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Functors $S^{-1}()$, $Hom_A(-,\mathcal{B})$, $-\otimes_A \mathcal{B}$ and $Ext_A^n(-,\mathcal{B})$ in Category of $A-Alg$

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Abstract: The purpose of this paper is to study some results of homological algebra in the category A-Alg (resp. Alg-A) of left (resp. right) A-algebra in the noncommutative case. In this paper A is a subring of B. So the main results of this paper are, if B is a noetherian duo-ring, S a central saturated multiplicatively closed subset of A, S_R the set of regular elements of S, $\mathcal{A}$ a finitely presented right A-algebra and $\mathcal{B}$ a (B-A)-bialgebra, then $Ext_A^n(\mathcal{A},\mathcal{B})\otimes_B S_R^{-1}B$ is isomorphic to $S_R^{-1}Ext_A^n(\mathcal{A},\mathcal{B})$, also $S_R^{-1}Ext_A^n(\mathcal{A},\mathcal{B})$ is isomorphic to $Ext_{S^{-1}A}^n(S^{-1}\mathcal{A},S^{-1}\mathcal{B})$, for any integer n.

Keywords: Algebra, Multiplicatively Closed Subset, Ore Condition of Multiplicatively Closed Subset, Localized, Category, Functor, Complex Projective Resolution

1. Introduction

In this paper, A is assumed unitary, associative and not necessarily commutative. $\mathcal{A}$ and $\mathcal{B}$ are algebras assumed unitary, associative and not necessarily commutative as a ring and unital as a A-module.

In general, the action of the functors $Hom_A(-,\mathcal{B})$, $-\otimes_A \mathcal{B}$ and $Ext_A^n(-,\mathcal{B})$ on a A-module M(resp. A-algebra $\mathcal{A}$) is not an algebra.

In this paper the conditions in which $Hom_A(M,\mathcal{B})$ and $Ext_A^n(M,\mathcal{B})$ have a structure of algebra are given. The conditions in which $Hom_A(\mathcal{A},\mathcal{B})$, $\mathcal{A}\otimes_A \mathcal{B}$ and $Ext_A^n(\mathcal{A},\mathcal{B})$ have a structure of algebra are also given, and the localization of its algebras with a multiplicatively closed subsets satisfying the left Ore condition is studied.

The main purpose of this paper is to study the functorial relations between the functor localization $S^{-1}()$ with the functors $Hom_A(-,\mathcal{B}): Alg-A\to B-Alg$, $-\otimes_A \mathcal{B}: Alg-B\to Alg-A$ and $Ext_A^n(-,\mathcal{B}): Alg-A\to B-Alg$.

In this paper the following main results are shown:

In the section 3:

If $\mathcal{A}$ is a left (resp. right) A-algebra and S a central saturated multiplicatively closed subset of A, then the left (resp. right) $S^{-1}A$-module $S^{-1}\mathcal{A}$ is a left (resp. right) $S^{-1}A$-algebra, furthermore $\mathcal{A}\otimes_A S^{-1}A\cong S^{-1}\mathcal{A}$.

If B is a duo-ring, A a subring of B, S a central saturated multiplicatively closed subset of A, S_R the set of regular elements of S, $\mathcal{A}$ right A-algebra and $\mathcal{B}$ a $(B-A)$-bialgebra then $Hom_{S^{-1}A}(S^{-1}\mathcal{A},S^{-1}\mathcal{B})$ is a left $S_R^{-1}B$-algebra.

If B is a duo-ring, A a subring of B, S a central saturated multiplicatively closed subset of A, S_R the set of regular elements of S, $\mathcal{A}$ a finitely presented right A-algebra and $\mathcal{B}$ a $(B-A)$-bialgebra, then there exists the isomorphisms of left $S_R^{-1}B$-algebras,

$$Hom_A(\mathcal{A},\mathcal{B})\otimes_B S_R^{-1}B\cong S_R^{-1}Hom_A(\mathcal{A},\mathcal{B})\cong Hom_{S^{-1}A}(S^{-1}\mathcal{A},S^{-1}\mathcal{B}).$$

furthermore if $\mathcal{P}$ is a prime ideal of B, $S=(B-\mathcal{P})\cap Z(B)$ and S_R the set of regular elements of S, then there exists the isomorphisms of left $S^{-1}B$-algebras,

$$Hom_B(\mathcal{A},\mathcal{B})\otimes_B S^{-1}B\cong S^{-1}Hom_B(\mathcal{A},\mathcal{B})\cong Hom_{S^{-1}B}(S^{-1}\mathcal{A},S^{-1}\mathcal{B}).$$

And furthermore also there exists the isomorphisms of left $S_R^{-1}B$-algebras,

$$Hom_B(\mathcal{A},\mathcal{B})\otimes_B S_R^{-1}B\cong S_R^{-1}Hom_B(\mathcal{A},\mathcal{B})\cong Hom_{S^{-1}B}(S^{-1}\mathcal{A},S^{-1}\mathcal{B}).$$

If B is a ring, S a saturated multiplicatively closed subset of $Z(B)$, $\mathcal{A}$ a finitely presented right $Z(B)$-algebra and $\mathcal{B}$ a $(B-Z(B))$-bialgebra, then there exists an isomorphisms of left $S^{-1}B$-algebras,

$$Hom_{Z(B)}(\mathcal{A},\mathcal{B})\otimes_B S^{-1}B\cong S^{-1}Hom_{Z(B)}(\mathcal{A},\mathcal{B})\cong Hom_{S^{-1}Z(B)}(S^{-1}\mathcal{A},S^{-1}\mathcal{B}).$$

Furthermore if $\mathcal{P}$ a prime ideal of $Z(B)$, then it exists the isomorphisms of left $Z(B)_{\mathcal{P}}$-algebras,

$$Hom_{Z(B)}(\mathcal{A},\mathcal{B})\otimes_B B_{\mathcal{P}}\cong Hom_{Z(B)}(\mathcal{A},\mathcal{B})_{\mathcal{P}}\cong Hom_{Z(B)_{\mathcal{P}}}(\mathcal{A}_{\mathcal{P}},\mathcal{B}_{\mathcal{P}}).$$

In the section 4:

If $\mathcal{A}$ is a right A-algebra, $\mathcal{B}$ a $(B-A)$-bialgebra the unitary left B-algebra, $Ext_A^n(\mathcal{A},\mathcal{B})$ is built as follows: if $P: P_n\xrightarrow{d_n}P_{n-1}\cdots\longrightarrow P_2\xrightarrow{d_2}P_1\xrightarrow{d_1}P_0\xrightarrow{\epsilon}\mathcal{A}\longrightarrow 0$ is a complex projective resolution of a right A-module $\mathcal{A}$, then

$$Ext_A^n(\mathcal{A},\mathcal{B})=[Kerd_{n+1}^*]/\langle Imd_n^*\rangle,$$

where

$$d_n^*=Hom_A(d_n,\mathcal{B}): Hom_A(P_{n-1},\mathcal{B})\longrightarrow Hom_A(P_n,\mathcal{B})$$

$$\varphi_n\longmapsto\varphi_n\circ d_n,$$

$[Kerd_{n+1}^*]$ is the unitary B-algebra generated by $Kerd_{n+1}^*$ and $\langle Imd_n^*\rangle$ is the ideal of $[Kerd_{n+1}^*]$ generated by Imd_n^*.

If B is a noetherian duo-ring, A a subring of B, S a central saturated multiplicatively closed subset of A, S_R the set of regular elements of S, $\mathcal{A}$ a finitely generated right A-algebra and $\mathcal{B}$ a $(B-A)$-bialgebra, then it exists the isomorphisms of left $S_R^{-1}B$-algebras,

$$Ext_A^n(\mathcal{A},\mathcal{B})\otimes_B S_R^{-1}B\cong S_R^{-1}Ext_A^n(\mathcal{A},\mathcal{B})\cong Ext_{S^{-1}A}^n(S^{-1}\mathcal{A},S^{-1}\mathcal{B}).$$

2. Preliminary Results

Definition 2.1

(1). A subset S of a ring A is called multiplicative if $1_A\in S$ and S is stable by multiplication i.e for all $x,\ t\in S,\ xt\in S$.

(2). A multiplicative subset S of a ring A is called closed if for all $s,\ s'\in A$ such that $ss'\in S\Rightarrow s\in S$ and $s'\in S$.

(3). Let S a multiplicatively closed subset of A. S satisfies the left Ore conditions.

If :

(a) $\forall a\in A$, $\forall s\in S\ \exists t\in S$ and $b\in A$ such that $ta=bs$

(b) $\forall a\in A$, $\forall\ s\in S$ such that $sa=0$, then it exists $t\in S$ such that $at=0$.

Theorem 2.1 *Let A a ring, M a left A-module and S a multiplicatively closed subset of A satisfying the left Ore conditions. The binary relation defined in $S\times M$ by*

$$(s,m)\mathcal{R}(s',m')\Leftrightarrow\exists x,y\in S/\begin{cases}xm=ym'\\ xx=ys'\end{cases}$$

is an equivalence relation.

Proof

See [14] ∎

Notation: $S^{-1}M$ is noted the set of equivalence classes modulo $\mathcal{R}$ $(S\times M/\mathcal{R})$.

If $(s,m)\in S\times M$, then the classe $\overline{(s,m)}$ is noted $\frac{m}{s}$, in particular if $M=A$ the classe $\overline{(s,a)}\in S^{-1}A$ is noted by $\frac{a}{s}$.

Theorem 2.2 *Let A a ring not necessary commutative, M a left A-module and S a multiplicatively closed subset of A satisfying the left Ore conditions, $S^{-1}A$ is a ring by the two following operations:*

1. $\frac{a}{t}+\frac{b}{s}=\frac{xa+yb}{ys}$, where $x,y\in S: xt=ys$
2. $\frac{a}{t}\times\frac{b}{s}=\frac{zb}{wt}$, where $(w,z)\in S\times A: wa=zs$.

$S^{-1}M$ is a left $S^{-1}A$-module by the two following operations:

1. $\frac{m}{s}+\frac{m\prime}{s\prime}=\frac{xm+ym\prime}{ys}$, where $x,y\in S$ such that $xs=ys$.
2. $\frac{a}{t}\cdot\frac{m}{s}=\frac{zm}{wt}$, where $(w,z)\in S\times A$ such that $wa=zs$.

Proof

See [13] and [14] ∎

Proposition 2.1 *Let A and $\mathcal{A}$ two rings, and $\theta: A\longrightarrow\mathcal{A}$ a morphism of rings. Then $\mathcal{A}$ has a structure of left (resp. right) A-module as follows:*

$$\bullet: A\times\mathcal{A}\longrightarrow\mathcal{A}$$

$$(a,x)\longmapsto a\bullet x=\theta(a)x$$

$$(\text{resp. } *:\mathcal{A}\times A\longrightarrow\mathcal{A}$$

$$(x,a)\longmapsto x*a=x\theta(a)\ )$$

Proof

Easy.∎

Definition 2.2 *Let A and $\mathcal{A}$ two rings, and $\theta: A\longrightarrow\mathcal{A}$ a morphism of rings. Then $(\mathcal{A},+,\times,\bullet)$ (resp. $(\mathcal{A},+,\times,*)$) is called a left (resp. right) A-algebra relatively to θ.*

Remark 1 *This definition shows that to provide $\mathcal{A}$ with a structure of left (or right) A-algebra it is enough to have a morphism of rings from A to $\mathcal{A}$.*

Definition 2.3 *Let A and $\mathcal{A}$ two rings, and $\theta: A \longrightarrow \mathcal{A}$ a morphism of rings. If $Im(\theta) \subseteq Z(\mathcal{A})$, then $\mathcal{A}$ is called a A-algebra relatively to θ.*

Proposition 2.2 *Let A and $\mathcal{A}$ two rings.*

(1). If $\mathcal{A}$ is a left A-module such that $a \cdot (xy) = (a \cdot x)y = x(a \cdot y)$, $\forall\, a \in A, \forall x,\ y \in \mathcal{A}$, then $\mathcal{A}$ is a left A-algebra.

(2). If $\mathcal{A}$ is a right A-module such that $(xy) \cdot a = x(y \cdot a) = (x \cdot a)y$, $\forall\, a \in A, \forall x,\ y \in \mathcal{A}$, then $\mathcal{A}$ is a right A-algebra.

Proof

$*$ Suppose that $\mathcal{A}$ is a left A-module such that $a \cdot (xy) = (a \cdot x)y = x(a \cdot y)$, $\forall\, a \in A$, $\forall x,\ y \in \mathcal{A}$ and show that $\mathcal{A}$ is a left A-algebra.

Consider the canonical morphism of left A-modules:

$$\theta_g: A \longrightarrow \mathcal{A}$$

$$a \longmapsto a \cdot 1_{\mathcal{A}}.$$

It is enough to show that θ_g is an isomorphism of rings.

It is clear that θ_g is a morphism of groups.

Let $a_1,\ a_2 \in A$ we have:

$$\theta_g(a_1 a_2) = (a_1 a_2) \cdot 1_{\mathcal{A}} = a_1 \cdot (a_2 \cdot 1_{\mathcal{A}})$$

$$= a_1 \cdot [(a_2 \cdot 1_{\mathcal{A}})1_{\mathcal{A}}] = a_1 \cdot [1_{\mathcal{A}}(a_2 \cdot 1_{\mathcal{A}})]$$

$$= (a_1 1_{\mathcal{A}})(a_2 \cdot 1_{\mathcal{A}}) = \theta_g(a_1)\theta_g(a_2).$$

So θ_g is a morphism of rings, so $(\mathcal{A}, +, \times, \bullet)$ is a left A-algebra where "$\bullet$" the law induced by θ_g.

$*$ Suppose that $\mathcal{A}$ is a left A-module such that $(xy) \cdot a = x(y \cdot a)$, $\forall\, a \in A$, $\forall x,\ y \in \mathcal{A}$ and show that $\mathcal{A}$ is a left A-algebra.

Consider the canonical morphism of left A-modules:

$$\theta_d: A \longrightarrow \mathcal{A}$$

$$a \longmapsto 1_{\mathcal{A}} \cdot a.$$

It is enough to show that θ_d is a morphism of rings.

It is clear that θ_d is a morphism of groups.

Let $a_1,\ a_2 \in A$

Then:

$$\theta_d(a_1 a_2) = 1_{\mathcal{A}} \cdot (a_1 a_2) = (1_{\mathcal{A}} \cdot a_1) \cdot a_2 = [(1_{\mathcal{A}} \cdot a_1)1_{\mathcal{A}}] \cdot a_2 = (1_{\mathcal{A}} \cdot a_1)(1_{\mathcal{A}} \cdot a_2) = \theta_d(a_1)\theta_d(a_2).$$

So θ_d is a morphism of rings, so $\mathcal{A}$ is a right A-algebra.■

3. Functor $S^{-1}()$ and Functor $Hom_A(-, \mathcal{B})$ in the Category A-Alg (resp. Alg-A)

3.1. Algebra of Fraction in the Non Commutative Case

Theorem 3.1 *Let $\mathcal{A}$ a left (resp. right) A-algebra and S a central saturated multiplicatively closed subset of A. Then the left (resp. right) $S^{-1}A$-module , $S^{-1}\mathcal{A}$ is a left (resp. right) $S^{-1}A$-algebra.*

Proof

Show that $S^{-1}\mathcal{A}$ is a left $S^{-1}A$-algebra.

Define a law of internal composition on $S^{-1}\mathcal{A}$.

Consider the following correspondence:

$$\widetilde{\times}: S^{-1}\mathcal{A} \times S^{-1}\mathcal{A} \longrightarrow S^{-1}\mathcal{A}$$

$$\left(\frac{x}{t}, \frac{x'}{t'}\right) \longmapsto \frac{x}{t} \widetilde{\times} \frac{x'}{t'} = \frac{xx'}{tt'}.$$

Let $\left(\frac{x_1}{t_1}, \frac{x_2}{t_2}\right)$, $\left(\frac{x'_1}{t'_1}, \frac{x'_2}{t'_2}\right) \in S^{-1}\mathcal{A} \times S^{-1}\mathcal{A}$ such that $\left(\frac{x_1}{t_1}, \frac{x_2}{t_2}\right) = \left(\frac{x'_1}{t'_1}, \frac{x'_2}{t'_2}\right)$.

So it exists $a,\ b,\ a',\ b' \in S$ such that

$$\begin{cases} \begin{cases} a \cdot x_1 = b \cdot x'_1 \ (1) \\ at_1 = bt'_1 \ (2) \end{cases} \\ \\ \begin{cases} a' \cdot x_2 = b' \cdot x'_2 \ (1)' \\ a't_2 = b't'_2 \ (2)' \end{cases} \end{cases}$$

Whereas:

$$\begin{cases} \dfrac{x_1}{t_1} \widetilde{\times} \dfrac{x_2}{t_2} = \dfrac{x_1 x_2}{t_1 t_2} \\ \\ \dfrac{x'_1}{t'_1} \widetilde{\times} \dfrac{x'_2}{t'_2} = \dfrac{x'_1 x'_2}{t'_1 t'_2} \end{cases}$$

Whereas:

$$(a \cdot x_1)(a' \cdot x_2) = a \cdot [x_1(a' \cdot x_2)] \text{ because } \mathcal{A} \text{ is a left } A\text{-algebra.}$$

$$= x_1(a \cdot (a' \cdot x_2)) \text{ because } \mathcal{A} \text{ is a left } A\text{-algebra}$$

$$= x_1((aa') \cdot x_2) \text{ because } \mathcal{A} \text{ is a } A\text{-module}$$

$$= (aa') \cdot (x_1 x_2) \text{ because } \mathcal{A} \text{ is a left } A\text{-algebra.}$$

Whereas also:

$$(b \cdot x'_1)(b' \cdot x'_2) = b \cdot [x'_1(b' \cdot x'_2)] \text{ because } \mathcal{A} \text{ is a left } A\text{-algebra}$$

$= x'_1(b\cdot(b'\cdot x'_2))$ because $\mathcal{A}$ is a left A-algebra

$= x'_1((bb')\cdot x'_2)$ because $\mathcal{A}$ is a left A-module

$= (bb')\cdot(x'_1x'_2)$ because $\mathcal{A}$ is a left A-algebra.

Or (1) and $(1)' \Longrightarrow (a\cdot x_1)(a'\cdot x_2) = (b\cdot x'_1)(b'\cdot x'_2)$.
So $(aa')\cdot(x_1x_2) = (bb')\cdot(x'_1x'_2)$ $(*)$.
Whereas: $(at_1)(a't_2) = (aa')(t_1t_2)$ because $S\subset Z(A)$.
Similarly we have $(bt'_1)(b't'_2) = (bb')(t'_1t'_2)$ because $S\subset Z(A)$

(2) and $(2)' \Longrightarrow (at_1)(a't_2) = (bt'_1)(b't'_2) \Rightarrow (aa')(t_1t_2) = (bb')(t'_1t'_2)$ $(**)$.

Pose $X = aa'$ and $Y = bb'$.

Whereas

$$\begin{cases} X(x_1x_2) = Y(x'_1x'_2) \\ X(t_1t_2) = Y(t'_1t'_2) \end{cases}$$

So

$$\frac{x_1}{t_1} \widetilde{\times} \frac{x_2}{t_2} = \frac{x'_1}{t'_1} \widetilde{\times} \frac{x'_2}{t'_2}.$$

Therefore $\widetilde{\times}$ is a law internal composition.

After [14] $S^{-1}\mathcal{A}$ is a left $S^{-1}A$-module, so to show that $S^{-1}\mathcal{A}$ is a left $S^{-1}A$-algebra, it is enough to show after the proposition 2.2 that:

$$\forall\ \frac{a}{s}\in S^{-1}A,\ \forall\ \frac{x}{t}, \frac{x'}{t'}\in S^{-1}\mathcal{A},\ \frac{a}{s}\cdot(\frac{x}{t}\widetilde{\times}\frac{x'}{t'}) = \frac{x}{t}\widetilde{\times}(\frac{a}{s}\cdot\frac{x'}{t'}) = (\frac{a}{s}\cdot\frac{x}{t})\widetilde{\times}\frac{x'}{t'}.$$

Whereas first:

$\frac{a}{s}\cdot(\frac{x}{t}\widetilde{\times}\frac{x'}{t'}) = \frac{a}{s}\cdot\frac{xx'}{tt'} = \frac{z\cdot(xx')}{ws}$ where $(w,z)\in S\times A$ such that $wa = ztt'$ (E_1).

Whereas second:

$\frac{x}{t}\widetilde{\times}(\frac{a}{s}\cdot\frac{x'}{t'}) = \frac{x}{t}\widetilde{\times}\frac{z'x'}{w's} = \frac{xz'x'}{tw's}$ where $(w',z')\in S\times A$ such that $w'a = z't'$ (E_2).

Since $(w,tw')\in S\times S\ \exists\ (p,q)\in S\times S$ such that $pw = qtw' \Rightarrow pwa = qtw'a$, so after (E_1) et (E_2) $pztt' = qtz't' \Rightarrow pztt' - qtz't' = 0 \Rightarrow (pzt - qtz')t' = 0$,

so after the seconde Ore condition $\exists\ r_1\in S$ such that $r_1(pzt - qtz') = 0 \Rightarrow r_1pzt = r_1qtz' \Rightarrow r_1pzt\cdot(xx') = r_1qtz'\cdot(xx')$.

Calculate first $r_1pzt\cdot(xx')$:

Whereas:

$(r_1pzt)\cdot(xx') = (r_1ptz)\cdot(xx')$ because $zt = tz$ since $S\subset Z(A) = ((r_1pt)z)\cdot(xx') = (r_1pt)\cdot(z\cdot(xx'))$ because $\mathcal{A}$ is a left A-module.

Then calculate $[(r_1qt)z']\cdot(xx')$

Whereas:

$[(r_1qt)z']\cdot(xx') = (r_1qt)\cdot(z'\cdot(xx'))$ because $\mathcal{A}$ is a left A-module $= (r_1qt)\cdot[x(z'\cdot x)]$ because $\mathcal{A}$ is a left A-algebra.

Also since $pw = qtw'$ then $r_1tpw = r_1tqtw'$ and since $S\subset Z(A)$ then $(r_1pt)(ws) = (r_1qt)(tw')$.

Pose $X = r_1pt$ and $Y = r_1qt$

Whereas:

$$\begin{cases} X(z\cdot(xx')) = Y(x(z'\cdot x')) \\ X(ws) = Y(tw') \end{cases} \Leftrightarrow \frac{z\cdot(xx')}{ws} = \frac{x\cdot(z'a)}{wt't}.$$

So

$$\frac{a}{s}\cdot(\frac{x}{t}\widetilde{\times}\frac{x'}{t'}) = \frac{x}{t}\widetilde{\times}(\frac{a}{s}\cdot\frac{x'}{t'}).$$

In the same way we show that $\frac{a}{s}\cdot(\frac{x}{t}\widetilde{\times}\frac{x'}{t'}) = (\frac{a}{s}\cdot\frac{x}{t})\widetilde{\times}\frac{x'}{t'}$.

So

$$\frac{a}{s}\cdot(\frac{x}{t}\widetilde{\times}\frac{x'}{t'}) = (\frac{a}{s}\cdot\frac{x}{t})\widetilde{\times}\frac{x'}{t'} = \frac{x}{t}\widetilde{\times}(\frac{a}{s}\cdot\frac{x'}{t'}),\ \forall\ \frac{a}{s}\in S^{-1}A,\ \forall\ \frac{x}{t}, \frac{x'}{t'}\in S^{-1}\mathcal{A}.$$

Therefore $S^{-1}\mathcal{A}$ is a left $S^{-1}A$-algebra. ■

Proposition 3.1 *Let B and A two rings.*

(1). If M is a right A-module and $\mathcal{B}$ is a $(B-A)$-bialgebra, then the group $Hom_A(M,\mathcal{B})$ is a unitary left B-algebra by the following operations:

$$(b\cdot f)(m) = bf(m), \forall m\in M, \forall f\in Hom_A(M,\mathcal{B}), \forall b\in B,$$

$$(f\ g)(m) = f(m)g(m), \forall m\in M, \forall f,g\in Hom_A(M,\mathcal{B}).$$

(2). If M is a left A-module and $\mathcal{B}$ is a $(A-A)$-bialgebra, then the group $Hom_A(M,\mathcal{B})$ is a unitary left A-algebra by the following operations:

$$(a\cdot f)(m) = af(m), \forall m\in M, \forall f\in Hom_A(M,\mathcal{B}), \forall a\in A,$$

$$(f\ g)(m) = f(m)g(m), \forall m\in \mathcal{A}, \forall f,g\in Hom_A(M,\mathcal{B}).$$

Proof

- the group $Hom_A(M,\mathcal{B})$ is a unitary left B-algebra.

$*$ After [18] $Hom_A(M,\mathcal{B})$ is a left B-module.

$*$ Show that $Hom_A(M,\mathcal{B})$ has a structure of ring via the operation defined above.

Let $f,g,h\in Hom_A(M,\mathcal{B})$, $m\in M$, we have:

$$[(fg)h](m) = (fg)(m)h(m) = (f(m)g(m))h(m)$$

$= f(m)(g(m)h(m))$, because $\mathcal{B}$ is a ring

$$= f(m)(gh)(m) = [f(gh)](m).$$

So $(fg)h = f(gh)$, so the internal composition law is associative.

Whereas also:

$$\begin{aligned}[f(g+h)](m) &= f(m)(g+h)(m) \\ &= f(m)(g(m)+h(m))\end{aligned}$$

$$= f(m)g(m) + f(m)h(m) = (fg + fh)(m).$$

So $f(g + h) = fg + fh$, so the internal composition law is distributive compared with the addition.

Therefore, $Hom_A(M, \mathcal{B})$ has a structure of ring of unit element,

$1: M \longrightarrow \mathcal{B}$.

$$m \longmapsto 1_{\mathcal{B}}$$

* Show that the two laws are compatible

Let $b \in B$, $f, g \in Hom_A(M, \mathcal{B})$, $m \in M$

whereas $[b \cdot (fg)](m) = b(fg)(m) = b(f(m)g(m)) = (bf(m))g(m)$

because $\mathcal{B}$ is a left B-algebra $= (b \cdot f)(m)g(m) = [(b \cdot f)g](m)$

whereas also $[b \cdot (fg)](m) = b(fg)(m) = b(f(m)g(m))$

$= f(m)(bg(m))$ because $\mathcal{B}$ is a left B-algebra.

$= f(m)(b \cdot g)(m) = [f(b \cdot g)](m)$.

So $b \cdot (fg) = (b \cdot f)g = f(b \cdot g))$.

So after the proposition 2.2 $Hom_A(M, \mathcal{B})$ is a left B-algebra.

- The item 2. is showing the same way. ■

Corollary 3.1

(1). If $\mathcal{A}$ is a right A-algebra and $\mathcal{B}$ is a $(B - A)$-bialgebra, then the group $Hom_A(\mathcal{A}, \mathcal{B})$ is

a unitary left B-algebra by the following operations:

$$(b \cdot f)(x) = bf(x), \forall x \in \mathcal{A}, \forall f \in Hom_A(\mathcal{A}, \mathcal{B}), \forall b \in B,$$

$$(f\ g)(x) = f(x)g(x), \forall x \in \mathcal{A}, \forall f, g \in Hom_A(\mathcal{A}, \mathcal{B}).$$

(2). If $\mathcal{A}$ is a left A-algebra and $\mathcal{B}$ is a $(A - A)$-bialgebra, then group $Hom_A(\mathcal{A}, \mathcal{B})$ is a left A-algebra by the following operations:

$$(a \cdot f)(x) = af(x), \forall x \in \mathcal{A}, \forall f \in Hom_A(\mathcal{A}, \mathcal{B}), \forall a \in A,$$

$$(f\ g)(x) = f(x)g(x), \forall f, g \in Hom_A(\mathcal{A}, \mathcal{B}), x \in \mathcal{A}.$$

Proof

Just consider the A-algebra $\mathcal{A}$ as a A-module and you have the results 1 and 2.■

Proposition 3.2 *Let B and A two rings.*

(1). If $\mathcal{A}$ is a right B-algebra and $\mathcal{B}$ is a $(B - A)$-bialgebra, then the group $\mathcal{A} \otimes_B \mathcal{B}$ is a

right A-algebra of unit element $1_{\mathcal{A}} \otimes 1_{\mathcal{B}}$ by the following operations:

$$(x \otimes y).a = x \otimes (y.a), \forall x \in \mathcal{A}, \forall y \in \mathcal{B}, \forall a \in A$$

$$(x \otimes y)(x' \otimes y') = (xx' \otimes yy'), \forall x, x' \in \mathcal{A}, \forall y, y' \in \mathcal{B}.$$

(2). If $\mathcal{A}$ is a left A-algebra and $\mathcal{B}$ is a $(B - A)$-bialgebra, then the group $\mathcal{A} \otimes_A \mathcal{B}$ is a

unitary left B-algebra by the following operations:

$$b \cdot (x \otimes y) = x \otimes (b.y), \forall x \in \mathcal{A}, \forall y \in \mathcal{B}, \forall b \in B,$$

$$(x \otimes y)(x' \otimes y') = (xx' \otimes yy'), \forall x, x' \in \mathcal{A}, \forall y, y' \in \mathcal{B}.$$

Proof

- $\mathcal{A} \otimes_B \mathcal{B}$ is a right A-algebra.

* After [1] $\mathcal{A} \otimes_B \mathcal{B}$ is a right A-module.

* It's not difficult to show that $\mathcal{A} \otimes_B \mathcal{B}$ has a structure of ring.

* Show that the two laws are compatible.

Let $x \otimes y$, $x' \otimes y' \in \mathcal{A} \otimes_B \mathcal{B}$ and $a \in A$.

whereas $[(x \otimes y)(x' \otimes y')].a = (xx' \otimes yy').a = (xx') \otimes ((yy').a) = (xx') \otimes (y(y'.a))$ because $\mathcal{B}$ is a right A-algebra. $= (x \otimes y)(x' \otimes y'.a) = (x \otimes y)[(x' \otimes y').a]$

whereas $[(x \otimes y)(x' \otimes y')].a = (xx' \otimes yy').a = (xx') \otimes ((yy').a) = (xx') \otimes ((y.a)y')$ because $\mathcal{B}$ is a right A -algebra. $= (x \otimes (y.a))(x' \otimes y') = [(x \otimes y).a](x' \otimes y')$

So $[(x \otimes y)(x' \otimes y')].a = (x \otimes y)[(x' \otimes y').a] = [(x \otimes y).a](x' \otimes y')$.

So after the proposition 2.2 $\mathcal{A} \otimes_B \mathcal{B}$ is a right A-algebra.

- The item 2. is showing the same way.■

Theorem 3.2 *Let B a duo-ring, A a subring of B, $\mathcal{B}$ a left B-algebra relatively to θ, S a central saturated multiplicatively closed subset of A and S_R the set of regular elements of S. Then $S^{-1}\mathcal{B}$ is a left $S_R^{-1}B$-algebra.*

Proof

After the theorem 3.1 $S^{-1}\mathcal{B}$ is a left $S^{-1}A$-algebra $\Rightarrow$ $S^{-1}\mathcal{B}$ is a ring.

Since S_R is the set of regular elements of the duo-ring B, so S_R is a saturated multiplicatively closed subset of B satisfying the Ore conditions, so after [13] $S_R^{-1}B$ is also a ring.

So it is enough to show that this following correspondence is a morphism of rings:

$$\theta': S_R^{-1}B \longrightarrow S^{-1}\mathcal{B}$$

$$\frac{b}{s} \longmapsto \frac{\theta(b)}{s}$$

So after the remark 1 $S^{-1}\mathcal{B}$ is a left $S_R^{-1}B$-algebra.■

Theorem 3.3 *Let B a ring, $\mathcal{B}$ a left B-algebra relatively to θ and S a saturated multiplicatively closed subset of $Z(B)$. Then $S^{-1}\mathcal{B}$ is a left $S^{-1}B$-algebra.*

Proof

The proof is similar to the previous one.■

Theorem 3.4 *Let B a duo-ring, A a subring of B, S a central saturated multiplicatively closed subset of A, S_R the set of regular elements of S, $\mathcal{A}$ right A-algebra and $\mathcal{B}$ a $(B - A)$-bialgebra. Then $Hom_{S^{-1}A}(S^{-1}\mathcal{A}, S^{-1}\mathcal{B})$ is a left $S_R^{-1}B$-algebra,*

Proof

After the theorem 3.1, $S^{-1}\mathcal{A}$ is a right $S^{-1}A$-algebra and $S^{-1}\mathcal{B}$ is a right $S^{-1}A$-algebra.

After the theorem 3.2, $S^{-1}\mathcal{B}$ has a structure of left $S_R^{-1}B$-algebra.

So $S^{-1}\mathcal{B}$ is a $(S_R^{-1}B - S^{-1}A)$-bialgebra, so after the proposition 3.1, $Hom_{S^{-1}A}(S^{-1}\mathcal{A}, S^{-1}\mathcal{B})$ has a structure of left $S_R^{-1}B$-algebra.■

3.2. Relationship Between Functors $\boldsymbol{S^{-1}()}$, $\boldsymbol{Hom_A(-, \mathcal{B})}$ and $\boldsymbol{- \otimes \mathcal{B}}$ in the Category $\boldsymbol{A - Alg}$ (resp. $\boldsymbol{Alg - A}$)

Proposition 3.3 *Let $\mathcal{B}$ a $(B - A)$-bialgebra.Then the* *correspondence*

$$Hom_A(-,\mathcal{B}): Mod-A \to B-Alg$$

(1). who has any right A-module M, we associate the left B-algebra $Hom_A(M,\mathcal{B})$,

(2). who has any morphism of right A-modules $f: M\to M'$, we associate $f^*Hom_A(f,\mathcal{B}): Hom_A(M',\mathcal{B})\to Hom_A(M,\mathcal{B})$ is a contravariant functor.

Proof

$*$ After the proposition 3.1 $M\in Ob(Mod-A) \Rightarrow Hom_A(M,\mathcal{B})\in Ob(B-Alg)$.

$*$ Let $f: M\to M'$ a morphism of right A-modules, show that

$f^* = Hom_A(f,\mathcal{B}): Hom_A(M',\mathcal{B})\to Hom_A(M,\mathcal{B})$ is a morphism of left B-algebras..

Let $\varphi,\psi\in Hom_A(M,\mathcal{B})$, $x\in M$.

We have

$$[f^*(\varphi\psi)](x) = [(\varphi\psi)\circ f](x) = \varphi(f(x))\psi(f(x))$$

$$= (\varphi\circ f)(x)(\psi\circ f)(x)$$

$$= [(\varphi\circ f)(\psi\circ f)](x)$$

$$= [f^*(\varphi)f^*(\psi)](x).$$

So we have $f^*(\varphi\psi) = f^*(\varphi)f^*(\psi)$.

Let $\varphi\in Hom_A(M,\mathcal{B}), b\in\mathcal{B}$, $x\in M$.

We have

$$[f^*(b\varphi)](x) = [(b\varphi)\circ f](x) = (b\varphi)(f(x)) = b(\varphi\circ f)(x)$$

$$= [b(f^*(\varphi)](x).$$

So we have $f^*(b\varphi) = b(f^*(\varphi)$.

Therefore f^*, for all $n\geq 0$, is a morphism of left B-algebras.

$*$ Let $f: M\to M'$ and $g: M'\to M''$ two morphism of right A-modules, $x\in M$, we have:

$$\begin{aligned} Hom_A(g\circ f,\mathcal{B})(\varphi)(x) &= [\varphi\circ(g\circ f)](x) \\ &= [Hom_A(f,\mathcal{B})\circ(\varphi\circ g)](x) \end{aligned}$$

$$= [Hom_A(f,\mathcal{B})\circ Hom_A(g,\mathcal{B})(\varphi)](x)$$

$$= [Hom_A(f,\mathcal{B})\circ Hom_A(g,\mathcal{B})](\varphi)(x).$$

So we have $Hom_A(g\circ f,\mathcal{B}) = Hom_A(f,\mathcal{B})\circ Hom_A(g,\mathcal{B})$.

$*$ Let $\varphi: M\to\mathcal{B}$ a morphism of left B-algebras, $x\in M$, we have:

$$\begin{aligned} [Hom_A(1_M,\mathcal{B})(\varphi)](x) = \varphi\circ 1_M(x) = \varphi(x) \Rightarrow \\ Hom_A(1_M,\mathcal{B})(\varphi) = \varphi. \end{aligned}$$

So $Hom_A(1_M,\mathcal{B}) = 1_{Hom_A(M,\mathcal{B})}$ and therefore $Hom_A(-,\mathcal{B})$ is a contravariant functor.■

Theorem 3.5 *Let $\mathcal{A}$ a A-bialgebra and S a central saturated multiplicatively closed subset of A. Then it exists an isomorphism of left $S^{-1}A$-algebras,*

$$\mathcal{A}\otimes_A S^{-1}A \cong S^{-1}\mathcal{A}.$$

Proof

Consider $\psi_{\mathcal{A}}: \mathcal{A}\times S^{-1}A \longrightarrow S^{-1}\mathcal{A}$

$$(x,\tfrac{a}{s}) \longmapsto \tfrac{xa}{s}.$$

We have $\psi_{\mathcal{A}}$ is A-bilinear, so according to the universal property of the tensor product it exists a map A-linear $\delta_{\mathcal{A}}: \mathcal{A}\otimes S^{-1}A \longrightarrow S^{-1}\mathcal{A}$ such that

$$\delta_{\mathcal{A}}(\sum x_i\otimes\frac{a_i}{s_i}) = \sum\frac{x_ia_i}{s_i}.$$

Let $\varphi_{\mathcal{A}}: S^{-1}\mathcal{A}\longrightarrow\mathcal{A}\otimes S^{-1}A$

$$\tfrac{x}{x} \longmapsto \boldsymbol{x}\otimes\tfrac{\mathbf{1}}{s}.$$

It is clear that $\varphi_{\mathcal{A}}$ is well defined.

We have

$$\varphi_{\mathcal{A}}\circ\delta_{\mathcal{A}}(\sum x_i\otimes\tfrac{a_i}{s_i}) = \varphi_{\mathcal{A}}(\sum\tfrac{x_ia_i}{s_i}) = \sum\varphi_{\mathcal{A}}(\tfrac{x_ia_i}{s_i}) = \sum x_ia_i\otimes\tfrac{1}{s_i} = \sum x_i\otimes\tfrac{a_i}{s_i}.$$

So $\varphi_{\mathcal{A}}\circ\delta_{\mathcal{A}} = 1_{\mathcal{A}\otimes S^{-1}A}$, $\varphi_{\mathcal{A}}$ and $\delta_{\mathcal{A}}$ are inverse of each other.

Therefore $\mathcal{A}\otimes_A S^{-1}A\cong S^{-1}\mathcal{A}$.■

Theorem 3.6 *Let B a duo-ring, A a subring of B, S a central saturated multiplicatively closed subset of A, S_R the set of regular elements of S, $\mathcal{A}$ a finitely presented right A-algebra and $\mathcal{B}$ a $(B-A)$-bialgebra. Then there exists the isomorphisms of left $S_R^{-1}B$-algebras,*

$$\begin{aligned} Hom_A(\mathcal{A},\mathcal{B})\otimes_B S_R^{-1}B &\cong S_R^{-1}Hom_A(\mathcal{A},\mathcal{B}) \\ &\cong Hom_{S^{-1}A}(S^{-1}\mathcal{A},S^{-1}\mathcal{B}). \end{aligned}$$

Proof

- Since $\mathcal{A}$ is a right A-algebra and $\mathcal{B}$ is a $(B-A)$-bialgebra, then after the proposition 3.1, $Hom_A(\mathcal{A},\mathcal{B})$ is a left B-algebra, we have also $S_R^{-1}B$ is a $(S_R^{-1}B-B)$-bialgebra, so after the proposition 3.2 and the theorem 3.5 we have,

$$Hom_A(\mathcal{A},\mathcal{B})\otimes_B S_R^{-1}B\cong S_R^{-1}Hom_A(\mathcal{A},\mathcal{B}).$$

- Show that $S_R^{-1}Hom_A(\mathcal{A},\mathcal{B})\cong Hom_{S^{-1}A}(S^{-1}\mathcal{A},S^{-1}\mathcal{B})$ is an isomorphism of left $S_R^{-1}B$-algebras.

After [6] $S_R^{-1}Hom_A(\mathcal{A},\mathcal{B})\cong Hom_{S^{-1}A}(S^{-1}\mathcal{A},S^{-1}\mathcal{B})$ is an isomorphism of left $S_R^{-1}B$ -module and since $Hom_{S^{-1}A}(S^{-1}\mathcal{A},S^{-1}\mathcal{B})$ is a left $S_R^{-1}B$ -algebra, so by structure transport $S_R^{-1}Hom_A(\mathcal{A},\mathcal{B})$ is a left $S_R^{-1}B$-algebra.

Therefore $S_R^{-1}Hom_A(\mathcal{A},\mathcal{B})\longrightarrow Hom_{S^{-1}A}(S^{-1}\mathcal{A},S^{-1}\mathcal{B})$ is the isomorphisms of left $S_R^{-1}B$-algebras.■

Theorem 3.7 *Let B a ring, S a saturated multiplicatively closed subset of $Z(B)$, $\mathcal{A}$ a finitely presented right $Z(B)$-algebra and $\mathcal{B}$ a $(B-Z(B))$-bialgebra. Then there exists the isomorphisms of left $S^{-1}B$-algebras,*

$$Hom_{Z(B)}(\mathcal{A},\mathcal{B}) \otimes_B S^{-1}B \cong S^{-1}Hom_{Z(B)}(\mathcal{A},\mathcal{B}) \cong Hom_{S^{-1}Z(B)}(S^{-1}\mathcal{A},S^{-1}\mathcal{B}).$$

Proof

- The first isomorphism is shown in the same way as that of the previous theorem.
- We have $\mathcal{A}$ is a right $Z(B)$-algebra $\Rightarrow$ $S^{-1}\mathcal{A}$ is a right $S^{-1}Z(B)$-algebra.

Also $\mathcal{B}$ is a $(B-Z(B))$-bialgebra $\Rightarrow$ $S^{-1}\mathcal{B}$ a $(S^{-1}B-S^{-1}Z(B))$-bialgebra. So $Hom_{S^{-1}Z(B)}(S^{-1}\mathcal{A},S^{-1}\mathcal{B})$ is a left $S^{-1}B$-algebra and since there is a bijection between $S^{-1}Hom_{Z(B)}(\mathcal{A},\mathcal{B})$ and $Hom_{S^{-1}Z(B)}(S^{-1}\mathcal{A},S^{-1}\mathcal{B})$, then by structure transport $S^{-1}Hom_{Z(B)}(\mathcal{A},\mathcal{B}) \cong Hom_{S^{-1}Z(B)}(S^{-1}\mathcal{A},S^{-1}\mathcal{B})$ is an isomorphism of left $S^{-1}B$-algebras.■

Proposition 3.4 *Let B a duo-ring, $\mathcal{P}$ a prime ideal of B, $\mathcal{A}$ a finitely presented right B-algebra, $\mathcal{B}$ a $(B-B)$-bialgebra, $S=(B-\mathcal{P})\cap Z(B)$ and S_R the set of regular elements of S. Then it exists the isomorphisms of left $S^{-1}B$-algebras,*

$$Hom_B(\mathcal{A},\mathcal{B}) \otimes_B S^{-1}B \cong S^{-1}Hom_B(\mathcal{A},\mathcal{B}) \cong Hom_{S^{-1}B}(S^{-1}\mathcal{A},S^{-1}\mathcal{B}).$$

Furthermore it exists the isomorphisms of left $S_R^{-1}B$-algebras,

$$Hom_B(\mathcal{A},\mathcal{B}) \otimes_B S_R^{-1}B \cong S_R^{-1}Hom_B(\mathcal{A},\mathcal{B}) \cong Hom_{S^{-1}B}(S^{-1}\mathcal{A},S^{-1}\mathcal{B}).$$

Proof

Since B is a duo-ring, then $B-\mathcal{P}$ is saturated multiplicatively closed subset, so $S=(B-\mathcal{P})\cap Z(B)$ is a central saturated multiplicatively closed subset of B.

So after the theorem 3.6 we have,

$$Hom_B(\mathcal{A},\mathcal{B}) \otimes_B S_R^{-1}B \cong S_R^{-1}Hom_B(\mathcal{A},\mathcal{B}) \cong Hom_{S^{-1}B}(S^{-1}\mathcal{A},S^{-1}\mathcal{B}) \text{ and}$$

$$Hom_B(\mathcal{A},\mathcal{B}) \otimes_B S_R^{-1}B \cong S_R^{-1}Hom_B(\mathcal{A},\mathcal{B}) \cong Hom_{S^{-1}B}(S^{-1}\mathcal{A},S^{-1}\mathcal{B}).\ ■$$

Proposition 3.5 *Let B a ring, $\mathcal{P}$ a prime ideal of $Z(B)$, $\mathcal{A}$ a finitely presented right B-algebra and $\mathcal{B}$ a $(B-B)$-bialgebra. Then it exists the isomorphisms of left $Z(B)_{\mathcal{P}}$-algebras,*

$$Hom_{Z(B)}(\mathcal{A},\mathcal{B}) \otimes_B B_{\mathcal{P}} \cong Hom_{Z(B)}(\mathcal{A},\mathcal{B})_{\mathcal{P}} \cong Hom_{Z(B)_{\mathcal{P}}}(\mathcal{A}_{\mathcal{P}},\mathcal{B}_{\mathcal{P}}).$$

Proof

Since $\mathcal{A}$ is a right B-algebra and $\mathcal{B}$ a $(B-B)$-bialgebra, then in particular $\mathcal{A}$ is a right $Z(B)$-algebra and $\mathcal{B}$ a $(B-Z(B))$-bialgebra.

Pose $S=Z(B)-\mathcal{P}$ $\Rightarrow$ S is a saturated multiplicatively closed subset of $Z(B)$. So after the theorem 3.7 we have:

$$Hom_{Z(B)}(\mathcal{A},\mathcal{B})_{\mathcal{P}} = S^{-1}Hom_{Z(B)}(\mathcal{A},\mathcal{B}) \cong Hom_{S^{-1}Z(B)}(S^{-1}\mathcal{A},S^{-1}\mathcal{B}) = Hom_{Z(B)_{\mathcal{P}}}(\mathcal{A}_{\mathcal{P}},\mathcal{B}_{\mathcal{P}}) \text{ and}$$

$$Hom_{Z(B)}(\mathcal{A},\mathcal{B}) \otimes_B B_{\mathcal{P}} = Hom_{Z(B)}(\mathcal{A},\mathcal{B}) \otimes_B S^{-1}B \cong S^{-1}Hom_{Z(B)}(\mathcal{A},\mathcal{B}) = Hom_{Z(B)}(\mathcal{A},\mathcal{B})_{\mathcal{P}}.■$$

4. Functor $S^{-1}()$ and Functor $Ext_A(-,\mathcal{B})$ in the Category $A-Alg$ (resp. $Alg-A$)

4.1. Construction of the Derived Fonctor $Ext_A(-,\mathcal{B})$ in the Category $A-Alg$ (resp. $Alg-A$)

Theorem 4.1 *Let $\mathcal{A}$ a right A-algebra, $\mathcal{B}$ a $(B-A)$-bialgebra and*
$P: P_n \xrightarrow{d_n} P_{n-1} \cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} \mathcal{A} \longrightarrow 0$ a complex projective resolution of a right A-module $\mathcal{A}$. Then

$$[Kerd_{n+1}^*]/\langle Imd_n^*\rangle$$

is a unitary left B-algebra, where

$$d_n^* = Hom_A(d_n,\mathcal{B}): Hom_A(P_{n-1},\mathcal{B}) \longrightarrow Hom_A(P_n,\mathcal{B})$$

$$\varphi_n \longmapsto \varphi_n \circ d_n$$

$[Kerd_{n+1}^*]$ is the unitary algebra generated by $Kerd_{n+1}^*$ and $\langle Imd_n^*\rangle$ is the ideal of $[Kerd_{n+1}^*]$ generated by Imd_n^*.

Proof

By applying the contravariant functor we have,

$$Hom_A(P,\mathcal{B}): 0 \longrightarrow Hom_A(\mathcal{A},\mathcal{B}) \longrightarrow \cdots Hom_A(P_{n-1},\mathcal{B}) \xrightarrow{d_n^*} Hom_A(P_n,\mathcal{B}) \xrightarrow{d_{n+1}^*} Hom_A(P_{n+1},\mathcal{B}).$$

So after the proposition 3.1 $Hom_A(P_n,\mathcal{B})$ has a structure of left B-algebra, for all $n \geq 0$.

Show that d_n^*, for all $n \geq 0$, is a morphism of left B-algebras.

Let $\varphi,\psi \in Hom_A(P_{n-1},\mathcal{B})$, $x \in P_n$.
whereas

$$[d_n^*(\varphi\psi)](x) = [(\varphi\psi)\circ d_n](x) = \varphi(d_n(x))\psi(d_n(x))$$

$$= (\varphi\circ d_n)(x)(\psi\circ d_n)(x)$$

$$= [(\varphi\circ d_n)(\psi\circ d_n)](x)$$

$$= [d_n^*(\varphi)d_n^*(\psi)](x).$$

So $d_n^*(\varphi\psi) = d_n^*(\varphi)d_n^*(\psi)$.

Let $\varphi \in Hom_A(P_{n-1},\mathcal{B}), b \in B,\ x \in P_n$.
whereas

$$\begin{aligned}[d_n^*(b\varphi)](x) &= [(b\varphi)\circ d_n](x) = (b\varphi)(d_n(x)) \\ &= b(\varphi\circ d_n)(x) = [b(d_n^*(\varphi)](x).\end{aligned}$$

So $d_n^*(b\varphi) = b(d_n^*(\varphi))$.

Therefore d_n^*, for all $n \geq 0$, is a morphism of left B-algebras.

It should be noted that $Kerd_{n+1}^*$ is not in general a unitary algebra, so take $[Kerd_{n+1}^*]$ the subalgebra of the unitary left B-algebra $Hom_A(P_n,\mathcal{B})$ generated by $Kerd_{n+1}^*$.

On the other hand we have $Imd_n^* \subset Kerd_{n+1}^* \subset [Kerd_{n+1}^*]$.

It should be noted that Imd_n^* is not in general an ideal, so take the ideal $\langle Imd_n^*\rangle$ of $[Kerd_{n+1}^*]$ generated by Imd_n^*.

Therefore $[Kerd_{n+1}^*]/\langle Imd_n^*\rangle$ is a unitary left B-algebra.■

In this paper we note by $Ext_A^n(\mathcal{A},\mathcal{B}) = [Kerd_{n+1}^*]/\langle Imd_n^*\rangle$.

Proposition 4.1 *Let $\mathcal{B}$ a $(B-A)$-bialgebra.Then the correspondence*

$$Ext_A^n(-,\mathcal{B}): Mod-A \longrightarrow B-Alg$$

(1). who has any right A-module M, we associate the left B-algebra $Ext_A^n(M,\mathcal{B})$,

(2). who has any morphism of right A-modules $f: M \longrightarrow M'$, we associate $Ext_A^n(f,\mathcal{B}): Ext_A^n(M',\mathcal{B}) \rightarrow Ext_A^n(M,\mathcal{B})$ is a contravariant functor.

Proof

∗ After the theorem 4.1 we have $M \in Ob(Mod-A) \Rightarrow Ext_A^n(M,\mathcal{B}) \in Ob(B-Alg)$, so the action of $Ext_A^n(-,\mathcal{B})$ on the objects of $B-Alg$ makes sense.

∗ Let $f: M \rightarrow M'$ a morphism of right A-module. After the comparison theorem the following commutative diagram is obtained

$$\begin{tikzcd} P_M: \dots \arrow[r] \arrow[d, "\tilde{f}"] & P_n \arrow[r] \arrow[d, "\tilde{f}_n"] & P_{n-1} \cdots \arrow[r] \arrow[d, "\tilde{f}_{n-1}"] & P_0 \arrow[r] \arrow[d, "\tilde{f}_0"] & \mathscr{A} \arrow[r] \arrow[d, "f"] & 0 \\ P_{M'}: \dots \arrow[r] & P'_n \arrow[r] & P'_{n-1} \cdots \arrow[r] & P'_0 \arrow[r] & \mathscr{B} \arrow[r] & 0 \end{tikzcd}$$

By applying the contravariant functor $Hom_A(-,\mathcal{B})$ we have

$$\begin{tikzcd} Hom_A(P_M,\mathscr{B}): 0 \arrow[r] \arrow[d, "Hom_A(\tilde{f},\mathscr{B})"'] & Hom_A(M,\mathscr{B}) \arrow[r] \arrow[d, "Hom_A(f,\mathscr{B})"'] & Hom_A(P_0,\mathscr{B}) \cdots \arrow[d, "Hom_A(\tilde{f}_0,\mathscr{B})"'] \\ Hom_A(P_{M'},\mathscr{B}): 0 \arrow[r] & Hom_A(M',\mathscr{B}) \arrow[r] & Hom_A(P'_0,\mathscr{B}) \cdots \end{tikzcd}$$

So $Hom_A(\tilde{f},\mathcal{B}): Hom_A(P_M,\mathcal{B}) \longrightarrow Hom_A(P_{M'},\mathcal{B})$ is a morphism of chain complex.

Whereas

$$\begin{aligned}H_n(Hom_A(\tilde{f},\mathcal{B})): H_n(&Hom_A(P_M,\mathcal{B})) \\ &\longrightarrow H_n(Hom_A(P_{M'},\mathcal{B}))\end{aligned}$$

$$\overline{z_n} \longmapsto \overline{Hom_A(\tilde{f}_n,\mathcal{B})z_n}.$$

After the theorem 4.1 $Hom_A(f,\mathcal{B}) = f^*$ and $Hom_A(\tilde{f}_n,\mathcal{B}) = \tilde{f}_n^*$ are morphism of left B-algebras.

So $H_n(Hom_A(\tilde{f},\mathcal{B})) = Ext_A^n(f,\mathcal{B})$ is a morphism of left B-algebra, so the action of $Ext_A^n(-,\mathcal{B})$ on the arrow makes sense.

∗ Whereas

$$\begin{aligned}Ext_A^n(g\circ f,\mathcal{B}) &= H_n(Hom_A(\tilde{g}\circ f,\mathcal{B})) \\ &= H_n(Hom_A(\tilde{g}\circ\tilde{f},\mathcal{B}))\end{aligned}$$

$$= H_n[Hom_A(\tilde{f},\mathcal{B})\circ Hom_A(\tilde{g},\mathcal{B})]$$

$$= H_n(Hom_A(\tilde{f},\mathcal{B}))\circ H_n(Hom_A(\tilde{g},\mathcal{B}))$$

$$= Ext_A^n(f,\mathcal{B})\circ Ext_A^n(g,\mathcal{B}).$$

∗ Whereas

$$\begin{aligned}Ext_A^n(1_M,\mathcal{B})(\overline{z_n}) &= \overline{Hom_A((\tilde{1}_M)_n,\mathcal{B})(z_n)} \\ &= \overline{1_{Hom_A(M,\mathcal{B})}(z_n)} = \overline{z_n}.\end{aligned}$$

So $Ext_A^n(1_M,\mathcal{B}) = 1_{Ext_A^n(M,\mathcal{B})}$.

Therefore $Ext_A^n(-,\mathcal{B}): Mod-A \longrightarrow B-Alg$ is a contravariant functor.■

Theorem 4.2 *Let B a duo-ring, A a subring of B, S a central saturated multiplicatively closed subset of A, S_R the set of regular elements of S and $\mathcal{B}$ a $(B-A)$-bialgebra. Then the correspondence*

$$Ext_{S^{-1}A}^n(-,S^{-1}\mathcal{B}): Mod-S^{-1}A \longrightarrow S_R^{-1}B-Alg$$

is a contravariant functor.

Proof

just see that after the theorem 3.2, $S^{-1}\mathcal{B}$ is a $(S_R^{-1}B-S^{-1}A)$-bialgebra .■

Corollary 4.1 *Let B a duo-ring, A a subring of B, S a central saturated multiplicatively closed subset of A, S_R the set of regular elements of S and $\mathcal{B}$ a $(B-A)$-bialgebra. Then the correspondence*

$$Ext_{S^{-1}A}^n(-,S^{-1}\mathcal{B}): Alg-S^{-1}A \longrightarrow S_R^{-1}B-Alg$$

is a contravariant functor.

Proof

Whereas $S^{-1}\mathcal{A} \in Ob(Alg-S^{-1}A) \Rightarrow S^{-1}\mathcal{A} \in Ob(Mod-S^{-1}A)$, so the conditions of the theorem 4.2 are verified.■

4.2. Relationship Between Functor $S^{-1}()$ and Functor $Ext_A(-,\mathcal{B})$ in Category $A-Alg$ (resp. $Alg-A$)

Theorem 4.3 *Let B a noetherian duo-ring, A a subring of B, S a central saturated multiplicatively closed subset of A, S_R the set of regular elements of S, $\mathcal{A}$ a finitely generated right A-algebra and $\mathcal{B}$ a $(B-A)$-bialgebra. Then it exists the isomorphisms of left $S_R^{-1}B$-algebras,*

$$\begin{aligned}Ext_A^n(\mathcal{A},\mathcal{B})\otimes_B S_R^{-1}B &\cong S_R^{-1}Ext_A^n(\mathcal{A},\mathcal{B}) \\ &\cong Ext_{S^{-1}A}^n(S^{-1}\mathcal{A},S^{-1}\mathcal{B}).\end{aligned}$$

Proof

After the theorem 3.5 we have the following isomorphism of left $S_R^{-1}B$ -algebra $Ext_A^n(\mathcal{A},\mathcal{B}) \otimes_B S_R^{-1}B \cong S_R^{-1}Ext_A^n(\mathcal{A},\mathcal{B})$.

$*$ Since B is a noetherian ring and $\mathcal{A}$ is a finitely generated left B -algebra, then $\mathcal{A}$ admits a projective resolution P.

Also since $\mathcal{A}$ is a finitely generated left algebra over a noetherian ring B, then $\mathcal{A}$ is finitely presented.

So after the theorem 3.6 we have the following isomorphism of left $S_R^{-1}B$-algebra,

$$S_R^{-1}Hom_A(\mathcal{A},\mathcal{B}) \cong Hom_{S^{-1}A}(S^{-1}\mathcal{A},S^{-1}\mathcal{B}).$$

So we can deduce the following complex isomorphism:

$$S_R^{-1}Hom_A(P_{\mathcal{A}},\mathcal{B}) \cong Hom_{S^{-1}A}(S^{-1}(P_{\mathcal{A}}),S^{-1}\mathcal{B}).$$

So,

$$H_n(S_R^{-1}Hom_A(P_{\mathcal{A}},\mathcal{B})) \cong H_n(Hom_{S^{-1}A}(S^{-1}(P_{\mathcal{A}}),S^{-1}\mathcal{B})).$$

Since the homological functor H_n commutes with the functor $S^{-1}()$, then we have on the one hand,

$$\begin{aligned} H_n(S_R^{-1}Hom_A(P_{\mathcal{A}},\mathcal{B})) &\cong S_R^{-1}H_n(Hom_A(P_{\mathcal{A}},\mathcal{B})) \\ &= S_R^{-1}Ext_A^n(\mathcal{A},\mathcal{B}). \end{aligned}$$

On the other hand we have,

$$H_n(Hom_{S^{-1}A}(S^{-1}(P_{\mathcal{A}}),S^{-1}\mathcal{B}))) = Ext^n_{S^{-1}A}(S^{-1}\mathcal{A},S^{-1}\mathcal{B}).$$

Therefore $S_R^{-1}Ext_A^n(\mathcal{A},\mathcal{B}) \cong Ext^n_{S^{-1}A}(S^{-1}\mathcal{A},S^{-1}\mathcal{B})$. ■

Corollary 4.2 *Let B a noetherian duo-ring, $\mathcal{P}$ a prime ideal of B, $\mathcal{A}$ a finitely generated right B-algebra, $\mathcal{B}$ a $(B-B)$-bialgebra, $S=(B-\mathcal{P})\cap Z(B)$ and S_R the set of regular elements of $B-\mathcal{P}$. Then it exists the isomorphisms of left $S^{-1}B$-algebras,*

$$S_R^{-1}Ext_B^n(\mathcal{A},\mathcal{B}) \cong Ext^n_{S^{-1}B}(S^{-1}\mathcal{A},S^{-1}\mathcal{B}).$$

Proof

Since B is a duo-ring, then $B-\mathcal{P}$ is saturated multiplicatively closed subset, so $S=(B-\mathcal{P})\cap Z(B)$ is a central saturated multiplicatively closed subset of B.

So after the theorem 4.3 we have, $S_R^{-1}Ext_B^n(\mathcal{A},\mathcal{B}) \cong Ext^n_{S^{-1}B}(S^{-1}\mathcal{A},S^{-1}\mathcal{B})$. ■

5. Conclusion

In this work the functors$Hom_A(-,\mathcal{B}): Alg-A \to B-Alg$, $-\otimes_A \mathcal{B}: B-Alg \to Alg-A$ and $Ext_A^n(-,\mathcal{B}): Alg-A \to B-Alg$ are built and their relationship with the functor $S^{-1}()$ has been studied. The following algebra structures: $Hom_A(M,\mathcal{B})$, $Ext_A^n(M,\mathcal{B})$, $Hom_A(\mathcal{A},\mathcal{B})$, $\mathcal{A}\otimes_A \mathcal{B}$ and $Ext_A^n(\mathcal{A},\mathcal{B})$ has been defined where M is a module and $\mathcal{A}$ and $\mathcal{B}$ are algebras.

The covariant functors $Hom_A(\mathcal{A},-)$, $\mathcal{A}\otimes_A-$ and $Ext_A^n(\mathcal{A},-)$ will be studied in the category $A-Alg$ (resp. $Alg-A$) and their relationship with the functor $S^{-1}()$ will be studied. In the future the result in [6] on the adjunction between the functors $Ext_A^n(\mathcal{A},-)$ and $Tor_n^A(\mathcal{A},-)$ (resp. $Ext_A^n(-,\mathcal{B})$ and $Tor_n^A(-,\mathcal{B})$) done in the category of $A-Mod$ will be generalized in the category $A-Alg$.

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