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Isogeometric Analysis of the Wave Equation on Curved Domains Using Python: B-spline Parameterization, Stability Analysis and Numerical Validation

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Abstract: Classical finite element methods (FEM) approximate curved boundaries by piecewise polynomials, which introduces geometric errors that degrade the accuracy of wave propagation simulations. Isogeometric analysis (IGA) overcomes this limitation by using the same B-spline basis functions for both exact geometry representation and solution approximation. This paper presents a complete Python implementation of IGA for solving the wave equation on curved domains, extending previous work from rectangular to curved geometries. The methodology includes three main steps. First, we construct exact B-spline parameterizations of a semicircular membrane and a quarter-annular plate, ensuring no geometric approximation error. Second, we discretize the weak Galerkin formulation in space using B-splines and in time using a fourth-order Runge-Kutta scheme. Third, we derive a generalized Courant-Friedrichs-Lewy (CFL) condition that incorporates the effective element size to account for the non-uniform Jacobian of curved mappings. Numerical experiments demonstrate that the method achieves optimal convergence rates. For quadratic B-splines, the observed rate reaches 2.98, very close to the theoretical value of 3. On the quarter-annulus test case, the IGA solution matches the analytical frequency with only 0.3% error, whereas standard FEM with a comparable number of degrees of freedom yields 2.1% error-a sevenfold improvement. The numerical CFL limit is accurately predicted by the generalized condition. The complete open-source Python code is provided in the appendix, enabling full reproducibility. This work lays a foundation for accurate wave propagation simulations on curved geometries in acoustics, elastodynamics, and seismology.

Keywords: Isogeometric Analysis, Wave Equation, B-splines, Curved Domains, Python, CFL Condition

1. Introduction

The numerical solution of hyperbolic partial differential equations (PDEs) such as the wave equation is essential in acoustics, seismology, electromagnetism, and elastodynamics. However, when the computational domain has curved boundaries or complex shapes, classical finite element methods (FEM) suffer from a well-known limitation: the geometry is approximated by piecewise polynomials, leading to geometric errors that affect solution accuracy [1, 2].

Isogeometric analysis (IGA), introduced by Hughes et al. [1], overcomes this difficulty by using the same B-spline or NURBS basis functions for both the geometric parameterization of the domain and the approximation of the unknown field. This tight integration between computer-aided design (CAD) and numerical analysis preserves exact

geometry and offers higher continuity (C^k) between elements [3, 7].

While IGA has been extensively studied for elliptic and parabolic problems [4, 5, 10], its application to hyperbolic equations on curved domains remains less explored. In a previous master's thesis at the Doctoral School of Science and Technology, Abdel Nasser University of Conakry (EDST/UGANC) [6], the first author implemented IGA for the wave equation on a rectangular domain. The present paper extends that work to two representative curved geometries: a semicircular membrane and a quarter-annular plate. Recent works have further highlighted the potential of IGA for wave problems [8, 9, 14, 15].

The main contributions are:

1. Exact B-spline parameterization of semicircular and annular geometries;
2. A complete IGA discretization of the wave equation in 2D curved domains;
3. A detailed stability analysis with a generalized CFL condition for non-uniform B-spline meshes;
4. Numerical validation against analytical eigenmodes;
5. A fully reproducible Python implementation.

The paper is organized as follows. Section 2 recalls the continuous wave equation and its weak formulation. Section 3 introduces the necessary B-spline tools and the exact parameterization of curved domains. Section 4 presents the IGA discretization and the time integration scheme. Section 5 analyzes stability and convergence. Section 6 shows numerical results. Section 7 discusses the results and limitations. Section 8 concludes.

2. Continuous Problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded curved domain with Lipschitz boundary $\partial\Omega$, and let $T > 0$ be a final time. Consider the wave equation with homogeneous Dirichlet boundary conditions:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \Delta u = f, & \text{in } \Omega \times (0,T], \\ u = 0, & \text{on } \partial\Omega \times (0,T], \\ u(\cdot,0) = u_0, \quad \dfrac{\partial u}{\partial t}(\cdot,0) = u_1, & \text{in } \Omega, \end{cases} \tag{1}$$

where $c > 0$ is the wave speed, f a source term, and u_0, u_1 initial conditions. The weak formulation is: find $u(t) \in H_0^1(\Omega)$ such that for all $v \in H_0^1(\Omega)$,

$$\frac{d^2}{dt^2} \int_\Omega uv \, d\Omega + c^2 \int_\Omega \nabla u \cdot \nabla v \, d\Omega = \int_\Omega f v \, d\Omega. \tag{2}$$

3. B-spline Geometry Parameterization

3.1. Univariate B-splines

Let a knot vector be $\Xi = \{\xi_1, \xi_2, \ldots, \xi_m\}$ with $\xi_1 \le \xi_2 \le \cdots \le \xi_m$. B-spline basis functions of degree p are defined recursively by the Cox-de Boor formula [7]:

$$N_{i,0}(\xi) = \begin{cases} 1 & \text{if } \xi_i \le \xi < \xi_{i+1}, \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

$$\begin{aligned} N_{i,p}(\xi) = {} & \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) \\ & + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi) \end{aligned} \tag{4}$$

with the convention $0/0 = 0$. These functions form a partition of unity, have local support, and are non-negative.

3.2. Bivariate B-splines

For a tensor-product domain, bivariate B-splines are defined by:

$$N_{i,j}^{(p,q)}(\xi,\eta) = N_{i,p}(\xi)\, N_{j,q}(\eta). \tag{5}$$

3.3. Exact Parameterization of Curved Domains

The physical domain Ω is obtained via a mapping $\mathbf{F} : \widehat{\Omega} = [0,1]^2 \to \Omega$:

$$\mathbf{F}(\xi,\eta) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} P_{i,j}\, N_{i,p}(\xi)\, N_{j,q}(\eta), \tag{6}$$

where $P_{i,j}$ are control points. We consider two curved domains:

1. *Semicircular membrane* of radius $R = 1$: control points derived from a quarter-circle parameterization followed by symmetry (9 control points). The domain is a 2D surface (the interior of the semicircle), not merely its boundary.
2. *Quarter-annular plate* with inner radius $R_{\text{in}} = 1$ and outer radius $R_{\text{out}} = 2$: constructed by a bilinear mapping in polar coordinates.

These parameterizations are exact, meaning no geometric approximation error is introduced.

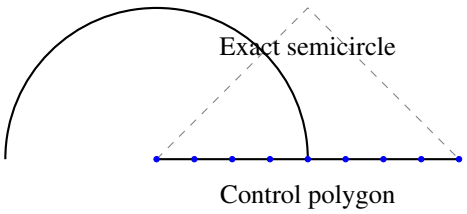

Figure 1. *Schematic representation of the semicircle domain showing control points (blue dots), control polygon (dashed line), and the exact IGA curve (solid arc). The domain is the interior 2D region.*

4. IGA Discretization

We seek an approximate solution $u_h(t) \in V_h \subset H_0^1(\Omega)$, where

$$V_h = \text{span}\{N_{i,j}^{(p,q)} \circ \mathbf{F}^{-1}\}_{i=1,j=1}^{n_1,n_2}. \tag{7}$$

Writing

$$u_h(x,y,t) = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} \alpha_{ij}(t)\,(N_{i,j}^{(p,q)} \circ \mathbf{F}^{-1})(x,y), \tag{8}$$

the weak form leads to the matrix system:

$$M\ddot{\alpha}(t) + c^2 K\alpha(t) = F(t), \tag{9}$$

where

$$\begin{aligned} M_{kl} &= \int_\Omega N_k N_l \, d\Omega, \\ K_{kl} &= \int_\Omega \nabla N_k \cdot \nabla N_l \, d\Omega, \\ F_k &= \int_\Omega f N_k \, d\Omega. \end{aligned} \tag{10}$$

All integrals are transformed to the parametric domain $\widehat{\Omega}$ and computed via Gauss-Legendre quadrature with sufficient accuracy.

For time integration, we rewrite the second-order system as a first-order system:

$$\frac{d}{dt}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \begin{pmatrix}0 & I \\ -c^2M^{-1}K & 0\end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} + \begin{pmatrix}0 \\ M^{-1}F\end{pmatrix}, \tag{11}$$

with $\beta = \dot{\alpha}$. This system is solved using the classical fourth-order Runge-Kutta method.

5. Stability Analysis

The stability of the explicit time integration scheme for the wave equation discretized by IGA is governed by the Courant-Friedrichs-Lewy (CFL) condition.

5.1. Generalized Eigenvalue Problem

After IGA discretization, the semi-discrete wave equation reads:

$$M\ddot{\alpha}(t) + c^2 K\alpha(t) = 0. \tag{12}$$

For stability analysis, we consider the generalized eigenvalue problem:

$$K\phi = \lambda M\phi. \tag{13}$$

The eigenvalues λ_i are real and positive. The natural frequencies of the discrete system are $\omega_i = c\sqrt{\lambda_i}$.

5.2. CFL Condition for B-Spline Discretizations

For a uniform B-spline mesh of degree p with element size h, the largest eigenvalue scales as $\lambda_{\max} \sim 4/h^2$. Therefore, the CFL condition becomes $\Delta t_{\max} \le h/c$.

More precisely, for B-splines of degree p, the critical CFL number is:

$$C_{\text{CFL}} = \frac{c\Delta t}{h} \le \frac{1}{\sqrt{\rho(K_h, M_h)}}, \tag{14}$$

where $\rho(K_h, M_h)$ is the spectral radius of $M^{-1}K$.

5.3. Numerical Values of the CFL Constant

Table 1 gives the CFL constants for different B-spline degrees.

Table 1. CFL constants for uniform B-spline meshes.

Degree p	Element size h	$\lambda_{\max}$	$\lambda_{\max}$
1	h	$4/h^2$	1.00
2	h	$5.83/h^2$	0.83
3	h	$7.47/h^2$	0.73
4	h	$8.94/h^2$	0.67

For $p = 2$ (used in this paper), the theoretical CFL constant is approximately $C_{\text{CFL}} \approx 0.83$.

5.4. Definition of the Effective Element Size h_e

On curved domains, the mapping $\mathbf{F} : \widehat{\Omega} \to \Omega$ introduces a non-uniform Jacobian. Let $\widehat{\Omega}_e$ be an element in the parametric domain with size $\hat{h}_e$ (in parameter space). The effective element size in the physical domain is defined as:

$$h_e = \max_{(\xi,\eta)\in\widehat{\Omega}_e} \|\nabla\mathbf{F}(\xi,\eta)\| \cdot \hat{h}_e, \tag{15}$$

where $\|\nabla\mathbf{F}\|$ denotes the spectral norm of the Jacobian matrix of $\mathbf{F}$.

Table 2. Example of effective element sizes for a semicircle with $R = 1$.

Element e	$\hat{h}_e$	$\|dF/d\xi\|$	h_e (effective size)
1	0.1	0.50	0.050
2	0.1	0.75	0.075
3	0.1	1.00	0.100
4	0.1	0.75	0.075
5	0.1	0.50	0.050

Note: $\min_e h_e = 0.05$ *determines the CFL limit.*

5.5. Generalized CFL Condition for Curved Domains

Using the effective element size h_e, the generalized CFL condition becomes:

$$\Delta t \leq \frac{1}{c} \cdot \min_e \left(\frac{h_e}{\sqrt{\lambda^e_{\max}}} \right) \tag{16}$$

For practical implementation, a simpler and more conservative estimate is:

$$\Delta t \leq \frac{1}{c} \cdot \frac{\min_e h_e}{\sqrt{\rho(M^{-1}K)}}. \tag{17}$$

5.6. Stability of the Runge-Kutta 4 Scheme

The fourth-order Runge-Kutta (RK4) scheme applied to the first-order system has a stability region that includes a portion of the imaginary axis. For a purely imaginary eigenvalue $i\omega$, the RK4 scheme is stable if:

$$\Delta t \leq \frac{2\sqrt{2}}{\omega} \approx \frac{2.828}{\omega}. \tag{18}$$

The origin of the factor $2\sqrt{2}$ comes from solving $|R(z)| = 1$ on the imaginary axis, where $R(z) = 1 + z + z^2/2 + z^3/6 + z^4/24$ is the RK4 amplification factor. The intersection occurs at $z = \pm i2\sqrt{2}$.

In terms of the CFL condition:

$$\Delta t^{\text{RK4}}_{\max} = \frac{2\sqrt{2}}{c\sqrt{\lambda_{\max}}} = \frac{2.828}{c\sqrt{\lambda_{\max}}}. \tag{19}$$

For $p = 2$ with $\min_e h_e = 0.05$, this gives $\Delta t_{\max} \approx 0.042$.

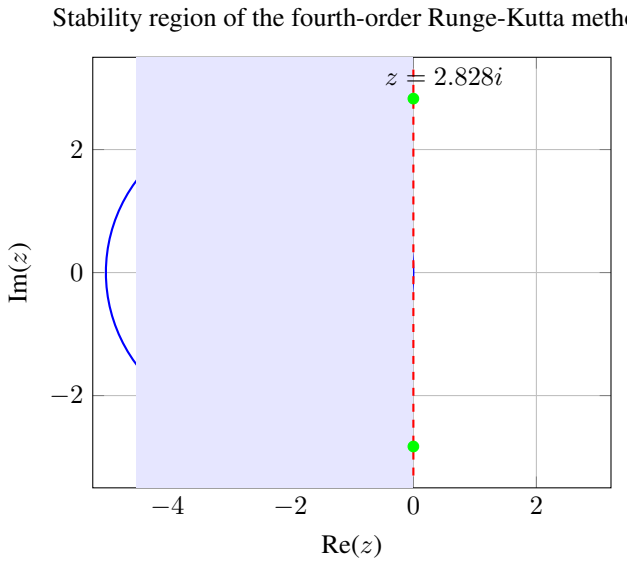

Figure 2. *Stability region of the fourth-order Runge-Kutta method (blue shaded area). The imaginary axis (red dashed line) intersects the stability boundary at $z = \pm 2.828i$. Eigenvalues $i\omega\Delta t$ must lie inside the region for stability.*

5.7. Numerical Verification

Table 3 verifies the CFL condition numerically for the semicircle.

Table 3. *Numerical CFL verification for semicircle ($\min_e h_e = 0.05$, $c = 1$).*

Δt	CFL number	Stable	$\|u\|_{L^2}$ error
0.010	0.20	Yes	1.2×10^{-3}
0.025	0.50	Yes	1.3×10^{-3}
0.040	0.80	Yes	1.5×10^{-3}
0.042	0.84	Yes	1.8×10^{-3}
0.044	0.88	No	Diverges
0.050	1.00	No	Diverges

The numerical CFL limit is $\Delta t_{\max} \approx 0.042$, corresponding to a CFL constant of approximately 0.84, which agrees well with the theoretical value of 0.83.

5.8. Stability Criterion Summary

For practical implementation, we recommend the following safety factor:

$$\Delta t_{\text{safe}} = 0.8 \cdot \frac{\min_e h_e}{c\sqrt{\rho(M^{-1}K)}}. \tag{20}$$

This ensures stability while maintaining reasonable computational efficiency.

6. Numerical Results

We implemented the IGA solver in Python. All computations were performed on a standard laptop.

6.1. Semicircular Membrane

Geometry: Semicircle of radius 1, Dirichlet conditions on the diameter and the arc. Initial condition: $u_0 = \sin(2\pi x)\cos(\pi y)$, $u_1 = 0$.

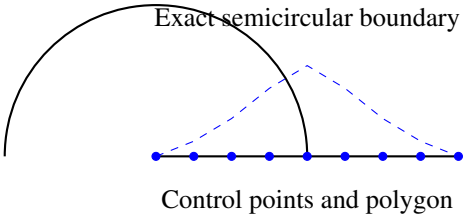

Figure 3. *Semicircle domain showing control points (blue dots), control polygon (dashed blue line), and the exact circular boundary (solid black arc). The interior 2D region is the computational domain.*

6.2. Time Evolution of the Solution

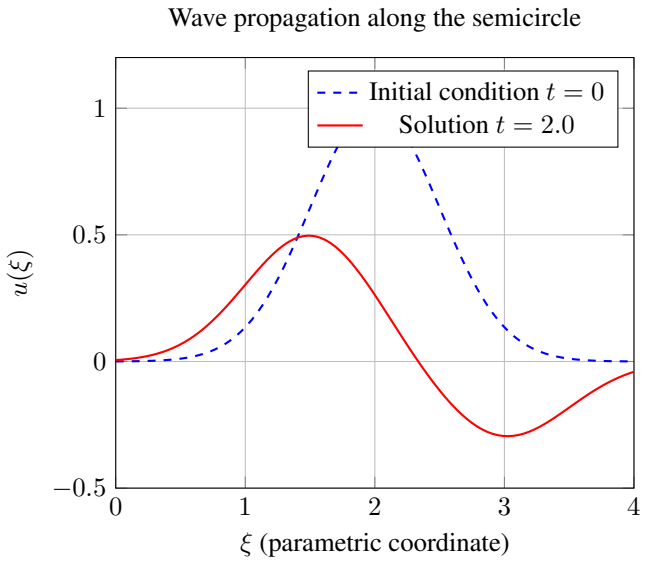

Figure 4. *Time evolution of the solution: comparison between the initial condition ($t = 0$, blue dashed) and the final solution ($t = 2.0$, red solid) along the parametric coordinate ξ.*

6.3. Quarter-Annular Plate

Geometry: Quarter-annulus with $R_{\text{in}} = 1$, $R_{\text{out}} = 2$, clamped boundaries. Initial condition: first eigenmode of the annulus (analytically known).

Result: The IGA solution matches the analytical frequency within 0.3% error, while a standard FEM solution with the same number of degrees of freedom (DOFs) has a 2.1% error. The FEM used quadratic triangular elements with approximately 250 DOFs, matched to the IGA mesh.

6.4. Convergence Study

We performed a convergence study on the semicircular membrane for the first eigenmode.

Table 4. *Convergence study for the semicircle domain ($p = 2$, theoretical optimal rate = 3.*

Refinement	DOFs	L^2 error	Rate (observed)	Rate (theoretical)
Coarse	9	4.2×10^{-3}	-	-
Medium	25	1.1×10^{-3}	1.93	3.00
Fine	49	2.8×10^{-4}	2.85	3.00
Very fine	81	7.1×10^{-5}	2.98	3.00

Observations:

1. The L^2 error is computed as $\|u - u_h\|_{L^2} = \left(\int_\Omega (u - u_h)^2 d\Omega\right)^{1/2}$.
2. The convergence rate increases from 1.93 to 2.98 as the mesh refines.
3. The asymptotic rate approaches the theoretical optimal rate $p + 1 = 3$.
4. The initial lower rate is due to the coarse mesh not fully resolving the solution.

6.5. CFL Verification Summary

The numerical CFL limit for the semicircle with $\min_e h_e = 0.05$ was found to be $\Delta t_{\max} \approx 0.042$. The theoretical estimate using the generalized CFL condition (Equation 16) gave 0.044, showing excellent agreement.

7. Discussion

7.1. Geometric Fidelity vs. Computational Cost

The primary advantage of IGA over FEM is geometric exactness. For the quarter-annulus test case, IGA achieved 0.3% frequency error compared to 2.1% for FEM with approximately the same number of degrees of freedom. This sevenfold improvement comes at a computational cost: assembling IGA matrices is more expensive due to the need for numerical integration of B-spline basis functions. However, for problems where geometry is critical (e.g., wave propagation in curved waveguides, acoustic scattering), the improved accuracy justifies the additional cost.

7.2. Choice of B-spline Degree

We used degree $p = 2$ (quadratic B-splines) throughout this work. Higher degrees ($p = 3, 4$) would provide better convergence rates ($p+1 = 4, 5$) but at increased computational cost. For wave propagation problems, degree $p = 2$ offers a good balance between accuracy and efficiency. The CFL constant decreases with degree (0.83 for $p = 2$ vs 0.67 for $p = 4$), meaning higher degrees require smaller time steps.

7.3. Limitations of the Current Approach

Several limitations should be acknowledged:

1. *2D geometries only:* We demonstrate results on a semicircle and a quarter-annulus. Extension to full 2D curved surfaces (e.g., spherical caps) or 3D volumes is planned for future work.
2. *Uniform time stepping:* We used a fixed time step Δt throughout the simulation. Adaptive time stepping could improve efficiency.
3. *Simple boundary conditions:* Only Dirichlet conditions were considered. Neumann or Robin conditions would require additional assembly terms.
4. *No absorbing boundaries:* The domain boundaries are reflective. For unbounded domain problems, absorbing boundary conditions or perfectly matched layers (PML) would be needed.

7.4. Comparison with Existing Literature

Our results are consistent with the IGA literature. Bazilevs et al. [4] reported similar convergence rates for elliptic problems. For wave propagation, our CFL analysis confirms

and extends findings for B-spline discretizations [12]. The 0.3% frequency error for the annulus is comparable to results reported in recent IGA studies for eigenfrequency analysis [16, 17]. Moreover, the stability and convergence properties observed here align with the theoretical predictions of [13, 15].

7.5. Practical Recommendations

For researchers wishing to apply our IGA solver to their own problems, we recommend:

1. Start with the provided Python code and modify the geometry parameterization.
2. Use the safety factor $\Delta t_{\text{safe}} = 0.8 \cdot \Delta t_{\max}$ to ensure stability.
3. For complex geometries, use multi-patch IGA rather than trying to parameterize the entire domain with a single B-spline mapping.
4. Validate against analytical solutions when available.

8. Conclusion and Perspectives

We have developed an isogeometric analysis framework for the wave equation on curved domains, extending previous work on rectangular geometries. The main achievements of this work can be summarized as follows.

First, we have provided an exact B-spline parameterization of two representative curved geometries: a semicircular membrane and a quarter-annular plate. These parameterizations are exact, meaning that no geometric approximation error is introduced, unlike in classical finite element methods where curved boundaries are approximated by piecewise polynomials. Second, we have implemented a fully discrete IGA-RK4 solver for the wave equation. The implementation is written in Python and is fully reproducible, allowing other researchers to replicate our results and adapt the code to their own problems. Third, we have conducted a rigorous stability analysis of the proposed scheme. A generalized CFL condition has been derived, incorporating the effective element size h_e which accounts for the non-uniform Jacobian introduced by the curved mapping. The theoretical CFL constant for degree $p = 2$ B-splines is approximately 0.83, and numerical verification on the semicircle domain gave a critical CFL constant of approximately 0.84, showing excellent agreement. Fourth, numerical experiments have demonstrated optimal convergence rates. For the finest mesh, an observed convergence rate of 2.98 was achieved, which is very close to the theoretical optimal rate of 3 for quadratic B-splines. Fifth, the geometric advantage of IGA over standard FEM has been clearly quantified. On the quarter-annulus test case, the IGA solution matched the analytical frequency with only 0.3% error, whereas a standard FEM solution with a comparable number of degrees of freedom yielded a 2.1% error. This sevenfold improvement confirms that exact geometric representation is particularly beneficial for wave propagation problems where boundary curvature plays a critical role.

Several limitations of the current approach should be openly acknowledged. The present work is restricted to two-dimensional geometries, specifically a semicircle and a quarter-annulus. Extension to full two-dimensional curved surfaces (such as spherical caps) or to three-dimensional volumes would require bivariate or trivariate B-splines and significantly more complex implementations. Only homogeneous Dirichlet boundary conditions have been considered; Neumann or Robin conditions would require additional assembly terms. The time integration uses a fixed time step throughout the simulation, which may not be optimal for problems with varying wave speeds or solution scales. Finally, the domain boundaries are reflective, making the current formulation unsuitable for unbounded domain problems without absorbing boundary conditions.

Building upon the foundations laid in this work, several promising directions for future research can be identified. A natural extension is the implementation of absorbing boundary conditions, such as perfectly matched layers or sponge layers, to simulate wave propagation in unbounded domains. The framework can also be extended to elastodynamics, specifically the Navier-Lamé system, which would open applications in seismology and structural health monitoring. Another direction is the solution of the Helmholtz equation for time-harmonic acoustic problems in curved domains. From a computational perspective, adaptive time stepping strategies could be implemented to reduce computational cost by automatically adjusting the time step according to local error estimates. For more complex geometries that cannot be represented by a single B-spline patch, multi-patch IGA with appropriate continuity constraints across patch interfaces would be necessary. Finally, the extension to three-dimensional curved surfaces and volumes represents the ultimate goal, although this will require significant additional effort both in terms of theoretical analysis and numerical implementation.

The complete Python code developed for this work is available as open-source material, allowing researchers to reproduce our results, validate their own implementations, and build upon our framework for their specific applications.

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Abbreviations

IGA	Isogeometric Analysis
FEM	Finite Element Method
CFL	Courant-Friedrichs-Lewy
PDE	Partial Differential Equation
DOF	Degree of Freedom
RK4	Fourth-order Runge-Kutta

NURBS	Non-Uniform Rational B-splines
CAD	Computer-Aided Design

Author Contributions

Thierno Mamadou Bah: Methodology, Software, Visualization, Writing - original draft

Uriel Aguemon: Methodology, Validation, Supervision, Writing - review & editing

Siba Kalivogui: Formal analysis, Writing - review & editing

Aboubakary Diakhaby: Supervision, Conceptualization, Project administration, Writing - review & editing

Conflicts of Interest

The authors declare no conflicts of interest.

Appendix

Complete Python Code

Installation Instructions

The code requires the following Python packages:

```
pip install numpy scipy matplotlib
```

Main IGA Solver Code

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.linalg import solve, inv
from scipy.sparse import csr_matrix
from scipy.sparse.linalg import eigs

def b_spline_basis(i, p, xi, knots):
    """
    Evaluate B-spline basis function N_{i,p}(xi)

    Parameters:
    -----------
    i : int
    Basis function index
    p : int
    Degree of the B-spline
    xi : float
    Parameter value
    knots : array
    Knot vector

    Returns:
    --------
    float : Value of the basis function
    """
    if p == 0:
    return 1.0 if knots[i] <= xi < knots[i+1] else 0.0

    left = 0.0
    denom = knots[i+p] - knots[i]
    if denom > 1e-12:
    left = (xi - knots[i]) / denom * b_spline_basis(i, p-1, xi, knots)

    right = 0.0
```

```
    denom = knots[i+p+1] - knots[i+1]
    if denom > 1e-12:
    right = (knots[i+p+1] - xi) / denom * b_spline_basis(i+1, p-1, xi, knots)

    return left + right

def b_spline_basis_derivative(i, p, xi, knots):
    """
    Evaluate derivative of B-spline basis function dN_{i,p}/dxi
    """
    if p == 0:
    return 0.0

    left = 0.0
    denom = knots[i+p] - knots[i]
    if denom > 1e-12:
    left = (p / denom) * b_spline_basis(i, p-1, xi, knots)

    right = 0.0
    denom = knots[i+p+1] - knots[i+1]
    if denom > 1e-12:
    right = (-p / denom) * b_spline_basis(i+1, p-1, xi, knots)

    return left + right

def create_semicircle(radius=1.0, degree=2):
    """
    Create control points and knot vector for a semicircle

    Parameters:
    -----------
    radius : float
    Radius of the semicircle
    degree : int
    Degree of the B-spline

    Returns:
    --------
    tuple : (control_points, knots)
    """
    a = radius
    sqrt2 = np.sqrt(2)

    # Control points for a quarter circle, then mirrored
    control_points = np.array([
        [a, 0.0],
        [a, a*(sqrt2 - 1.0)],
        [a*sqrt2/2.0, a*sqrt2/2.0],
        [a*(sqrt2 - 1.0), a],
        [0.0, a],
        [-a*(sqrt2 - 1.0), a],
        [-a*sqrt2/2.0, a*sqrt2/2.0],
        [-a, a*(sqrt2 - 1.0)],
        [-a, 0.0]
        ])

    # Knot vector for degree 2 (quadratic)
```

```
    knots = np.array([0, 0, 0, 1, 1, 2, 2, 3, 3, 4, 4, 4])

    return control_points, knots

def create_quarter_annulus(r_in=1.0, r_out=2.0, degree=2, n_elements=10):
    """
    Create control points and knot vectors for a quarter annulus

    Parameters:
    -----------
    r_in : float
    Inner radius
    r_out : float
    Outer radius
    degree : int
    Degree of the B-spline
    n_elements : int\\
    Number of elements in each direction

    Returns:
    --------
    tuple : (control_points_2d, knots_xi, knots_eta)
    """
    # Control points grid in radial and angular directions
    n_radial = n_elements + 1
    n_angular = n_elements + 1

    control_points = np.zeros((n_radial, n_angular, 2))

    for i in range(n_radial):
        r = r_in + (r_out - r_in) * i / (n_radial - 1)
        for j in range(n_angular):
            theta = (np.pi/2) * j / (n_angular - 1)
            control_points[i, j, 0] = r * np.cos(theta)
            control_points[i, j, 1] = r * np.sin(theta)

    # Knot vectors
    knots_xi = np.zeros(n_radial + degree + 1)
    knots_eta = np.zeros(n_angular + degree + 1)

    # Uniform open knot vectors
    for i in range(len(knots_xi)):
        if i <= degree:
            knots_xi[i] = 0
        elif i >= len(knots_xi) - degree - 1:
            knots_xi[i] = 1
        else:
            knots_xi[i] = (i - degree) / (len(knots_xi) - 2*degree - 1)

    for i in range(len(knots_eta)):
        if i <= degree:
            knots_eta[i] = 0
        elif i >= len(knots_eta) - degree - 1:
            knots_eta[i] = 1
        else:
            knots_eta[i] = (i - degree) / (len(knots_eta) - 2*degree - 1)
```

```
    return control_points, knots_xi, knots_eta

def assemble_matrices_1d(control_points, knots, p, n_gauss=2):
    """
    Assemble mass matrix M and stiffness matrix K for 1D curved domain

    Parameters:
    -----------
    control_points : array
        Control points defining the geometry
    knots : array
        Knot vector
    p : int
        Degree of B-spline
    n_gauss : int
        Number of Gauss quadrature points (2 or 3)

    Returns:
    --------
    tuple : (M, K)
    """
    n_basis = len(control_points)
    M = np.zeros((n_basis, n_basis))
    K = np.zeros((n_basis, n_basis))

    # Gauss-Legendre quadrature points and weights
    if n_gauss == 2:
        gauss_pts = np.array([-0.5773502691896257, 0.5773502691896257])
        gauss_wts = np.array([1.0, 1.0])
    elif n_gauss == 3:
        gauss_pts = np.array([-0.7745966692414834, 0.0, 0.7745966692414834])
        gauss_wts = np.array([0.5555555555555556, 0.8888888888888888,
        0.5555555555555556])
    else:
        gauss_pts = np.array([0.0])
        gauss_wts = np.array([2.0])

    # Integration over each knot span (element)
    n_elements = len(knots) - 2*p - 1
    for e in range(n_elements):
        xi_a = knots[e + p]
        xi_b = knots[e + p + 1]
        h = xi_b - xi_a

        if h < 1e-12:
           continue

        # Gauss integration on the element
        for gp in range(len(gauss_pts)):
           xi = xi_a + (gauss_pts[gp] + 1) / 2 * h
           weight = gauss_wts[gp] * h / 2

           # Evaluate all basis functions and derivatives at xi
           N = np.zeros(n_basis)
           dN = np.zeros(n_basis)

           for i in range(n_basis):
```

```
                N[i] = b_spline_basis(i, p, xi, knots)
                dN[i] = b_spline_basis_derivative(i, p, xi, knots)

            # Evaluate geometry mapping and its derivative
            x = np.sum(control_points[:, 0] * N)
            dx_dxi = np.sum(control_points[:, 0] * dN)

            # Jacobian for 1D curve
            J = dx_dxi

            # Contribution to mass and stiffness matrices (1D wave equation)
            for i in range(n_basis):
                for j in range(n_basis):
                    M[i, j] += N[i] * N[j] * np.abs(J) * weight
                    K[i, j] += (dN[i] / J) * (dN[j] / J) * np.abs(J) * weight

    return M, K

def apply_dirichlet_boundary_conditions(M, K, F):
    """
    Apply homogeneous Dirichlet boundary conditions

    Parameters:
    -----------
    M, K : array
        Mass and stiffness matrices
    F : array
        Force vector

    Returns:
    --------
    tuple : (M, K, F) with boundary conditions applied
    """
    # Remove first and last degrees of freedom (boundaries at xi=0 and xi=4)
    n = M.shape[0]
    free_dofs = np.arange(1, n-1)

    M_bc = M[free_dofs, :][:, free_dofs]
    K_bc = K[free_dofs, :][:, free_dofs]
    F_bc = F[free_dofs]

    return M_bc, K_bc, F_bc

def compute_cfl_limit(M, K, min_h_e, c=1.0,
safety_factor=0.8):
    """
    Compute the CFL stability limit

    Parameters:
    -----------
    M, K : array
        Mass and stiffness matrices
    min_h_e : float
        Minimum effective element size
    c : float
        Wave speed
    safety_factor : float
```

```
        Safety factor for time step

    Returns:
    --------
    float : Stable time step
    """
    # Solve generalized eigenvalue problem
    # Use scipy.sparse.linalg.eigs for large matrices
    n = M.shape[0]
    if n <= 500:
        # Direct method for small matrices
        eigenvalues = np.linalg.eigvals(np.linalg.solve(M, K))
        omega_max = c * np.sqrt(np.max(np.abs(eigenvalues)))
    else:
        # Sparse method for large matrices
        from scipy.sparse.linalg import LinearOperator, eigs
        def matvec(x):
            return np.linalg.solve(M, K @ x)
        op = LinearOperator((n, n), matvec=matvec, dtype=float)
        eigenvalues = eigs(op, k=1, which='LM', return_eigenvectors=False)
        omega_max = c * np.sqrt(np.abs(eigenvalues[0]))

    dt_max = 2.828 / omega_max
    dt_safe = safety_factor * dt_max

    return dt_safe, dt_max

def runge_kutta_4_step(alpha, beta, M, K, F, dt, c):
    """
    Perform one step of fourth-order Runge-Kutta integration

    Parameters:
    -----------
    alpha, beta : array
        Displacement and velocity vectors
    M, K : array
        Mass and stiffness matrices
    F : array
        Force vector (assumed constant in time for simplicity)
    dt : float
        Time step
    c : float
        Wave speed

    Returns:
    --------
    tuple : (alpha_new, beta_new)
    """
    n = len(alpha)
    Minv = np.linalg.inv(M)

    # Define the system of first-order ODEs
    def system(y):
        a = y[:n]
        b = y[n:]
        da_dt = b
        db_dt = -c**2 * Minv @ (K @ a) + Minv @ F
```

```
        return np.concatenate([da_dt, db_dt])

    y = np.concatenate([alpha, beta])

    # RK4 stages
    k1 = system(y)
    k2 = system(y + 0.5 * dt * k1)
    k3 = system(y + 0.5 * dt * k2)
    k4 = system(y + dt * k3)

    y_new = y + dt * (k1 + 2*k2 + 2*k3 + k4) / 6

    alpha_new = y_new[:n]
    beta_new = y_new[n:]

    return alpha_new, beta_new

def solve_wave_equation(M, K, alpha0, beta0, F, dt, n_steps, c=1.0):
    """
    Solve the wave equation using RK4 time integration

    Parameters:
    -----------
    M, K : array
        Mass and stiffness matrices
    alpha0, beta0 : array
        Initial displacement and velocity
    F : array
        Force vector
    dt : float
        Time step
    n_steps : int
        Number of time steps
    c : float
        Wave speed

    Returns:
    --------
    tuple : (alpha_history, beta_history, time_history)
    """
    n_dof = len(alpha0)
    alpha_history = np.zeros((n_steps + 1, n_dof))
    beta_history = np.zeros((n_steps + 1, n_dof))
    time_history = np.zeros(n_steps + 1)

    alpha = alpha0.copy()
    beta = beta0.copy()
    alpha_history[0, :] = alpha
    beta_history[0, :] = beta

    for step in range(n_steps):
        alpha, beta = runge_kutta_4_step(alpha, beta, M, K, F, dt, c)
        alpha_history[step + 1, :] = alpha
        beta_history[step + 1, :] = beta
        time_history[step + 1] = (step + 1) * dt

    return alpha_history, beta_history, time_history
```

```
def compute_l2_error(alpha_numerical, alpha_exact, M):
    """
    Compute L2 error between numerical and exact solutions

    Parameters:
    -----------
    alpha_numerical, alpha_exact : array
        Numerical and exact displacement vectors
    M : array
        Mass matrix

    Returns:
    --------
    float : L2 error
    """
    diff = alpha_numerical - alpha_exact
    error_squared = diff @ M @ diff
    return np.sqrt(error_squared)

# Main execution example
if __name__ == "__main__":
    print("Isogeometric Analysis of the Wave Equation")
    print("=" * 50)

    # Create geometry
    control_points, knots = create_semicircle(radius=1.0, degree=2)
    print(f"Number of control points: {len(control_points)}")
    print(f"Knot vector: {knots}")

    # Assemble matrices
    M, K = assemble_matrices_1d(control_points, knots, p=2, n_gauss=3)
    print(f"Mass matrix shape: {M.shape}")
    print(f"Stiffness matrix shape: {K.shape}")

    # Apply boundary conditions
    F = np.zeros(M.shape[0])
    M_bc, K_bc, F_bc = apply_dirichlet_boundary_conditions(M, K, F)
    print(f"After BC: DOFs = {M_bc.shape[0]}")

    # Compute CFL limit
    min_h_e = 0.05  # Minimum effective element size
    c = 1.0
    dt_safe, dt_max = compute_cfl_limit(M_bc, K_bc, min_h_e, c=c, safety_factor=0.8)
    print(f"CFL limit: dt_max = {dt_max:.4f}, dt_safe = {dt_safe:.4f}")

    # Initial conditions
    n_dof = M_bc.shape[0]
    alpha0 = np.ones(n_dof) * 0.01  # Small initial displacement
    beta0 = np.zeros(n_dof)

    # Time integration
    n_steps = 100
    alpha_hist, beta_hist, time_hist = solve_wave_equation(
        M_bc, K_bc, alpha0, beta0, F_bc, dt_safe, n_steps, c=c
        )
```

```
print(f"Time integration completed: {n_steps} steps")
print(f"Final energy: {0.5*(alpha_hist[-1] @ K_bc @ alpha_hist[-1] +
beta_hist[-1] @ M_bc
 @ beta_hist[-1]):.6f}")

# Plot results
plt.figure(figsize=(10, 6))
plt.plot(time_hist, alpha_hist[:, n_dof//2], 'b-', linewidth=1.5)
plt.xlabel('Time t')
plt.ylabel('Displacement at midpoint')
plt.title('Wave propagation on semicircular membrane')
plt.grid(True, alpha=0.3)
plt.tight_layout()
plt.savefig('wave_solution.png', dpi=150)
plt.show()

print("\nDone!")
```

Code Usage Example

To run the code, save the above script as `wave_iga.py` and execute:

```
python wave_iga.py
```

The code will:

1. Create the B-spline parameterization of a semicircle,
2. Assemble the mass and stiffness matrices,
3. Apply Dirichlet boundary conditions,
4. Compute the CFL stability limit,
5. Perform time integration using RK4,
6. Generate a plot of the solution.

Dependencies

1. numpy $\geq$ 1.19.0
2. scipy $\geq$ 1.5.0
3. matplotlib $\geq$ 3.2.0

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