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On An Illustrative Examples of a Studied Noetherity Dirac-Delta Extensions for a Noether Operator

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Abstract: The purpose of this work is to illustrate by clear examples the noetherity nature of a finite Dirac-delta Extensions of a studied noether operator. Previously in our published papers, we have investigated in different two cases, the noetherization of a Dirac-delta extensions of a noether linear integro-differential operator defined by a third kind integral equation in some specific well chosen functional spaces. Our various already published researches were connected with such topic widely studied and clearly presenting different specific approaches, applied when establishing fundamentaly noether theory for some kind of integro-differential operators to reach the noetherization. The initial considered noether operator A has been extended with some finite dimensional spaces of Dirac-delta functions, and the noetherization of the two cases of extensions has been established depending with the parameters of the third kind integral equation defining A. The previous lead us to set the problem of the construction of practical examples clearly illustrating the relationship between theory and practise. For this aim, we based on an established wellknown noether theory and, we construct in this work step by step, two illustrative examples to show the interconnexion between the theory and pratise related to the investigation of the construction of noether theory for the considered extended noether operator denoted $\bar{A}$, defined by a third kind linear singular integral equation in some generalized functional spaces. The extended operator $\bar{A}$ of the initial noether operator A is verified being also noether and therefore we deduce the index of the extended operator $\bar{A}$.

Keywords: Noether Theory, Noetherization, Third Kind Integral Equation, Singular Linear Integro-Differential Operator, Deficient Numbers, Index of the Operator

1. Introduction

Using well known noether theory devoted to various integro-differential operators defined by some integral equations of the third kind studied in some scientific researches, it has been established the conditions of solvency of the considered integral equations in terms of the conditions of the orthogonality of linear dependance of the solutions of the homogeneous associated equation in the associated space. Full details for example can be found in the following references [11, 12, 21, 23].

The process of the construction of noether theory for integro-differential operators defined by some types of third kind integral equations in some specific functional spaces, which we may call as noetherization, lead us to indicated the conditions of the solvability of the considered integral equation besides the establishment of the noetherity of the investigated operator. In some special studied cases, we faced difficulties related to the investigation of the solvability of integral equations of the third kind while constructing noether theory for operator defined by such integral equations. Many times, we showed the necessity to well choose the necessary approaches that lead us to the noetherization of the studied operator. We recall that different wellknown scientists in their researches have undertook specific and needed approaches when realizing the noetherization of an integro-differential operators as mentioned. We can enumerate among others, the normalization method, the method of hypersingular integrals and the method of approximative inverses operators. The illustration of such approaches with full details can be found in the following references [1, 2, 6, 7, 8, 10-13, 18].

In the recent published papers [32, 33], Abdourahman investigated the noetherization of a more general case of an integro-differential operator defined by a third kind integral equation having a main part as a linear differential operator denoted L. It has been applied the approach indicating the necessity of the construction of the continuity of the regularizators to reach the noetherity of the considered operator devoted to the researches undertook by scientist Yurko are illustrated in the papers [30, 31].

Some mathematicians as Bart G. R, Warnock R. L, Shulaia D, and Gobbassov N. S devoted various papers related to some linear integral equations of the third kind winthin the framework of investigation of conditions of their solvency or noetherisation of the considered operators in their works [4, 5, 8, 9, 14].

Among many others also, we can note a special theorem dedicated to the investigation for solvency of certains types of third kind integral equations which has been formulated by Picard E in Comptes Rendus 150, 489–491 (1910). Full details on such work may be found in the reference [3].

Besides analytical methods of investigations of some types of integral equations, we do not forget to bring out the special approach in connexion with the approximative methods of solving Fredholm integral equations in some specific functional spaces, which has been used by the russian mathematician Gobbassov N. S. in his researches, conducted within the Metody Resheniya integral'nykh uravnenii Fredgol'ma v prostranstvakh obobshchennykh funktsii (Methods for Solving Fredholm Integral Equations in Spaces of Distributions). The previous described method is located with the full explanations in [16].

A special work dedicated to the Collocation Method for Integral Equations of the Third Kind executed by Gobbassov N. S was illustrated in the field of differential equations which is located in [15].

In our previous published papers, we have already established and constructed the noether theory for the finite dimensional Dirac-delta extensions of the following integro-differential operator defined by a third kind linear singular integral equation of the following form:

$$(A\varphi)(x) = x^p\varphi'(x) + \int_{-1}^{1} K(x,t)\,\varphi(t)dt = f(x); x \in [-1,1],$$

with the unknown function $\varphi \in C_{-1}^{1}[-1,1]$, the second right hand side $f(x) \in C_0^{\{p\}}[-1,1]$ and the kernel $K(x,t) \in C_0^{\{p\}}[-1,1]$ X C $[-1,1]$ extending the space $C_{-1}^{1}[-1,1]$ to the spaces $D_m = C_{-1}^{1}[-1,1] \oplus \left\{\sum_{k=0}^{m} \alpha_k \delta^{\{k\}}(x)\right\};\ 0 \le m \le p-2$ or $D_m = C_{-1}^{1}[-1,1] \oplus \left\{\sum_{k=0}^{m} \alpha_k \delta^{\{k\}}(x)\right\};\ m > p-2$ depending of cases. For full details of such researches, see [21, 28].

Therefore, it is interesting for us to illustrate the obtained theoretical results by constructing practical examples which are to be investigated within this work by applying theory.

In the present paper, we construct two illustrative examples of a noetherized operator into the two mentioned upstairs functional spaces.

Namely here, we consider for investigation the integro-differential operator defined by a third kind integral equation of the following type for which we construct two illustrative examples.

$$(A\varphi)(x) = x^p\varphi'(x) + \int_{-1}^{1} K(x,t)\,\varphi(t)dt = f(x); x \in [-1,1] \quad (1)$$

where $f(x) \in C_0^{\{p\}}[-1,1]$ and $\varphi(t) \in \varphi \in C_{-1}^{1}[-1,1], K(x,t) \in C_0^{\{p\}}[-1,1]$ X C $[-1,1]$.

Theoretically, we have studied and established the noetherity of the extended integro-differential defined by (1), see our published papers [21, 28].

Recall that the theoretical investigation for noetherity conducted for such operator lead us to focus our approaches to reach the the construction of the continuity of the regularisators, while moving from the interval [0,1], with the goal to cover for noetherity the whole closed interval [-1,1]. See papers [12, 30, 31, 32, 33].

Within this work, we calculate and determine the deficient numbers $\alpha(\bar{A})$ and $\beta(\bar{A})$ and also the index $\varkappa(\bar{A})$ of the operator $\bar{A}$ depending of the first case studied. The second case may be obtained analogically.

The paper is organized as follow: firstly, we present in section 2 all the necessary preliminaries related to the concept and the notions of well known noether theory widely used from various books in connexion with operators theory. Section 3 is properly devoted to the presentation of the two considered examples to be investigated depending of the cases $0 \le m \le p-2$ and $m > p-2$. Next we then and lastly, summarize the content of the work in section 4 titled conclusion, followed by some necessary recommendations for the follow-up or future scientific works to undertake, stated in section 5.

2. Preliminaries

Before presenting in details our main results, we recall the following definitions and concepts well known, and previously used, from the noether theory of operators which can be found also in the following references [6, 10, 11, 12, 21, 22].

By the way, we briefly review this important notions of Taylor derivatives, which is widely used when constructing noether theory of some integro-differential operators in specific generalized functional spaces. They will appear within the whole paper dealing with the investigated considered examples.

Definition 2.1 Function $\varphi(x) \in C[-1,1]$ admits at the point $x = 0$ Taylor derivative up to the order $p \in \mathbb{N}$ if there exists recurrently for $k = 1,2, \dots, p$, the following limits:

$$\varphi^{\{k\}}(0) = k! \lim_{x\to 0} x^{-k} \left[\varphi(x) - \sum_{j=0}^{k-1} \frac{\varphi^{\{j\}}(0)}{j!} x^j\right]. \quad (2)$$

The class of such functions $\varphi(x)$ is noted $C_0^{\{p\}}[-1,1]$.

Related to the kernel of the integro-differential operator let us mention that the kernel $k(x,t) \in C_0^{\{p\}}[-1,1] \, X C[-1,1]$, if and only if $k(x,t) \in C[-1,1] \, X \, C[-1,1]$ and admits Taylor derivatives according to the variable *x* at the point (0,*t*) whatever $t \in [-1,1]$.

The following important concepts are widely used while investigating the noetherity nature of an integro-differential operator defined by third kind integral equation in some specific classes of generalized functions.

A) Associated operator and associated space.

Definition 2.2. We state that the Banach space $E' \subset E^*$ is called associated space with the space E, if

$$|(f,\varphi)| \leq c\|f\|_{E'} \|\varphi\|_E \quad (3)$$

for every $\varphi \in E, f \in E'$.

We note that the initial space E can be considered associated with the space E'. Moreover, the norm $\|f\|_{E'}$ is not obliged to be equivalent to the norm $\|f\|_{E^*}$.

Let be noted $\mathcal{L}(E_1, E_2)$ the Banach algebra of all linear bounded operators from E_1 into E_2.

Definition 2.3. Let $E_j, j = 1,2$ two Banach spaces and E'_j their associated spaces. The operators $A \in \mathcal{L}(E_1, E_2)$ and $A' \in \mathcal{L}(E'_2, E'_1)$ are called associated, if

$$(A'f, \varphi) = (f, A\varphi) \quad (4)$$

for all $f \in E'_2$ and $\varphi \in E_1$.

By defining the concept of associated space and associated operator, we used the work [22].

Further we will also use some concepts of noether theory and wellknown results on some classes of singular equations with their particularities took from the references [17, 18, 19, 20, 27, 29].

For the operator $A \in \mathcal{L}(E_1, E_2)$ we put $\alpha(A) =$ dimker A $-$ the number (of linearly independent) zero of the operator A; and $\beta(A) =$ dimcokerA $-$ the number of zero of the conjugate operator in the conjugate space; $\chi(A) = \alpha(A) - \beta(A)$ $-$ the index of the operator.

In the case when $\alpha(A)$ and $\beta(A)$ are finite, and the image of the operator A closed in E_2, then the operator A is called noether operator.

It seems that, we can formalise the noetherity in terms of associated operator and associated space. See [22, 26].

Lemma 2.1 Let $E_j, j = 1,2$ two Banach spaces and E'_j their associated spaces and, let $A \in \mathcal{L}(E_1, E_2)$ and $A' \in \mathcal{L}(E'_2, E'_1)$ be associated noether operators and more,

$\alpha(A) = -\alpha(A')$.

Then, for the solvency of the equation $A\varphi = f$ it is necessary and sufficient that $(f, \psi) = 0$ for all solutions of the homogeneous associated equation $A'\psi = 0$.

B) Pair of associated spaces

We give the following definition.

Definition 2.4 Let $x_0 \in [-1,1]$. Through $C^1_{x_0}[-1,1]$ we represent the set of functions from $C^1[-1,1]$ verifying the condition $\varphi(x_0) = 0$.

It is clear that, $C^1_{x_0}[-1,1]$ is a Banach subspace in the space $C^1[-1,1]$, if remarking, that for $\varphi_n(x_0) \in C^1_{x_0}[-1,1]$ the convergence by norm $C^1[-1,1]$ conducts $\varphi_n(x_0) \to \varphi(x_0), n \to \infty$, that, with respect to $\varphi_n(x_0) = 0$ for all $n \in \mathbb{N}$ leads us to $\varphi(x_0) = 0$.

Now let us state this important lemma.

Lemma 2.2 The space $C^1_{x_0}[-1,1]$ is associated to the space $C[-1,1]$.

Proof: It is sufficient to ensure that for the regular functional (f, φ), where $f \in C^1_{x_0}[-1,1]$ and $\varphi \in C[-1,1]$ it is taking place the approximation of the form (3):

$$|(f,\varphi)| \leq c\|f\|_{C^1_{x_0}[-1,1]} \|\varphi\|_{C[-1,1]}$$

with some constant $c > 0$. The last is obvious as

$$|(f,\varphi)| = \left|\int_{-1}^{1} f(x)\varphi(x)dx\right| \leq$$
$$2\max_{-1\leq x\leq 1}|f(x)| \cdot \max_{-1\leq x\leq 1}|\varphi(x)| \cdot \quad (5)$$

That is what was required.

Let make a remark. From the approximation (5) it can be seen that, the associated with the space $C[-1,1]$ should be the spaces $C^1[-1,1]$ and $C[-1,1]$, as in (5) it has not been used the approximation of the derivative and the value at the point x_0.

Therefore, we can, narrowing the associated space, pick up which one is convenient for our further goals. Namely for that reason in the lemma 2.2, it is featured the space $C^1_{x_0}[-1,1]$.

Definition 2.5 Through $P^1 = P^{1,\{p\}}_{1,0}[-1,1]$ we note the space of generalized functions $\psi(x)$ on the subspace of test functions $C_0^{\{P\}}[-1,1]$ such that,

$$\psi(x) = \frac{z(x)}{x^p} + \sum_{k=0}^{p-1} \beta_k \delta^{\{k\}}(x), \quad (6)$$

where $z(x) \in C_0^{\{P\}}[-1,1] \cap C^1_{-1}[-1,1]$, $\beta_k -$ arbitrary constants $\delta^{\{k\}}(x) -$ *k-th* Taylor derivative of Dirac delta function which can be understood in the following way

$$(\delta^{\{k\}}(x), \varphi(x)) = \int_{-1}^{1} \delta^{\{k\}}(x) \, \varphi(x) dx = (-1)^k \varphi^{\{k\}}(0).$$

In the space P^1 let introduce the norm

$$\|\psi\|_{P^1} = \|z\|_{C_0^{\{P\}}[-1,1]} + \|z\|_{C^1[-1,1]} + \sum_{k=0}^{p-1}|\beta_k|, \quad (7)$$

We note that under $p = 1$ the expression of the norm can be writing in a more simple way.

In fact, from the equality $z(x) = x^p(N^p z)(x) + \sum_{k=0}^{p-1} \frac{z^{\{k\}}(0)}{k!} x^k$ it follows, that (under $p \in \mathbb{N}$)

$$\|z(x)\|_{C[-1,1]} \leq \|N^p z\|_{C[-1,1]} + \sum_{k=0}^{p-1}|z^{\{k\}}(0)| = \|z(x)\|_{C_0^{\{P\}}[-1,1]}.$$

In the case $p = 1$ we have:

$$\|z\|_{C_0^{\{1\}}[-1,1]} = \left\|\frac{z(x)-z(0)}{x}\right\|_{C[-1,1]} + |z(0)| \leq \|z\|_{C^1[-1,1]},$$

so that under $p = 1$:

$$\|z\|_{C[-1,1]} \leq \|z\|_{C_0^{\{1\}}[-1,1]} \leq \|z\|_{C^1[-1,1]}. \quad (8)$$

From (7) it follows that under $p = 1$ the norm in (8) can be defined in the following way:

$$\|\psi\|_{P^1} = \|z\|_{C^1[-1,1]} + |\beta_0| \quad (9)$$

Theorem 2.1 The space P^1 is a Banach space associated with $C_0^{\{P\}}[-1,1]$.

Proof: The fact that P^1 is a Banach space follows from the definition (7) where the norm in P^1 is defined as sum of the norms in the Banach spaces $C_0^{\{P\}}[-1,1]$ and $C^1[-1,1]$ with addition of a norm of finite-dimensional space.

The fact that $|(f,\psi)| \leq c\|f\|_{C_0^{\{P\}}[-1,1]}\|\psi\|_{P^1}$ can be obtained analogously as done in [21].

Next, we note that, it is obvious to see that $\|\varphi\|_{C[-1,1]} \leq \|\varphi\|_{C_0^{\{p\}}[-1,1]}$.

Finally, note that from the definition 2.1 it follows the following fact, if $\varphi(x) \in C[-1,1]$, then $x^p\varphi(x) \in C_0^{\{p\}}[-1,1]$. This assertion may be generalized as follows.

Lemma 2.2. Let $p \in \mathbb{N}, s \in \mathbb{Z}_+$. If $\varphi(x) \in C_0^{\{s\}}[-1,1]$ then, $x^p\varphi(x) \in C_0^{\{p+s\}}[-1,1]$, and the formula holds

$$\left(x^p\varphi(x)\right)^{\{j\}}(0) = \begin{cases} 0, j = 0,1,\ldots,p-1, \\ \frac{j!}{(j-p)!}\varphi^{\{j-p\}}(0), j = p,\ldots,p+s. \end{cases} \quad (10)$$

Proof. Note that a stronger assumption on the function $\varphi(x)$, such that $\varphi(x) \in C_0^{\{p+s\}}[-1,1]$ would allow us to easily prove the lemma just by applying Leibniz formula.

Note that Lemmas 2.1 and 2.2 imply the next lemma.

Lemma 2.3. Let $f(x) \in C_0^{\{p\}}[-1,1], p \in \mathbb{N}$ and $f(0) = \ldots\ldots = f^{\{r-1\}}(0) = 0, 1 \leq r \leq p$. Then $\frac{f(x)}{x^r} \in C_0^{\{p-s\}}[-1,1]$.

It is also convenient to use an equivalent definition for the norm in $C_0^{\{p\}}[-1,1]$:

$$\|\varphi\|^1_{C_0^{\{p\}}[-1,1]} = \sum_{j=0}^{p}\left\|N^j\varphi\right\|_{C[-1,1]} \quad (11)$$

It is easy to verify the equivalence of the norms (7) and (11). Namely, we always have $\left|\varphi^{\{j\}}(0)\right| \leq j!\left\|N^j\varphi\right\|_{C[-1,1]}$ which gives the estimate $\|\varphi\| \leq c\|\varphi\|^1$. To obtain the inverse estimate, we use the next following equality below.

$$(N^j\varphi)(t) = t^{p-j}(N\varphi)(t) + \sum_{s=j}^{p-1}\frac{t^{s-j}}{s!}\varphi^{\{s\}}(0), \quad (12)$$

from which $\left\|N^j\varphi\right\|_{C[-1,1]} \leq \|\varphi\|, j = 0,1,\ldots\ldots,p-1$ and, then $\|\varphi\|^1 \leq c\|\varphi\|$.

Lemma 2.4. The operator $N^p: C_0^{\{p\}}[-1,1] \longrightarrow C[-1,1]$ has the following properties:

N^p is bounded, and $\|N^p\varphi\|_{C[-1,1]} \leq \|\varphi\|_{C_0^{\{p\}}[-1,1]}$;

N^p is right invertible;

$\alpha(N^p) = p$, where $\alpha(N^p)$ is the dimension of the null subspace for N^p.

For the proof of the previous lemma with full details we can refer to references [21, 23].

Next, in the following section, we will present the two models of situation of an extension of an already noetherized operator by adding finite dimensional Dirac-delta functions, to be investigated for noetherity respectively with consideration of the generalized functional spaces, in which investigations are are conducted.

3. Main Results

In this section, we undertake properly the illustration of the noetherity investigation of a realized extension with Dirac-delta functions of an already established noether operator in some generalized functional spaces. More details and explanations related to such finite dimensional extensions and construction of noetherity theory, are found in the papers [11, 12, 21, 23, 24, 25, 28].

Namely, here we consider as model already theoretically investigated, the integro-differential operator defined by the next formula (13).

I) Case when $0 \leq m \leq p-2$ and the space $D_m = D_0$.

$$(\bar{A}\,\varphi)(x) = x^2\varphi'(x) + \int_{-1}^{1} t\varphi(t)dt = f(x) \quad (13)$$

And we will suppose that $\varphi(x) \in D_0$ $f(x) \in C_0^{\{2\}}[-1,1]$so that$m = 0, p = 2,$ and $m = p-2$ which corresponds to the case of general studied noetherization investigated in our recent published following papers [21, 28].

In this case, we have:

$$\varphi(x) = \varphi_0(x) + \alpha\delta(x), \quad (14)$$

where we take the function $\varphi_0(x) \in C_{-1}^1[-1,1]$.

From the form of the function $\varphi(x)$ and next, substituting into (13) we obtain

$$x^2(\varphi'_0(x) + \alpha\delta'(x)) + \int_{-1}^{1} t(\varphi_0(t) + \alpha\delta(t))\,dt = f(x) \quad (15)$$

or the same as

$$x^2\varphi'_0(x) + \int_{-1}^{1} t\varphi_0(t)dt = f(x) \quad (16)$$

from where we see that it is necessary, that

$$f(0) = \int_{-1}^{1} t\varphi_0(t)dt = c \quad (17)$$

and next

$$\varphi'_0(x) = \frac{f(x) - f(0)}{x^2} \quad (18)$$

Taking into account that, $f(x) \in C_0^{\{2\}}[-1,1]$, we may represent $f(x)$ in the form of

$$f(x) = f(0) + xf^{\{1\}}(0) + x^2 g(x) \tag{19}$$

that is allowing us to rewrite (18) in the following way:

$$\varphi'_0(x) = \frac{f^{\{1\}}(0)}{x} + g(x), \tag{20}$$

and by the virtue of the continuity of $\varphi'_0(x)$ it is necessary to require that

$$f^{\{1\}}(0) = 0. \tag{21}$$

Under assumption of this condition $\varphi'_0(x) = g(x)$ and this lead us to

$$\varphi_0(x) = \int_{-1}^{x} g(t)dt + c_1 = \int_{-1}^{x} g(t)dt, \tag{22}$$

where it has been taking into account that $\varphi_0(-1) = 0$.

It remains for us to satisfy the arising within the process of the solution the conditions (17) and (21). We have the following results:

$$f(0) = c = \int_{-1}^{1} t\varphi_0(t)dt = \int_{-1}^{1} tdt \int_{-1}^{1} g(s)ds = \frac{1}{2}\int_{-1}^{1}(1-s^2)g(s)ds\,. \tag{23}$$

Recalling how are connected $f(x)$ and $g(x)$, we obtain the following:

$$f(0) = \frac{1}{2}\int_{-1}^{1} \frac{f(s) - f(0) - sf^{\{1\}}(0)}{s^2}(1-s^2)ds. \tag{24}$$

Considering (21), we finally find that:

$$2f(0) = \int_{-1}^{1} \frac{f(s)-f(0)}{s^2}(1-s^2)ds = \int_{-1}^{1} \frac{f(s)}{s^2}ds - f(0)\, F.p.\int_{-1}^{1} \frac{f(s)}{s^2}ds - \int_{-1}^{1} f(s)ds + f(0)\int_{-1}^{1} ds \tag{25}$$

From that with respect to the fact that $F.p.\int_{-1}^{1}\frac{ds}{s^2} = 2$, we have

$$(f, \frac{1}{s^2} - 2\delta(s) - 1) = 0. \tag{26}$$

Remark that, we may have obtained (26) using the associated operator and N^2:

$$(N^2 f, \psi) = (f, (N^2)'\psi), \text{ where } \psi(s) = 1 - s^2.$$

Therefore, we have two conditions of solvency (21) and (26).

All what has been done lead us therefore to conclude: $\beta(\bar{A}) = 2$. It is not difficult to see that from the previous, $\alpha(\bar{A}) = 1$. It follows from the fact that the homogeneous equation in (16) has only zero solution.

Additionally, from (16) it follows that $\varphi'_0(x) = \frac{c}{x^2}$ from which it is necessary $\varphi'_0(x) = 0$ or $\varphi_0(x) = c_1 = 0$ with respect to $\varphi_0(-1) = 0$.

Therefore, we deduce the index of the extended operator $\bar{A}$ defined by the formula $\chi(\bar{A}) = \alpha(\bar{A}) - \beta(\bar{A}) = -1$.

Now, let us consider the associated homogeneous equation of the following form:

$$(\bar{A}'\,\psi)(x) = -(x^2\psi)' + x\int_{-1}^{1}\psi(t)dt = 0, \tag{27}$$

where it is supposed that, $\psi(t) \in P^1$, i.e: $\psi(t) = \frac{z(t)}{t^2} + \sum_{i=0}^{1} w_i\, \delta^{\{i\}}(x)$ and $z(t) \in C_1^1[-1,1]$. Substitution in (27) gives us

$$-z'(x) + x(F.p.\int_{-1}^{1}\frac{z(t)}{t^2}dt + w_0) = 0, \tag{28}$$

If we denoted temporarily $c = F.p.\int_{-1}^{1}\frac{z(t)}{t^2}dt$, then $z'(x) = x(c + w_0)$

or $z(x) = \frac{x^2}{2}(c + w_0) + c_1$. With respect to the condition $z(1) = 0$, we find

$$z(x) = \frac{(c+w_0)}{2}(x^2 - 1) \tag{29}$$

Now, let us find the unknown c. We have the following:

$$c = F.p.\int_{-1}^{1}\frac{z(t)}{t^2}dt = \frac{(c+w_0)}{2}F.p.\int_{-1}^{1}\frac{t^2-1}{t^2}dt = 0\,. \tag{30}$$

Therefore, we obtain $z(x) = \frac{w_0}{2}(x^2 - 1)$.

So that, definitively the solution $\psi(x)$ of the homogeneous equation has the following form (with consideration to the arbitrariness in the choice of w_1)

$$\psi(x) = \frac{w_0}{2}\left(\frac{x^2-1}{x^2} + 2\delta(x)\right) + w_1\delta^{\{1\}}(x), \tag{31}$$

So therefore we see, that the conditions (21) and (26) are the conditions of the orthogonality of linearty independance of the solutions of the homogeneous equation $\bar{A}'\,\psi = 0$ in $\bar{P}^1$ of the form

$$\psi_1(x) = \delta^{\{1\}}(x) \text{ and } \psi_2(x) = \frac{x^2-1}{x^2} + 2\delta(x). \tag{32}$$

Note that in the presentation (26) obviously the function $z(x)$ is defined as follows $z(x) = x^2 - 1 \in C_1^1[-1,1]$.

Next, we move to the following illustrative example in section II°).

II) Case when $m > p - 2$ and the space $D_m = D_1$.

We now move to the following type of example when the extended operator $\bar{A}$ as previously has the form defined by (13) but at this time $\varphi(x) \in D_1$ and $f(x) \in C_{0,1}^{\{2\}}[-1,1]$.

This is meaning that:

$$\varphi(x) = \varphi_0(x) + \alpha_0\delta(x) + \alpha_1\delta^{\{1\}}(x),$$

$f(x) = f_0(x) + \beta\delta(x)$ and $f_0(x) \in C_0^{\{2\}}[-1,1]$.

Substitution of $\varphi(x)$ into (13) gives us the following result:

$$x^2(\varphi'_0(x) + \alpha_0\delta'(x) + \alpha_1\delta''(x)) + \int_{-1}^{1} t\varphi_0(t)dt - \alpha_1 = f_0(x) + \beta\delta(x) \tag{33}$$

or the same as:

$$x^2\varphi'_0(x) + 2\alpha_1\delta(x) + \int_{-1}^{1} t\varphi_0(t)dt - \alpha_1 = f_0(x) + \beta\delta(x). \tag{34}$$

From the linear dependence of the functions, it follows the disintegration of this equation into the system:

$$\begin{cases} x^2\varphi'_0(x) + c - \alpha_1 = f_0(x), \\ 2\alpha_1 = \beta \end{cases} \quad (35)$$

where it is denoted

$$c = \int_{-1}^{1} t\varphi_0(t)dt. \quad (36)$$

It is clear that, $\alpha_1 = \frac{\beta}{2}$ and turning to the first equation of the system we note that $f_0(0) = c - \alpha_1$. From that we find out:

$$\varphi'_0(x) = \frac{f_0(x)+\alpha_1-c}{x^2} = \frac{f_0(0)+xf_0^{\{1\}}(0)+x^2g(x)+\alpha_1-c}{x^2} = \frac{xf_0^{\{1\}}(0)+x^2g(x)}{x^2}. \quad (37)$$

By virtue of the continuity of $\varphi'_0(x)$, we should require that $f^{\{0\}}(0) = 0$ and that is giving $\varphi'_0(x) = g(x)$ or:

$$\varphi_0(x) = \int_{-1}^{x} g(t)dt, \quad (38)$$

where it is considered, that $\varphi_0(-1) = 0.$ It remains to turn down to the definition $c = \int_{-1}^{1} t\varphi_0(t)dt$.

We have the following result:

$$c = \int_{-1}^{1} tdt \int_{-1}^{t} g(s)ds = \frac{1}{2}\int_{-1}^{1}(1-s^2)g(s)ds = f_0(0) + \frac{\beta}{2}. \quad (39)$$

Recalling the connexion between $g(s)$ and $f(s)$ we find out:

$$2f_0(0) + \beta = \int_{-1}^{1} \frac{f_0(s)-f_0(0)}{s^2}(1-s^2)ds, \quad (40)$$

From that, after some computations already done when investigating the first example lead us to the following:

$$\beta = \left(f_0(s), \frac{1}{s^2} + 2\delta(s) - 1\right). \quad (41)$$

If taking into account the relationship 3.13 from [23] then the previous relationship can be writing in the form of the condition of orthogonality of such following type:

$$\left(f_0(s) + \beta\delta(s), \frac{1}{s^2} + 2\delta(s) - 1\right) = 0, \quad (42)$$

and similar condition $f_0^{\{1\}}(0) = 0$ we write in the following form:

$$\left(f_0(s) + \beta\delta(s), \delta'(s)\right) = 0. \quad (43)$$

So therefore, we see that the conditions (42) - (43) are the orthogonality conditions of the linear dependence of the solutions of the homogeneous associated equation $\bar{A}'\psi = 0$ in the space $\bar{P}^1$.

(Remark that for the associated equation in this example we can use the same spaces $\bar{P}^1$ and $C[-1,1]$.).

Finally, let us move to the conclusion of the work in the following next section.

4. Conclusion

This achieved scientific work presents in full details the interconnexion between the theory and practise related to the investigation of the construction of noether theory (noetherization) for an extended noether operator defined by a third kind linear singular integral equation in some generalized functional spaces.

We have completely illustrated within these constructed examples the two situations arising when realizing the Dirac-delta Extensions of a noether operator in the generalized functional spaces $D_m = C_{-1}^1[-1,1] \oplus \{\sum_{k=0}^{m} \alpha_k \delta^{\{k\}}(x)\};\ 0 \le m \le p-2$ and $D_m = C_{-1}^1[-1,1] \oplus \{\sum_{k=0}^{m} \alpha_k \delta^{\{k\}}(x)\};\ m > p-2.$ After presenting some concepts and notions related to the noether theory for integro-differential operators, we applied theoretical results to investigate the noetherity of the extended operator depending of the finite dimensional cases of the generalized functional spaces considered $C_{-1}^1[-1,1]$.

Based on the two cases of studied noetherity construction for an extended noether integro-differential operator of such mentioned kind, we bring out clearly from these illustrative examples all the theoretical obtained results from our previous published papers.

5. Recommendations

Following what already have been done in our previous published researches, we should encourage one first of all, to construct more illustrative examples of the investigation of noetherity nature (noetherization) of some extended noether integro-differential operators defined by third kind integral equations in various generalized functional spaces. By this mean, we should make through these illustrative examples a clear interconnexion between theory and practical once more. Lastly, we will generalize our results of the investigation on such topic but at this time to the integro-differential operator defined by an integral equation of the third kind of the higher order of the following type:

$$(A\varphi)(x) = x^p\varphi^{(n)}(x) + \int_{-1}^{1} K(x,t)\,\varphi(t)dt = f(x); x \in [-1,1].$$

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