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A Method of the Best Approximation by Fractal Function

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Abstract: We present a method constructing a function which is the best approximation for given data and satisfiesthe given self-similar condition. For this, we construct a space F of local self-similar fractal functions and show its properties. Next we present a computational scheme constructing the best fractal approximation in this space and estimate an error of the constructed fractal approximation. Our best fractal approximation is a fixed point of some fractal interpolation function.

Keywords: Fractal Interpolation, Fractal Approximation, Iterated Function System, Fractal Function Space

1. Introduction

Fractal approximation has been applied to model the objects which have fractal characteristics in nature. Fractal functions whose graphs are fractal sets have been widely used in approximation theory, signal processing, interpolation theory,computer graphics and so on. Hence, constructions of fractal functions and fractal approximation have been studied in many papers.

Constructions of fractal functions by fractal interpolation have been introduced by many researchers. A construction of one variable fractal interpolation functions by the iterated function system (IFS) with a data set on R was studied in [1, 2, 14], where the constructed fractal functions were self-similar ones. The construction was generalized in [3, 4, 17], which constructed local self-similar fractal functions. Constructions of bivariate fractal interpolation functions (BFIFs) have been studied in [5–11, 13, 17]. A construction of BFIFs by fractal interpolation on R was presented in [5, 17] and self-affine fractal interpolation functions were constructed by IFS with a data set on a triangular domain in [12]. Constructions of self-similar BFIFs in [9, 11] and self-affine BFIFs in [10, 13] by IFS with a data set on a rectangular grid were introduced. In [6], local self-similarBFIFs were constructed by the recurrent iterated function system(RIFS) on a rectangular grid. A construction of local self-similar fractal interpolation functions in R^n was studied in [4].

To construct fractal interpolation we need a data set $\{(x_i, y_i), i=0,1,\ldots,n\}$ and a set of scale parameters $\{s_i, i=1,\ldots,n\}$. The fractal property of the graph of the interpolation function is determinated by those data. Let a division of the interval and scale parameters be given, that is, a fractal property of the function be given. If the number of experimental data is more than the number of the interval division, then we can not construct the fractal interpolation for the data using fractal interpolation theory.

So we assume that a division of the interval and scale parameters be given (that is, a fractal property of the function) and study the problem constructing the best fractal approximation for the data set $\{(\bar{x}_i, \bar{z}_i), i=0,\ 1,\ldots,m\}$, where $m>n$ (n is the number of the interval division).

In [15, 18], constructionsofthebest approximation of functions by the fractal functions were presented, respectively. Butthe continuity of the approximation was not guaranteed then. The best fractal approximation of a continuous function in L^2space was introduced in [16]. In [7], a space of differentiable fractal interpolation functions was constructed and it was proved that the constructed space is the reproducing kernel Hilbert space.

We construct a space of fractal interpolation functions with a given division of the interval and scale parameters and find a function satisfying some approximation condition for data $\{(\bar{x}_i, z_i), i=0,\ 1,\ldots,m\}$, with $m>n$ in this space. We call it a local self-similar fractal approximation. The values of the function at nodes of division $\{y_i, i=0,\ 1,\ldots,n\}$ are unknown unlike interpolation function.

The rest of the article is organized as follows: Section 2 constructs a linear space $\mathcal{F}$ of local self-similar fractal functions and then a linear space $\mathcal{T}$ of contraction operators which is isomorphic to the constructed $\mathcal{F}$. Section 3 proves that there exists a unique contraction operator (RB operator) in $\mathcal{T}$ corresponding to LSFAin $\mathcal{F}$. We change the problem constructing this RB operator to the problem constructing $\{y_i, i=0, 1, \ldots, n\}$ and the fixed point of this operator is LSFA in $\mathcal{F}$ of a given data set. Section 4 estimates an error of the constructed fractal approximation and Section 5 gives examples of calculation of the least squares fractalapproximation of a coastline.

2. A Space of Local Self-Similar Fractal Functions and a Space of Contractive Operators

In this section, we construct a space of local self-similar fractal functions and a space of contraction operators which are isomorphic to each other.

Let

$$\Delta = \{x_i \in \mathrm{R} : i = 0,1,\cdots,n\},$$

$$a = x_0 < x_1 < \ldots < x_n = b,$$

$$I = [a, b], \quad I_i = [x_{i-1}, x_i],$$

$$\{s_i : |s_i| < 1, i = 1, 2, \cdots, n\}$$

be given.

Let $1 < q \le n$ ($q \in \mathrm{N}$),

$$J_k = [x_{s(k)}, x_{e(k)}], x_{s(k)}, x_{e(k)} \in \{x_0, x_1, \cdots, x_n\}$$

And $e(k) - s(k) \ge 2$, $k = 1,\cdots,q$. I_i is called a region and J_k a domain. We define a mapping $\gamma : \{1, \ldots, n\} \to \{1, \ldots, q\}$, which means that we relate every region to a domain. For each $i \in \{1, \ldots, n\}$, denote $k = \gamma(i)$.

For $i \in \{1, \ldots, n\}$, define a mapping $u_{i,k} : J_k \to I_i$ by

$$u_{i,k}(x) = a_i x + b_i \tag{1}$$

which satisfies

$$u_{i,k}(x_{s(k)}) = x_{i-1}, \; u_{i,k}(x_{e(k)}) = x_i. \tag{2}$$

Let $f (\in C(I))$ be a continuous function satisfying

$$f(x) = s_i \cdot f(u_{i,k}^{-1}(x)) + p_{i,k}(u_{i,k}^{-1}(x)), \quad x \in I_i \tag{3}$$

where functions $p_{i,k} : J_k \to I_i$, $i = 1, \ldots, n$ are defined by $p_{i,k}(x) = c_i x + d_i$ and satisfy the following conditions:

$$s_i f(x_{s(k)}) + p_{i,k}(x_{s(k)}) = f(x_{i-1}) \tag{4}$$

$$s_i f(x_{e(k)}) + p_{i,k}(x_{e(k)}) = f(x_i) \tag{5}$$

Define a space of functions satisfying the equations (4), (5) by $\mathcal{F}$. The graph of $f \in \mathcal{F}$ has a local self-similarity and we get $f(x) \equiv 0 \in \mathcal{F}$ which corresponds to $c_i = 0, d_i = 0$, $i \in \{1, \ldots, n\}$.

Lemma 1 $\mathcal{F}$ is a linear subspace of dimension $n+1$ of $C(I)$.

Proof. For $f, \tilde{f} \in \mathcal{F}$ and $\lambda \in R$, we have

$$f(x) = s_i \cdot f(u_{i,k}^{-1}(x)) + p_{i,k}(u_{i,k}^{-1}(x)),$$

$$\tilde{f}(x) = s_i \cdot \tilde{f}(u_{i,k}^{-1}(x)) + \tilde{p}_{i,k}(u_{i,k}^{-1}(x)), \quad x \in I_i. \tag{6}$$

Hence

$$(f + \tilde{f})(x) := s_i \cdot (f + \tilde{f})(u_{i,k}^{-1}(x)) + (p_{i,k} + \tilde{p}_{i,k})(u_{i,k}^{-1}(x)), \quad x \in I_i, \tag{7}$$

$$(\lambda f)(x) := s_i \cdot (\lambda f)(u_{i,k}^{-1}(x)) + (\lambda p_{i,k})(u_{i,k}^{-1}(x)), \quad x \in I_i. \tag{8}$$

Thus $f + \tilde{f} \in \mathcal{F}$ and $\lambda f \in \mathcal{F}$.

Because for $f \in \mathcal{F}$, $(f(x_0), f(x_1), \ldots, f(x_n)) \in R^{n+1}$ is uniquely determined, a mapping $\Psi : \mathcal{F} \to R^{n+1}$ is defined by

$$\Psi(f) = (f(x_0), f(x_1), \ldots, f(x_n)). \tag{9}$$

And for $(y_0, y_1, \ldots, y_n) \in R^{n+1}$, there exists a unique $f \in \mathcal{F}$ such that $f(x_i) = y_i, i = 1, \ldots, n$.

In fact, the existence and uniquenssof f areensured by the existence and uniquenss of the recurrent fractal interpolation function ([3]).

This shows that the mapping $\Psi : \mathcal{F} \to R^{n+1}$ is a bijection. We can easily check that the mapping Ψ is linear. Hence $\mathcal{F}$ and R^{n+1} are isomorphic. A basis of $\mathcal{F}$ is

$$\Psi^{-1}(e_i), e_i = (0, \ldots, 1, 0, \ldots, 0), \quad i = 1, \ldots, n+1. \tag{10}$$

The space $\mathcal{F}$ is a Banach space with the norm $\|\cdot\|_\infty$.

For a $y = (y_0, y_1, \ldots, y_n) \in R^{n+1}$, define a function space $\mathcal{F}_y$ by

$$\mathcal{F}_y = \{f \in C(I) : f(x_i) = y_i, \quad i = 1, \ldots, n\}. \tag{11}$$

Then $(\mathcal{F}_y, \|\cdot\|_\infty)$ is a complete space.

For $f \in \mathcal{F}_y$, define a function $T_y f : I \to R$ by

$$(T_y f)(x) := s_i \cdot f(u_{i,k}^{-1}(x)) + p_{i,k}^{T_y}(u_{i,k}^{-1}(x)), \quad x \in I_i, \tag{12}$$

where $p_{i,k}^{T_y}(x) = c_{i,k}^{T_y} x + d_{i,k}^{T_y}$ satisfies the following conditions:

$$(T_y f)(x_{i-1}) = y_{i-1}, (T_y f)(x_i) = y_i \tag{13}$$

i.e.

$$s_i y_{s(k)} + p_{i,k}^{T_y}(x_{s(k)}) = y_{i-1},\ s_i y_{e(k)} + p_{i,k}^{T_y}(x_{e(k)}) = y_i. \quad (14)$$

By the seconditions, $c_{i,k}^{T_y}$ and $d_{i,k}^{T_y}$ are uniquely givenby

$$c_{i,k}^{T_y} = \frac{(y_i - y_{i-1}) - s_i(y_{e(k)} - y_{s(k)})}{x_{e(k)} - x_{s(k)}}, \quad (15)$$

$$d_{i,k}^{T_y} = \frac{y_{i-1}x_{e(k)} - y_i x_{s(k)} + s_i(x_{s(k)}y_{e(k)} - y_{s(k)}x_{e(k)})}{x_{e(k)} - x_{s(k)}}. \quad (16)$$

Therefore, for a $y = (y_0, y_1, \ldots, y_n) \in R^{n+1}$, we get a unique operator $T_{\mathbf{y}}$. And because $T_{\mathbf{y}} f$ is continuous by (12), $T_y f \in \mathcal{F}_y$. Thustheoperator $T_y : \mathcal{F}_y \to \mathcal{F}_y$ is defined by (12). It is easy to verify that the operator $T_{\mathbf{y}}$ isacontraction with respect to $\|\cdot\|_\infty$. According to the fixed-point theorem in a complete space, there exists a unique $f_{T_y} \in \mathcal{F}_y$ such that

$$(f_{T_y}(x_0),\ f_{T_y}(x_1),\ \cdots,\ f_{T_y}(x_n)) = (y_0,\ y_1,\ \cdots,\ y_n) \quad (17)$$

Let $\mathcal{T}$ betheset of such operators. Define a mapping $\Phi : \mathbf{R}^{n+1} \to \mathcal{T}$ by $\Phi(y) = T_y \in \mathcal{T}$. Then the mapping Φ is a bijection.

Lemma 2. $\mathcal{T}$ is a linear space of dimensionn+1.

Proof. For $T_{y_1}, T_{y_2} \in \mathrm{T}$ and $\lambda \in R$, define $T_{\mathbf{y}_1} + T_{\mathbf{y}_2}$ on $\mathcal{F}_{y_1+y_2}$ by $f \in \mathcal{F}_{y_1+y_2}$,

$$(T_{\mathbf{y}_1} + T_{\mathbf{y}_2})(f)(x) := s_i \cdot f(u_i^{-1}(x)) + (p_i^{T_{y_1}} + p_i^{T_{y_2}})(u_i^{-1}(x)), \quad x \in I_i \quad (18)$$

and $\lambda T_{\mathbf{y}_1}$ on $\mathcal{F}_{\lambda y_1}$ by

$$\tilde{f} \in \mathcal{F}_{\lambda y_1},\ (\lambda T_{\mathbf{y}_1})(\tilde{f})(x) := s_i \cdot \tilde{f}(u_i^{-1}(x)) + \lambda p_i^{T_{y_1}}(u_i^{-1}(x)),$$
$$x \in I_i, \quad i = 1, \ \ldots, \ n \quad (19)$$

Note that we omit a subscript k after this because the domain and region are all fixed.

It is clear that $(T_{y_1} + T_{y_2})f \in \mathcal{F}_{y_1+y_2}$ and $(\lambda T_{y_1})\tilde{f} \in \mathcal{F}_{\lambda y_1}$. Therefore, $T_{y_1} + T_{y_2} \in \mathcal{T}$ and $\lambda T_{y_1} \in \mathcal{T}$, i.e. the linear operations are defined in the set $\mathcal{T}$. It is easy to prove that the set $\mathcal{T}$ is a linear space with respect to the linear operations.

The mapping $\Phi : R^{n+1} \to \mathcal{T}$ is linear. In fact, because for

$$\mathbf{y}_1 = (y_{1,0},\ y_{1,1},\ \cdots,\ y_{1,n}),$$
$$\mathbf{y}_2 = (y_{2,0},\ y_{2,1},\ \cdots,\ y_{2,n}) \in \mathrm{R}^{n+1}$$

and $\lambda \in \mathrm{R}$, by (15), (16)

$$p_i^{T_{y_1+y_2}}(x) = c_i^{T_{y_1+y_2}}x + d_i^{T_{y_1+y_2}} = (c_i^{T_{y_1}} + c_i^{T_{y_2}})x + (d_i^{T_{y_1}} + d_i^{T_{y_2}})$$
$$= c_i^{T_{y_1}}x + d_i^{T_{y_1}} + c_i^{T_{y_2}}x + d_i^{T_{y_2}} = p_i^{T_{y_1}}(x) + p_i^{T_{y_2}}(x), \quad (20)$$

$$p_i^{T_{\lambda y_1}}(x) = c_i^{T_{\lambda y_1}}x + d_i^{T_{\lambda y_1}} = \lambda(c_i^{T_{y_1}}x + d_i^{T_{y_1}}) = \lambda p_i^{T_{y_1}}(x) \quad (21)$$

and we get

$$(\Phi(\mathbf{y}_1 + \mathbf{y}_2))(f)(x) = (\Phi(\mathbf{y}_1) + \Phi(\mathbf{y}_2))(f)(x), \quad (22)$$

$$(\Phi(\lambda\mathbf{y}_1))(f)(x) = \lambda(\Phi(\mathbf{y}_1))(f)(x). \quad (23)$$

Hence, $\mathcal{T}$ and R^{n+1} are isomorphic, which means that the dimension of $\mathcal{T}$ is n+1.

By the isomorphic relation, $(0, 0, \cdots, 0) \in R^{n+1}$ correspondedstothe operator T defined by

$$(Tf)(x) = s_i \cdot f(u_i^{-1}(x)),\ x \in I_i, \quad (24)$$

whose fixed point is $f_T(x) \equiv 0$.

Theorem 1. Let $\mathcal{F}$ and $\mathcal{T}$ be the linear spaces constructed above. Then they are isomorphic.

Proof. This follows from Lemmas 1 and 2.

Denote the isomorphism of $\mathcal{F}$ to $\mathcal{T}$ by $\widetilde{\Psi}$. Note that for $f \in \mathcal{F}$, the fixed point of T with $\widetilde{\Psi}(f) = T$ is f.

3. Construction of LSFA of a Data Set

In this section, we prove that there exists the least squares fractal approximation f in $\mathcal{F}$ of a data setand present an algorithm for finding f by calculating approximately the contraction operator*T*in $\mathcal{T}$ corresponding to *f*.

Let P be a data set given by

$$P = \{(\bar{x}_i,\ z_i) : i = 0, 1, \ldots, m\}, \quad (25)$$
$$(\bar{x}_0 < \bar{x}_1 < \cdots < \bar{x}_m,\ \bar{x}_0 = x_0,\ \bar{x}_m = x_n)$$

where $m > n$.

An f^* is called the *least squares fractal approximation* (LSFA) if f^* is a solution of the following question:

$$\min_{f \in \mathcal{F}} \sum_{i=0}^{m} (f(\bar{x}_i) - \bar{z}_i)^2. \quad (26)$$

First, we consider theexistence and uniqeness of LSFA.

Theorem 2. If $\{x_0, x_1, \ldots, x_n\} \subset \{\bar{x}_0, \ldots, \bar{x}_m\}$, then there exist a unique solution $f^* \in \mathcal{F}$ of (26) and a unique $T_{f^*} \in \mathcal{T}$ whose fixed point is f^*.

Proof. Define an operator $B_m : \mathcal{F} \to R^{m+1}$ by

$$f \in \mathcal{F},\ B_m f = (f(\bar{x}_0), f(\bar{x}_1), \cdots, f(\bar{x}_m)) \quad (27)$$

And denote $B_m\mathcal{F}$ by $\mathfrak{D}$. Then B_m is a linear operator and $\mathfrak{D}$ is a linear subspace of R^{m+1}. The question (26) is represented by

$$\min_{f\in\mathcal{F}}\| B_m f-\hat{\mathbf{z}}\|_E^2, \tag{28}$$

where $\hat{\mathbf{z}}=\{\bar{z}_0, \bar{z}_1, \ldots, \bar{z}_m\}\in R^{m+1}$ and $\|\cdot\|_E$ is the Euclidean norm.

Therefore, (26) is equivalent to the following question:

$$\min_{z\in\mathfrak{D}}\|\mathbf{z}-\hat{\mathbf{z}}\|_E^2. \tag{29}$$

Because $(R^{m+1}, \|\cdot\|_E)$ is a Hilbert space and $\mathfrak{D}$ is a subspace of R^{m+1}, there exists a unique solution z^* of (29). If $B_m f\equiv 0$, then from the hypothesis of the theorem

$$(f(x_0),\ f(x_1),\ \cdots,\ f(x_n))=(0,0,\ldots,0) \tag{30}$$

And $f(x)\equiv 0$ by the construction of $\mathcal{F}$. Therefore, B_m is an injection and there exists a unique $f^*\in\mathcal{F}$ such that $B_m f^*=z^*$, i.e. there exists a unique $T^*=\widetilde{\Psi}^{-1}(f^*)\in\mathcal{T}$.

From Theorem 2, Equation (26) is equivalent to the following question:

$$\min_{T\in\mathcal{T}}\| B_m f_T-\hat{\mathbf{z}}\|_E^2. \tag{31}$$

Now, we consider a construction of the LSFA.

Let Ψ be the linear mapping defined by (9) and denote

$$v_i=\Psi^{-1}(\mathbf{e}_i),\ \mathbf{e}_i=(0,\ \ldots,\ 1,\ 0,\ \ldots,\ 0),\quad i=1,\ \ldots,\ n+1. \tag{32}$$

Then $\{v_i\}_{i=1}^{n+1}$ is a basis of $\mathcal{F}$ and there exist unique $\lambda_1,\ldots,\lambda_n\in\mathrm{R}$ such that $f^*=\sum_{k=0}^{n}\lambda_k v_k$. For $f, g\in\mathcal{F}$, define $<f,\ g>\in\mathrm{R}$ by $<f, g>=\sum_{k=0}^{m} f(\bar{x}_i)\cdot g(\bar{x}_i)$.

We get a normal equation

$$A\alpha=b, \tag{33}$$

$$A=(a_{ij}),\ a_{ij}=<v_i,\ v_j>,\ b=(b_i),\ b_i=\sum_{k=0}^{m} z_k v_i(\bar{x}_k) \tag{34}$$

to find $f^*\in\mathcal{F}$.

Since v_i, $i=1,\cdots,n$ are fractal functions in (34), it needs enormous operations. Therefore, we consider an algorithm for calculating the approximation of contractionoperator T_{f^*}. Wecalculateapproximately f^* as the fixed point of T_{f^*}.

Now, for $p_0=(\bar{x}_0,\ \bar{x}_1,\ \ldots,\ \bar{x}_m)$, let us denote $X_m=p_0\times R^{m+1}$. Define an operator T_m on X_m by

$$z=(z_0,\ z_1,\ \ldots,\ z_m)\in R^{m+1},\ T_m(p_0,\ z)=(p_0,\ \tilde{z}),$$
$$\tilde{z}=(\tilde{z}_0,\ \tilde{z}_1,\ \ldots,\ \tilde{z}_m), \tag{35}$$

where $\tilde{z}_i$, $i=1, 2,\cdots,m$ are defined as follows:for $\bar{x}_i$, there exist $l(\in\{1,\cdots,n\})$ and $k(\in\{0, 1,\ldots, m-1\})$ such that $\bar{x}_i\in I_l$ and $\bar{x}_k\le u_l^{-1}(\bar{x}_i)\le\bar{x}_{k+1}$. Then

$$\tilde{z}_i=s_l(z_k+z_{k+1})/2+c_l u_l^{-1}(\bar{x}_i)+d_l. \tag{36}$$

The operator T_m is given by s_i, c_i and d_i, $i=1, 2, \ldots, n$, where $|s_i|<1$, $i=1, 2, \ldots, n$ and c_i, d_i, $i=1, 2, \ldots, n$ are calculated by (15), (16) and represented by $y_0, y_1, \ldots, y_n$. Let us denote $(T_m(p_0, z))_2=\tilde{z}$.

We find a T_m^* such that

$$\|(T_m(p_0,\ \hat{z}))_2-\hat{z}\|_E=\|\tilde{\tilde{z}}-\hat{z}\|_E\to\min. \tag{37}$$

This problem is a minimization problem of a multi-variable function with unknown $y_0, y_1, \ldots, y_n$. We find $y_0, y_1, \ldots, y_n$ from this problem. Next we find the RB operator T_m^* using the method constructingthe fractal interpolation and its fixed point, that is the fractal interpolation with $\{(x_0,\ y_0\},\ \{x_1,\ y_1\},\ \ldots,\ \{x_n,\ y_n\}\}$ and scale parameters $s_1, s_2, \ldots, s_n$, is our best fractal approximation.

4. Estimation for Errors of the Approximation

In this section, we consider a relation between T_m and T and estimatean error betweenthe approximation solution f_{T^*} and given data.

For $(p_0,\ y), (p_0,\ g)\in X_m$ and $\lambda\in R$, define $(p_0,\ y)+(p_0,\ g)$, $\lambda(p_0,\ y)$ and $\|(p_0,\ y)\|$ as follows:

$$(p_0,\ y)+(p_0,\ g):=(p_0,\ y+g),\ \lambda(p_0, y):=(p_0,\ \lambda y) \tag{38}$$

$$\|(p_0,\ y)\|:=\max_{0\le i\le m}|y_i|. \tag{39}$$

Lemma 3. T_m is a contraction operator on X_m.

Proof. For

$$(p_0,\ y), (p_0,\ g)\in X_m,$$

we get

$$|\bar{y}_i - \bar{g}_i| = \frac{s_l}{2} | y_k + y_{k+1} - g_k - g_{k+1} |,$$

$$\| \mathrm{T_m}(\mathrm{p_0}, \mathrm{y}) - \mathrm{T_m}(\mathrm{p_0}, \mathrm{g}) \| = \| (\mathrm{p_0}, \bar{\mathrm{y}}) - (\mathrm{p_0}, \bar{\mathrm{g}}) \|$$
$$= \| (\mathrm{p_0}, \bar{\mathrm{y}} - \bar{\mathrm{g}}) \| = \max_i | \bar{\mathrm{y}}_i - \bar{\mathrm{g}}_i |,$$

$$\leq \frac{s_l}{2}(| y_k - g_k | + | y_{k+1} - g_{k+1} |)$$

$$\leq s_i \max_i | y_i - g_i | = s_i \| (p_0, y - g) \|,$$

where $c = \max\{|s_1|, \ldots, |s_n|\} < 1$. Therefore, T_m is a contraction operator with contraction constant c.

Because X_m is equivalent to R^{m+1}, we identify X_m with R^{m+1} and get a diagram that shows the relation between T and T_m (see Fig 1).

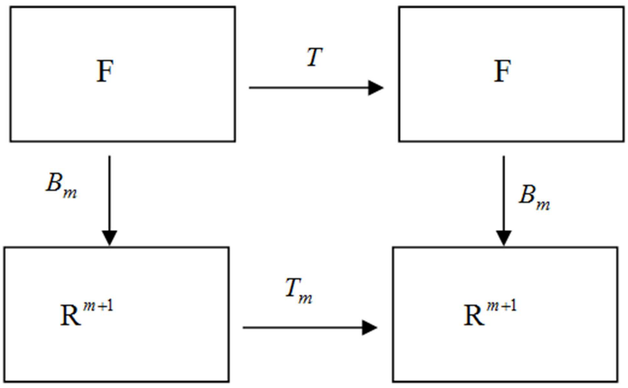

Fig. 1. *Relation between T and T_m.*

Lemma 4. Let $\bar{x}_i - \bar{x}_{i-1} = \dfrac{\bar{x}_m - \bar{x}_0}{m}$, for $i = 1, 2, \ldots, m$. Let T and T_m be defined by the same s_i, c_i d_i, $i = 1, 2, \ldots, n$. Then for $g \in \mathrm{F}$, we have

$$\| T_m B_m g - B_m T g \|_{X_m} \to 0 . \quad (40)$$

Proof. By the definitions of B_m and T_m, we get

$$B_m g = (g(\bar{x}_0), g(\bar{x}_1), \ldots, g(\bar{x}_m)), \quad T_m B_m g = (\bar{g}_0, \bar{g}_1, \ldots, \bar{g}_m) \quad (41)$$

$$\begin{aligned} &\bar{g}_i = s_l(g_k + g_{k+1})/2 + p_l(u_l^{-1}(\bar{x}_i)), \\ &\bar{x}_i \in I_l, \quad \bar{x}_k \leq u_l^{-1}(\bar{x}_i) \leq \bar{x}_{k+1}, \end{aligned} \quad (42)$$

and

$$Tg(x) = s_l g(u_l^{-1}(x)) + p_l(u_l^{-1}(x)), \; x \in I_l, \quad (43)$$

$$B_m T g = (Tg(\bar{x}_0), Tg(\bar{x}_1), \ldots, Tg(\bar{x}_m)). \quad (44)$$

Therefore, we have

$$\begin{aligned} |(T_m B_m g)_i - (B_m T g)_i| &= | s_l(g_k + g_{k+1})/2 + p_l(u_l^{-1}(\bar{x}_i)) \\ &\quad - s_l g(u_l^{-1}(\bar{x}_i)) - p_l(u_l^{-1}(\bar{x}_i)) | = \end{aligned}$$

$$= | s_l((g_k + g_{k+1})/2 - g(u_l^{-1}(\bar{x}_i))) | \to 0 \quad (m \to \infty),$$

which gives(40).

If T and T_m are defined by the same s_i, c_i and d_i, $i = 1, 2, \ldots, n$, then since contraction constants of T and T_m are given by s_i, $i = 1, 2, \ldots, n$, the elements of $\mathcal{T}$ and T_m have the same contraction constant c.

Theorem 3. Let T and T_m be defined by the same s_i, c_i and d_i, $i = 1, 2, \ldots, n$. If for $\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_m$, $u_l^{-1}(\bar{x}_i) \in \{\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_m\}$, $\bar{x}_i \in I_l$ and $l \in \{1, \cdots, n\}$, then we get $B_m f_T = T_m B_m f_T$ and

$$\| B_m f_T - \hat{z} \| \leq \frac{1}{1-c} \| T_m \hat{z} - \hat{z} \| . \quad (45)$$

Proof. Since f_T is the fixed point of T, we have

$$f_T(x) = Tf_T(x) = s_l \cdot f_T(u_l^{-1}(x)) + p_l(u_l^{-1}(x)), \quad x \in I_l, \quad (46)$$

and by the definitions of B_m and T_m, we get

$$(B_m f_T(x))_i = s_l \cdot f_T(u_l^{-1}(\bar{x}_i)) + p_l(u_l^{-1}(\bar{x}_i)) . \quad (47)$$

Since

$$B_m f_T = (f_T(\bar{x}_0), \ldots, f(\bar{x}_m)),$$

we have

$$T_m B_m f_T = T_m(f_T(\bar{x}_0), \ldots, f(\bar{x}_m))$$

and

$$(T_m B_m f_T)_i = s_l \cdot f_T(u_l^{-1}(\bar{x}_i)) + p_l(u_l^{-1}(\bar{x}_i)) .$$

Hence, we have $B_m f_T = T_m B_m f_T$ and

$$\begin{aligned} &\| \mathrm{B_m f_T} - \hat{\mathrm{z}} \| \leq \| \mathrm{B_m f_T} - \mathrm{T_m}\hat{\mathrm{z}} \| + \| \mathrm{T_m}\hat{\mathrm{z}} - \hat{\mathrm{z}} \| \\ &= \| \mathrm{T_m B_m f_T} - \mathrm{T_m}\hat{\mathrm{z}} \| + \| \mathrm{T_m}\hat{\mathrm{z}} - \hat{\mathrm{z}} \| \end{aligned}$$

$$\leq c \| B_m f_T - \hat{z} \| + \| T_m \hat{z} - \hat{z} \|$$

Thus $\| B_m f_T - \hat{z} \| \leq \dfrac{1}{1-c} \| T_m \hat{z} - \hat{z} \|$.

Lemma 5. [1] Let X be aBanach space and T a contraction operator on X with the contraction constant c. Let f_T be the fixed point of T. If for $f \in X$, $\| f - Tf \| < \varepsilon$, then

$$\| f_T - f \| < \varepsilon / (1 - c) . \quad (48)$$

Denote $\varepsilon_1 = \| T_m^* \hat{z} - \hat{z} \|$ and

$$\varepsilon_2 = \| T_m^* B_m f_{T^*} - B_m f_{T_m^*} \| = \| T_m^* B_m f_{T^*} - B_m T^* f_{T_m^*} \| . \quad (49)$$

Theorem 4. Let f_{T^*} be the fixed point of T^* defined by the solution $(y_0^*, \ldots, y_n^*)$ of (37). Then we have

$$\| B_m f_{T^*} - \hat{z} \| \le (\varepsilon_1 + \varepsilon_2)/(1-c), \quad (50)$$

where c is the contraction constant of the contraction operator T_m^*. Especially, $\varepsilon_2 = 0$ under the conditions of Theorem 3.

Proof. We caneasily see that

$$\| B_m f_{T^*} - \hat{z} \| \le \| B_m f_{T^*} - f_{T_m^*} \| + \| f_{T_m^*} - \hat{z} \| . \quad (51)$$

From Lemma 4, we have $\| T_m^* B_m f_{T^*} - B_m f_{T^*} \| \to 0.$

Let us denote $p := T_m^* B_m f_{T^*} - B_m f_{T^*}$, $p = (p_1, \ldots, p_m)$, For $\bar{x}_i \in I_l$, there exists a $k \in \{0, 1, \cdots, m\}$ such that $\bar{x}_k \le u_l^{-1}(\bar{x}_i) \le \bar{x}_{k+1}$. Then we get

$$\begin{aligned} p_i &= s_l (f_{T^*}(\bar{x}_k) + f_{T^*}(\bar{x}_{k+1}))/2 + p_l(u_l^{-1}(\bar{x}_i)) \\ &\quad - s_l f_{T^*}(u_l^{-1}(\bar{x}_i)) - p_l(u_l^{-1}(\bar{x}_i)) \\ &= s_l((f_{T^*}(\bar{x}_k) + f_{T^*}(\bar{x}_{k+1}))/2 - f_{T^*}(u_l^{-1}(\bar{x}_i))). \quad (52) \end{aligned}$$

From Lemma 5, we have

$$\| B_m f_{T^*} - f_{T_m^*} \| \le \varepsilon_2/(1-c), \| f_{T_m^*} - \hat{z} \| \le \varepsilon_1/(1-c), \quad (53)$$

where $c = \max \; \{ | s_i |, \; i = 1, \; \ldots, \; n \}$.

By (51) and(53),we get(50).

5. Examples of Calculation

Example 1. Let P be a data set given by $P = \{(\bar{x}_i, \; z_i) \in R^2 : i = 0, \; 1, \; \ldots, 10\}$ ={(0, 3.6), (0.1, 5.1), (0.2, 5.6), (0.3, 6.3), (0.4, 6.0), (0.5, 5.4), (0.6, 5.6), (0.7, 5.0), (0.8, 4.2), (0.9, 3.2), (1, 1.7)}.

Let $\{x_0, \; x_1, \; x_2, x_3, x_4\} = \{0, \; 0.2, \; 0.5, \; 0.7, \; 1\}$,

$$\Delta = \{(0, \; y_0), \; (0.2, \; y_1), \; (0.5, \; y_2), \; (0.7, \; y_3), \; (1, \; y_4)\},$$

$$S = \{s_1, \; s_2, \; s_3, \; s_4\} = \left\{\frac{1}{3}, \frac{2}{5}, \frac{1}{6}, \frac{1}{7}\right\}.$$

Then, $I_1 = [0, \; 0.2]$, $I_2 = [0.2, \; 0.5]$, $I_3 = [0.5, \; 0.7]$ and $I_4 = [0.7, \; 1]$.

Let $J_1 = [0, \; 1]$, $J_2 = [0, \; 1]$, $J_3 = [0, \; 1]$, $J_4 = [0, \; 1]$.

By (1) and (3), we have $u_1(x) = 0.2x$, $u_2(x) = 0.3x + 0.2$, $u_3(x) = 0.2x + 0.5$, $u_4(x) = 0.3x + 0.7$ and by (15) and(16), c_i, d_i, $i = 1, \cdots, 4$ are given by

c ={-0.666667 $y_0 + y_1$ -0.333333 y_4,
0.4 y_0 - y_1 + y_2 -0.4 y_4,

0.166667 y_0 - y_2 + y_3 -0.166667 y_4,
0.142857 y_0 - y_3 +0.857143 y_4 }

d={0.666667 y_0 ,-0.4 $y_0 + y_1$,-0.166667 $y_0 + y_2$,-0.142857 $y_0 + y_3$ }.

Then, we have $(y_0, y_1, y_2 \; y_3, y_4)$ =(3.16738, 4.97274, 5.05272, 4.84987, 1.66452) from the problem (37).

Hence, we getc =(2.30631, 0.681127, 0.0476312, -2.97066), d =(2.11159, 3.70579, 4.52482, 4.39739).

The attractor of IFS $\{R^2 : w_1, w_2, w_3, w_4\}$, $w_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u_i(x) \\ s_i y + c_i x + d_i \end{pmatrix}$, $i = 1, 2, 3, 4$, is the graph of the found least squares fractal approximation(see Fig 2).

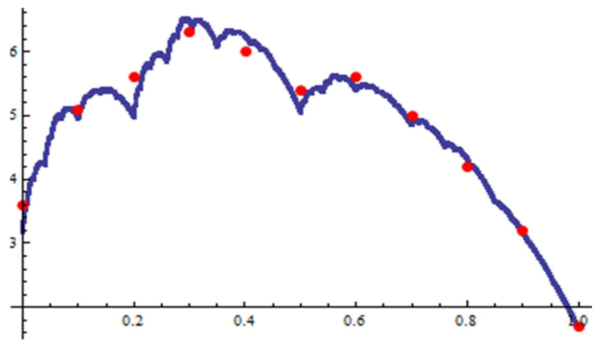

Fig. 2. *LSFA of a data set. The points are one of the data set.*

Example 2. Calculate LSFA of a coastline with a data set taken from the coastline in Fig 3.

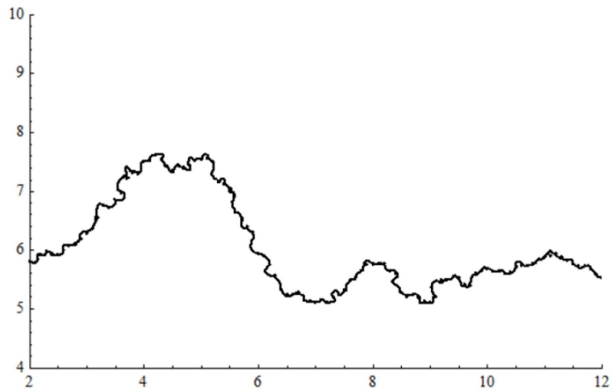

Fig. 3. *The coastline.*

We take the following data set: P={(2.0, 5.82), (2.1, 5.86), (2.2, 5.9), (2.3, 5.92), (2.4, 5.92), (2.5, 5.92), (2.6, 6.0), (2.7, 6.22), (2.8, 6.29), (2.9, 6.31), (3.0, 6.29), (3.1, 6.27), (3.2, 6.39), (3.3, 6.63), (3.4, 7.15), (3.5, 7.29), (3.6, 7.47), (3.7, 7.49), (3.8, 7.55), (3.9, 7.59), (4.0, 7.61), (4.1, 7.54), (4.2, 7.42), (4.3, 7.26), (4.4, 7.28), (4.5, 7.36), (4.6, 7.36), (4.7, 7.40), (4.8, 7.40), (4.9, 7.36), (5.0, 7.32), (5.1, 7.42), (5.2, 7.38), (5.3, 7.38), (5.4, 7.28), (5.5, 7.01), (5.6, 6.75), (5.7, 6.54), (5.8, 6.23), (5.9, 5.52), (6.0, 5.48), (6.1, 5.44), (6.2, 5.36), (6.3, 5.46),(6.4, 5.34), (6.5, 5.18), (6.6, 5.24), (6.7, 5.28), (6.8, 5.16), (6.9, 5.14), (7.0, 5.12), (7.1, 5.16), (7.2,5.12}, (7.3, 5.10), (7.4, 5.06), (7.5, 5.1), (7.6, 5.48), (7.7, 5.82), (7.8, 5.98), (7.9, 5.98), (8.0, 5.84), (8.1, 5.70), (8.2, 5.62), (8.3, 5.58), (8.4, 5.4), (8.5, 5.28), (8.6, 5.20), (8.7, 5.24), (8.8, 5.2), (8.9, 5.14), (9.0, 5.12), (9.1, 5.20), (9.2, 5.50), (9.3, 5.50), (9.4, 5.54), (9.5, 5.56),(9.6, 5.42), (9.7, 5.40), (9.8, 5.6), (9.9, 5.64), (10.0, 5.68), (10.1, 5.64), (10.2, 5.64), (10.3, 5.60), (10.4, 5.62), (10.5, 5.70), (10.6, 5.74), (10.7, 5.74), (10.8, 5.78),(10.9, 5.86), (11.0, 5.88), (11.1, 5.88), (11.2, 5.70), (11.3, 5.76), (11.4, 5.84), (11.5, 5.78), (11.6, 5.74), (11.7, 5.74), (11.8, 5.68), (11.9, 5.58), (12.0, 5.52)}

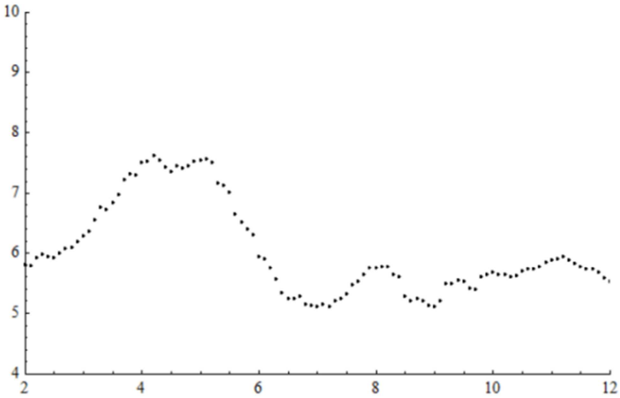

Fig. 4. *A data set of the coastline.*

Then, we have I=[2, 12], $\Delta = \{x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$, $I_1 = [2, 3]$, $I_2 = [3, 4]$, $I_3 = [4, 5]$, $I_4 = [5, 6]$, $I_5 = [6, 7]$, $I_6 = [7, 8]$, $I_7 = [8, 9]$, $I_8 = [9, 10]$, $I_9 = [10, 11]$, $I_{10} = [11, 12]$, $(y_0, y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10})$ = (5.71008, 6.22253, 7.4376, 7.58819, 5.95495, 4.91328, 5.74059, 5.12894, 5.59619, 5.83781, 5.6046),c= (0.0522994, 0.122562, 0.0161142, -0.162269, -0.103113, 0.0837858, -0.0601096, 0.0477794, 0.0252167, -0.0222661), d= (5.03448, 5.4064, 6.83436, 7.34172, 5.59017, 4.1747, 5.2898, 4.46238, 4.97475, 5.31133).

The attractor of IFS $\{\mathrm{R}^2 : w_1, w_2, \cdots, w_{10}\}$, $w_i\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u_i(x) \\ s_i y + c_i x + d_i \end{pmatrix}$, $i = 1, 2, \cdots, 10$ is the graph of the found least squares fractal approximation (see Fig 5).

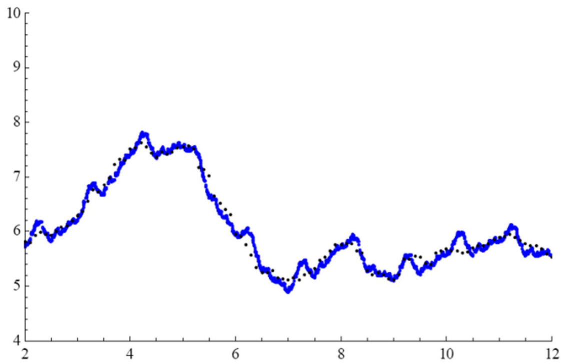

Fig. 5. *The fractal approximation of the coastline.*

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