2(D)$, $\left|a_{i2}^{(1)}(x,t)\right|^2 \leq M\left|k_2(x)\right|$ are fulfilled. Then there exists unique solution of problem (2.1), (2.2) ,(2.3) from the space $u(x,t) \in H_{2,L}(D) \cap L_{\rho_1+2,|c_1|}(D)$, $\upsilon(x,t) \in W_2^2(D) \cap L_{\rho_2+2}(D)$.

Proof. Multiplying (6.2) by the vector $\overline{\eta_1} = (\alpha u_t, -\upsilon_t)$ in domain D, after integration by parts and using the Cauchy inequality, allowing for boundary condition (by analogically action to Theorems 3.1, 4.1, and 5.1) we get the following estimates

$$\left\|L^*\bar{u}\right\|_{L_2(D)} \geq m^* \left\|\bar{u}\right\|_{H_1(D)\cap L_{\rho_1+2,|q|}(D)\cap L_{\rho_2+2}(D)} \quad \text{or}$$

$$\left\|L^*\bar{u}\right\|_{L_2(D)} \geq m^* \left\|\bar{u}\right\|_{H_{1,L}(D)\cap W_2^1(D)\cap L_{\rho_2+2}(D)} . \qquad (6.3)$$

Now, let $H_{t,0}$ - is the space of vector function $\bar{\phi} = (\phi_1, \phi_2)$ such that $\phi_1, \phi_{1t}, \phi_2 \in L_2(D)$ and $\phi_1(x,-T) = 0$ .The norm of space $H_{t,0}$ is defined by $\|\phi\|_{t,0}^2 = \|\phi_{1t}\|_0^2 + \|\phi_2\|_0^2$. From the results of the Theorems 3.1, 4.1, 5.1, 5.2, 5.3, 5.4, 6.1, 6.2, it follows the following a prior estimates

$$\left\|\bar{u}\right\|_{H_{2,L}(D)\cap L_{\rho^*}(D)} \leq m_6^* \left\|M_1^*\bar{u}\right\|_{t,0} \text{ or}$$

$$\left\|\bar{u}\right\|_{H_{2,L}(D)\cap W_2^2(D)\cap L_{\rho^*(d)}} \leq m_7^* \left\|M_1^*\bar{u}\right\|_{t,0} \qquad (6.4)$$

where $\rho^* = \max(\rho_1, \rho_2)$ and the constants m^* , m_6^* , m_7^* are not dependent from $\bar{u}(x,t)$ .We must to show that, analogical estimates(6.3), (6.4) are also have to for operator $L^*\bar{u}$. Indeed, we may rewrite $M_1^*\bar{u} = L^*\bar{u} - N\bar{u}$, then $\left\|\bar{u}\right\|_{H_{2,L}(D)\cap L_{\rho^*}(D)} \leq m_8^*\left(\left\|L^*\bar{u}\right\|_{t,0} + \left\|N\bar{u}\right\|_{t,0}\right)$ or $\left\|\bar{u}\right\|_{H_{2,L}(D)\cap W_2^2(D)\cap L_{\rho^*}(D)} \leq m_9^*\left(\left\|L^*\bar{u}\right\|_{t,0} + \left\|N\bar{u}\right\|_{t,0}\right)$ are valid. Now, we consider the set of equations: $L_1^*\bar{u} = M_1^*\bar{u} + \tau N\bar{u}$ where $0 \leq \tau \leq 1$. Obviously, the following a prior estimate is uniformly bounded respect to parameter of τ :

$$\left\|\bar{u}\right\|_{H_{2,L}(D)\cap W_2^2(D)\cap L_{\rho^*}(D)} \leq m_{10}^* \left\|L_\tau^*\bar{u}\right\|_{t,0}$$

where the constants m_8^*, m_9^*, m_{10}^* are independent from parameter τ and $\bar{u}(x,t)$. Other side for $\tau = 0$ we have $L_0^*\bar{u} = M_1^*\bar{u}$. In this case considered problem is solvable. Notice that, if $\tau = 1$ then $L_1^* = L^*$. Then as well as known method of continuation by parameter, with the standard approaches, the solvability of problem (2.1),(2.2) ,(2.3) can be proved. But the uniqueness of problem (2.1), (2.2),(2.3) can be proved by a similar way as Theorem3.1 The proof of this theorem is completed.

7. Conclusion

The existence and uniqueness of the boundary value problem (2.1), (2.2),(2.3) for semi-linear systems of the mixed hyperbolic-elliptic Keldysh type in the multivariate domain with the changing time direction were studied. The existence and uniqueness of generalized and regular solutions of a boundary value problem were established in a weighted Sobolev space. In this case applying the method of result of the work (e.g.,[20]) and with aid Theorem 6.1. (Gluing solutions in the spaces) shown that weak and strong (e.g., [17]) solutions of the boundary value problem for weakly nonlinear systems equations of the mixed hyperbolic-elliptic type in the multivariate domain with the changing time direction are identity. Finally, the solvability of the boundary value problem (2.1)-(2.3) was proved.

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