Applied and Computational Mathematics
2026; 15(4): 133-143
http://www.sciencepublishinggroup.com/j/acm
doi: 10.11648/j.acm.20261504.12

A Crank-nicolson Finite Element Treatment of Time Singularities of the One Dimensional Heat Equation

Jake Leonard Nkeck*

Department of Mathematics, University of Buea, Buea, Cameroon

Email address:
nkeck.jake@ubuea.cm (Jake Leonard Nkeck)

To cite this article:
Jake Leonard Nkeck. (2026). A Crank-nicolson Finite Element Treatment of Time Singularities of the One Dimensional Heat Equation. *Applied and Computational Mathematics, 15*(4), 133-143. https://doi.org/10.11648/j.acm.20261504.12

Received: 4 June 2026; **Accepted:** 15 June 2026 **Published:** 12 August 2026

Abstract: The Heat Equation is a well-known partial differential equation that can be solved numerically by finite difference methods in time coupled with finite element methods in space. Crank-Nicolson methods can therefore be applied as finite difference methods and coupled with linear Lagrange finite element methods in order to solve the Heat equation and obtain an efficient convergence rate, the Heat equation is said to be solved by a Crank-Nicolson finite element method. However, the convergence rate of the Crank-Nicolson finite element method for the Heat equation can be affected if the exact solution entails time singularities; in that case the lack of smoothness of the solution though it is local in time, affects the convergence of the finite element method in the whole domain. This paper presents a Crank-Nicolson finite element method coupled to a Predictor-corrector algorithm to recover the optimal convergence rate when the solution has time singularities. The finite element method presented is based on the approximation of the time singular functions using a Fourier decomposition of the exact solution that leads to computable formulas of the time dependent coefficients of singularities that reduce the smoothness of the solution; so removing those coefficients ameliorate the efficiency of the Crank-Nicolson finite element method. Numerical experiments are presented to show the efficiency of the method.

Keywords: Heat Equation, Crank-nicolson Finite Element Methods, Time Singularities

1. Introduction

The Heat problem is an initial boundary value problem of particular importance in mechanical and chemical engineering (see [1]). Most of the problems involving partial differential equations are confronted to singularities (see [2–7]) due to the geometry of the domain of study, the boundary or initial conditions or the source term and the Heat problem is not an exception. Finite Element Methods (FEM) are well known and mostly used for elliptic boundary value problems (see [8, 9]) but are also efficient for problems of parabolic type like the Heat equation. One can cite the Space-time FEM (see [10]) and the well known method of lines that is a FEM coupled with some finite difference σ-weighted schemes like the Backward Euler ($\sigma = 1$), the Forward Euler ($\sigma = 0$) or the Crank-Nicolson ($\sigma = 1/2$) (see [11, 12]). The efficiency of those methods of lines is also related to the convergence rate, reason why the Crank-Nicolson scheme, which has a convergence rate 2 is preferred to the Backward and Forward Euler schemes that have convergence rates 1. However, the convergence rate of the Crank-Nicolson FEM can be affected when a singularity due to the smoothness of the source term occurs and then affects the convergence to the exact solution (by the Shift Theorem from [13]).

Predictor-corretor methods are well known adaptive methods to ameliorate and recover the convergence rate of solutions of differential equations (see [14–16]) in the case of geometric singularities. This paper studies the one-dimensional Heat equation with time singularities and presents a Crank-Nicolson FEM based on a Fourier decomposition of the solution, the approximation of the time singular term in the Fourier decomposition and a predictor-corrector algorithm to approximate the solution.

This article is organized as follows: in section 2 the initial value problem is presented with the existence and uniqueness of the solution and its Fourier decomposition. Section 3 presents the Crank-Nicolson finite element scheme, the predictor-corrector algorithm with some error estimates and section 4 is devoted to some numerical experiments to show the efficiency of the method.

2. The Model Problem; Fourier Decomposition

2.1. The Initial Value Problem

This paper consists of solving the Problem: Find $u = u(x,t)$ such that

$$\begin{aligned} \frac{\partial u}{\partial t} - c^2 \frac{\partial^2 u}{\partial x^2} = f \qquad \text{in} \quad (0,L)\times(0,T] \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = g_0(x) \end{aligned} \tag{1}$$

where $c \in \mathbb{R}$, $L > 0$, $T > 0$, f and g_0 are given real valued functions.

In order to prove the existence and uniqueness of solutions the following function spaces are defined.

1) Given an interval or a product of intervals I, the space $L^2(I)$ of classes of Lebesgue-measurable and square integrable functions over I with the norm

$$||u||_0 = \left(\int_I |u(x)|^2 dx\right)^{1/2}$$

whre x can be a vector or a scalar depending on the interval I.

2) Given an interval I and the Hilbert space X, $C^k(I,X)$ will denote the space of bounded k-continuously differentiable functions of the form $t \mapsto u(\cdot,t) \in X$ equipped with the norm

$$||u||_{C^k} = \sup_{t\in I, |\alpha|\le k} ||\partial^\alpha u(t)||_X.$$

3) The space $H^m(0,L)$, $m \in \mathbb{N}$, $m \geq 1$ will denote the space of measurable classes of functions u such that $u \in L^2(0,L)$ and $\dfrac{d^k u}{dx^k} \in L^2(0,L)$ for $0 < k \leq m$ with the norm

$$||u||_m = \left(\int_0^L \left(|u|^2 + \sum_{k=1}^m \left|\frac{d^k u}{dx^k}\right|^2\right) dx\right)^{\frac{1}{2}}$$

4) The space $V := H_0^1(0,L) = \{u \in H^1(0,L) : u(L) = u(0) = 0\}$ equipped with the norm $||\cdot||_1$.
5) The space $V' := H^{-1}(0,L)$ is the dual space of V in the sense of distributions.
6) The space $L^2((0,T),V)$ is the completion of $C((0,T),V)$ with respect to the norm

$$||u||_{L^2((0,T),V)} = \left(\int_0^T \int_0^L |u|^2 + \left|\frac{\partial u}{\partial x}\right|^2 dxdt\right)^{\frac{1}{2}}.$$

7) The space $H^1((0,T),V')$ with the norm

$$||u||_{H^1((0,T),V')} = \left(\int_0^T ||u||_{V'}^2 + \left|\left|\frac{\partial u}{\partial t}\right|\right|_{V'}^2 dt\right)^{\frac{1}{2}},$$

where

$$||u||_{V'} = \sup_{v\in V}\left(\frac{|u(v)|}{||v||_{H^1(0,L)}}\right)$$

The equivalent variational formulation to Problem (1) is: Find u such that

$$\begin{aligned} \int_0^L \frac{\partial u}{\partial t} v + c^2 \nabla u \cdot \nabla v dx = \int_0^L f v dx \quad \forall v \in H_0^1(0,L) \\ \int_0^L u(\cdot,0) v dx = \int_0^L g_0 v dx \qquad \forall v \in H_0^1(0,L) \end{aligned}. \tag{2}$$

In this paper it is assumed that $f \in L^2((0,T),V')$ and $g_0 \in L^2(0,L)$ with $g_0(0) = g_0(L) = 0$. It is assumed that the solution of Problem (1) has only one time singularity located at $t = t_0$, $t_0 \in [0,T]$. The existence and uniqueness of Problem (2) is given by the following theorem.

Theorem 2.1 ([17, p. 382 (11.3)]).

Assume that the functions $f \in L^2((0,T),V')$ and $g_0 \in L^2(0,L)$ are given. Then (1) has a unique solution $u \in L^2((0,T),V) \cap H^1((0,T),V')$.

By the Sobolev imbedding theorem (see [18], page 98) the solution u of (1) belongs to $C([0,T],V')$. The following Theorem from [17] (Lemma 11.4, page 383) gives a better continuity results for u.

Theorem 2.2 ([17, p. 383 (11.4)]).

Suppose that $u \in L^2((0,T),V) \cap H^1((0,T),V')$, then $u \in C([0,T],L^2(0,L))$.

Remark 2.1.

$L^2((0,T),V)$ is imbedded in $\subset L^2((0,L)\times(0,T))$ since V is imbedded in $L^2(0,L)$.

2.2. Decomposition of the Solution

Considering the homogeneous equation associated to the first equation of (1) and setting $u(x,t) = v(x)\varphi(t)$ one has $\dfrac{\varphi'(t)}{\varphi(t)} = c^2\dfrac{v''(x)}{v(x)}$. The operator $-\dfrac{d^2(\cdot)}{dx^2} : L^2(0,L) \to L^2(0,L)$ is positive and self-adjoint then it is compact and from the Hilbert -Schmidt theory (see [19] Theorem 2.34 page 47), its eigenvalues are positive and discrete. Hence there exist $\alpha_k \in \mathbb{R}$, $k \in \mathbb{N}$ such that $\dfrac{v''(x)}{v(x)} = -\dfrac{\alpha_k^2}{c^2}$, this leads to $v(x) = A_k \cos(\frac{\alpha_k}{c}x) + B_k \sin(\frac{\alpha_k}{c}x)$, $A_k, B_k \in \mathbb{R}$.

The condition $u(0,t)=0$ implies that $A_k=0$ for any $k\in\mathbb{N}$ while the condition $u(L,t)=0$ implies that $\alpha_k=\dfrac{kc\pi}{L}$, $k\in\mathbb{N}\setminus\{0\}=\mathbb{N}^\star$. The case $k=0$ is excluded since the eigenvalues α_k^2 are positive.

$\{\sin(\frac{k\pi}{L})\}_{k\in\mathbb{N}^\star}$ is an orthogonal basis of $L^2(0,L)$ then the decomposition

$v(x)=\displaystyle\sum_{k=1}^{\infty}B_k\sin\left(\frac{\alpha_k}{c}x\right)$ holds and for a certain $t_0\geq 0$ one can assume the decomposition

$$u(x,t)=\begin{cases}\displaystyle\sum_{k=1}^{\infty}v_k(t^\alpha)s_k(x) & \text{if}\quad t_0=0\\ \displaystyle\sum_{k=1}^{\infty}v_k((t_0-t)^\alpha)s_k(x) & \text{if}\quad t_0>0\end{cases}\tag{3}$$

where $s_k(x)=\sin\left(\frac{k\pi}{L}x\right)$ and α is a given positive real number that will be determined by the regularity of the solution u. Using (3), the change of variables $y=t^\alpha$ in (1) if $t_0=0$ and $y=(t_0-t)^\alpha$ in(1) if $t_0>0$ together with the othogonality of the basis $\{s_k\}_{k\in\mathbb{N}^\star}$ in $L^2(0,L)$ one obtains the systems

$$\begin{matrix}\alpha y^{\frac{\alpha-1}{\alpha}}\dfrac{dv_k}{dy}+\beta_k^2v_k=f_k(y^{1/\alpha}) \quad ,y\geq 0\\ v_k(0)=g_{0k}\end{matrix}\qquad \text{if } t_0=0$$

and

$$\begin{matrix}-\alpha y^{\frac{\alpha-1}{\alpha}}\dfrac{dv_k}{dy}+\beta_k^2v_k=f_k(t_0-y^{1/\alpha}) \quad ,y\geq t^\alpha\\ v_k(t_0^\alpha)=g_{0k}\end{matrix}\qquad \text{if } t_0>0.$$

where $\beta_k=kc\pi/L$, $f_k(t)=\dfrac{2}{L}\displaystyle\int_0^L f(x,t)s_k(x)dx$ and $g_{0k}=\dfrac{2}{L}\displaystyle\int_0^L g_0(x)s_k(x)dx$. The solution is

$$v_k(y)=\begin{cases}\left(g_{0k}+\displaystyle\int_0^{t^\alpha}\frac{1}{\alpha}f_k(\tau^{1/\alpha})\tau^{\frac{1-\alpha}{\alpha}}e^{\beta_k^2\tau^{1/\alpha}}d\tau\right)e^{-\beta_k^2t}\\ \text{if } y=t^\alpha\\ \left(\displaystyle\int_{(t_0-t)^\alpha}^{t^\alpha}\frac{1}{\alpha}f_k(t_0-\tau^{1/\alpha})\tau^{\frac{1-\alpha}{\alpha}}e^{-\beta_k^2\tau^{1/\alpha}}d\tau\right)e^{\beta_k^2(t_0-t)}\\ +g_{0k}e^{-\beta_k^2t}\text{ if } y=(t_0-t)^\alpha\end{cases}.$$

So by setting

$$\gamma_k(t)=\begin{cases}\left(g_{0k}+\displaystyle\int_0^{t^\alpha}\frac{1}{\alpha}f_k(\tau^{1/\alpha})\tau^{\frac{1-\alpha}{\alpha}}e^{\beta_k^2\tau^{1/\alpha}}d\tau\right)e^{-\beta_k^2t}\\ \text{if } t_0=0\\ g_{0k}e^{-\beta_k^2t}+\\ \left(\displaystyle\int_{(t_0-t)^\alpha}^{t^\alpha}\frac{1}{\alpha}f_k(t_0-\tau^{1/\alpha})\tau^{\frac{1-\alpha}{\alpha}}e^{-\beta_k^2\tau^{1/\alpha}}d\tau\right)e^{\beta_k^2(t_0-t)}\\ \text{if } t_0>0\end{cases}\tag{4}$$

one has

$$\frac{d\gamma_k}{dt}(t)=f_k(t)-\beta_k^2\gamma_k(t).\tag{5}$$

and $u(x,t)=\displaystyle\sum_{k=1}^{\infty}\gamma_k(t)s_k(x)$. Using the Taylor expansion Theorem, $e^{-\beta_k^2(t-t_0)}=\displaystyle\sum_{n=0}^{\infty}\frac{(-\beta_k^2(t-t_0))^n}{n!}$ then $\gamma_k(t)=\varphi_k(t)+w_k(t)$ where if $t_0=0$ one has

$$\begin{aligned}\varphi_k(t)&=\left[g_{0k}+\int_0^{t^\alpha}f_k(\tau^{1/\alpha})\tau^{\frac{1-\alpha}{\alpha}}e^{\beta_k^2\tau^{1/\alpha}}d\tau\right]\sum_{n=0}^{3}\frac{(-\beta_k^2t)^n}{\alpha n!}\\ w_k(t)&=\left[g_{0k}+\int_0^{t^\alpha}f_k(\tau^{1/\alpha})\tau^{\frac{1-\alpha}{\alpha}}e^{\beta_k^2\tau^{1/\alpha}}d\tau\right]\sum_{n=4}^{\infty}\frac{(-\beta_k^2t)^n}{\alpha n!}\end{aligned}\tag{6}$$

$\dfrac{d\varphi_k}{dt}(t)=\left[g_{0k}+\displaystyle\int_0^{t^\alpha}f_k(\tau^{1/\alpha})\tau^{\frac{1-\alpha}{\alpha}}e^{\beta_k^2\tau^{1/\alpha}}d\tau\right]\displaystyle\sum_{n=1}^{3}\frac{(-\beta_k^{2n})t^{n-1}}{\alpha(n-1)!}$
$+f_k(t)\displaystyle\sum_{n=0}^{3}\frac{(-\beta_k^2t)^n}{n!}$ for any $k\in\mathbb{N}, k>0$, and if $t_0>0$,

$$\begin{gathered}\varphi_k(t) = g_{0k}e^{-\beta_k^2 t}+\\ \left[\int_{(t_0-t)^\alpha}^{t^\alpha}\frac{1}{\alpha}f_k(t_0-\tau^{1/\alpha})\tau^{\frac{1-\alpha}{\alpha}}e^{-\beta_k^2\tau^{1/\alpha}}d\tau\right]\sum_{n=0}^{3}\frac{(\beta_k^2(t_0-t))^n}{n!}\\ w_k(t) = g_{0k}e^{-\beta_k^2 t}+\\ \left[\int_{(t_0-t)^\alpha}^{t^\alpha}\frac{1}{\alpha}f_k(t_0-\tau^{1/\alpha})\tau^{\frac{1-\alpha}{\alpha}}e^{-\beta_k^2\tau^{1/\alpha}}d\tau\right]\sum_{n=4}^{\infty}\frac{(\beta_k^2(t_0-t))^n}{n!}\\ \frac{d\varphi_k}{dt}(t) = -\beta_k^2 g_{0k}e^{-\beta_k^2 t}+f_k(t)\sum_{n=0}^{3}\frac{(\beta_k^2(t_0-t))^n}{n!}\\ -\left[\int_{(t_0-t)^\alpha}^{t^\alpha}\frac{1}{\alpha}f_k(t_0-\tau^{1/\alpha})\tau^{\frac{1-\alpha}{\alpha}}e^{-\beta_k^2\tau^{1/\alpha}}d\tau\right]\sum_{n=1}^{3}\frac{\beta_k^{2n}(t_0-t)^{n-1}}{(n-1)!}\end{gathered}\tag{7}$$

for any $k \in \mathbb{N}, k > 0$.

Hence $u(x,t) = w(x,t) + \sum_{k=1}^{\infty}\varphi_k(t)s_k(x)$ with

$$w(x,t) = \sum_{k=1}^{\infty} w_k(t)s_k(x) \tag{8}$$

where φ_k and w_k are defined in (6) and (7).

Remark 2.2.

1) The value $t_0 \in [0,T]$ is a time singular point of the solution u.
2) The function w defined in (8) is an element of $C^3([0,T], H^2(0,L))$.

3. A Predictor-corrector Algorithm; Error Estimates

3.1. A Crank-nicolson Scheme

The numerical treatment of Problem (1) can be performed by a finite difference in the time variable t coupled with a finite element scheme in the space variable x. Let us consider a quasi-uniform partition (x_i, x_{i+1}), $i = 0, \cdots, N_x$ of $(0,L)$ into N_x intervals of length h and let us set $V_h := \left\{v_h \in H^1(0,L) : v_h|_{[x_i,x_{i+1}]} \in \mathcal{P}^1(x_i,x_{i+1}), i = 1,\cdots,N_x\right\}$ Let us also define

$$V_{0h} := \{v_h \in V_h : v_h(0,t) = v_h(L,t) = 0\},$$

the finite element discretization of (2) consist of finding u_h such that $u_h(\cdot,t) \in V_{0h}$ such that $\frac{\partial u_h}{\partial t}(\cdot,t) \in L^2(0,L)$ for any $t \in (0,T)$ and

$$\begin{aligned}\int_0^L \frac{\partial u_h}{\partial t}v_h + c^2\nabla u_h\cdot\nabla v_h dx &= \int_0^L f v_h dx \quad \forall v_h \in V_{0h}\\ \int_0^L u_h(\cdot,0)v_h dx &= \int_0^L g_0 v_h dx \quad \forall v_h \in V_{0h}.\end{aligned}\tag{9}$$

Let $(\varphi_j)_{j=1,\cdots,N_x}$ be a basis of V_h and set

$u(x,t) = \sum_{j=1}^{N_x} u_j(t)\varphi_j(x) = U_h(t)^T\Phi(x)$ where

$U_h(t) = (u_j(t))_{j=1,\cdots,N_x}^T$ and $\Phi(x) = (\varphi_j(x))_{j=1,\cdots,N_x}^T$. The discretized system (9) is equivalent to the system

$$\begin{aligned}M_h\frac{dU_h}{dt}(t) + A_hU_h(t) &= F_h(t), \quad t \in (0,T)\\ M_hU_h(0) &= F_{0h}\end{aligned}\tag{10}$$

where $M_h = \left(\int_0^L \varphi_j\varphi_i dx\right)_{1\le i,j\le N_x}$,

$A_h = \left(c^2\int_0^L \nabla\varphi_j\cdot\nabla\varphi_i dx\right)_{1\le i,j\le N_x}$,

$F_h(t) = \left(\int_0^L f(\cdot,t)\varphi_j dx\right)_{1\le j\le N_x}$,

$F_{0h} = \left(\int_0^L g_0\varphi_j dx\right)_{1\le j\le N_x}$.

The finite difference scheme consists of subdividing the interval $(0,T)$ into a partition of N_t subinterval (t_i, t_{i+1}) of the same length τ, $i = 1,\cdots,N_t$. Setting $U_h^i = U_h(t_i)$, $F_h^i = F_h(t_i)$, $0 \le i \le N_t$, using the σ-weighted scheme,

$$\frac{dU_h}{dt}(t_{i+\sigma}) \approxeq \frac{U_h^{i+\sigma} - U_h^i}{\sigma\tau} \approxeq \frac{U_h^{i+1} - U^{i+\sigma}}{(1-\sigma)\tau}$$

then $U_h^{i+\sigma} \approxeq \sigma U_h^{i+1} + (1-\sigma)U_h^i$. The system (10) becomes

$$\begin{aligned}(M_h + \sigma\tau A_h)U_h^{i+1} &= (M_h + (1-\sigma)\tau A_h)U_h^i + \sigma\tau F_h^{i+1}\\ &+ (1-\sigma)\tau F_h^i,\\ &i = 1,\cdots,N_tM_hU_h^0 = F_{0h}.\end{aligned}\tag{11}$$

The Crank-Nicolson scheme is considered when $\sigma = 1/2$ and it is shown to be unconditionnally stable and consistent of order 2 when the solution is smooth enough, then convergent with order $\mathcal{O}(h^2+\tau^2)$ (see [20] or [12], page 562).

3.2. The Predictor-corrector Algorithm

It is well known (from [21] Theorem 13.7 page 162 or [22], Chapter 18) that if the solution of (1) u belongs to $C([0,T], L^2(0,L))$ and $\frac{\partial u}{\partial t}$ is piecewise continuous then for any $N \in \mathbb{N},\ N > 0$,

$$\left\|u - \sum_{k=1}^{N}\gamma_k s_k\right\|_0 \le \frac{C}{\sqrt{N}} \tag{12}$$

where $s_k(x) = \sin(\frac{k\pi}{L}x)$, γ_k is computed using the formula (4) and $C > 0$ is an arbitrary constant. So for the Fourier series to converge at the order $\mathcal{O}(h^2+\tau^2)$ it suffices to have $N \geq C(h^2+\tau^2)^{-2}$. Moreover if $h=\tau$ one has $N \geq h^{-4}$.

The predictor-corrector method presented is based on the decomposition of the approximate solution $u_{h\tau N}(x,t) = w_{h\tau}(x,t) + \sum_{k=1}^{N} \varphi_k(t)s_k(x)$ where $u_{h\tau N}$ is the solution of (11), $s_k(x) = \sin(\frac{k\pi}{L}x)$, γ_k is computed using the formula (4), $N \in \mathbb{N}$, $N \geq (h^2+\tau^2)^{-2}$ and $w_{h\tau} \in C^3([0,T], H^2(0,L))$.

Predictor-Corrector Algorithm

1) Solve the problem: Find $u_{h\tau}$ such that

$$\begin{array}{lll} \dfrac{\partial u_{h\tau}}{\partial t} - c^2\dfrac{\partial^2 u_{h\tau}}{\partial x^2} = f & \text{in} & (0,L)\times(0,T] \\ u_{h\tau}(0,t) = u_{h\tau}(L,t) = 0 & & t \in [0,T] \\ u_{h\tau}(x,0) = g_0(x) & & \end{array}$$

2) For $N \in \mathbb{N}$ fixed, do

a) For $k = 1,\cdots,N$, compute $\varphi_k(t)$ using the formula in (6) or (7) and find an interpolation $\varphi_{k\tau}(t)$ at the points $t_i,\ i = 0,\cdots,N_t$ of the triangulation of $[0,T]$ by polynomials of degree 2.

b) For $k = 1,\cdots,N$, compute $\dfrac{d\varphi_k}{dt}(t)$ using the formula in (6) or (7) and find an interpolation $\varphi^p_{k\tau}(t)$ at the points $t_i,\ i = 1,\cdots,N_t$ of the triangulation of $[0,T]$ by polynomials of degree 2.

c) If $V_{1h} := \left\{ v_h \in V_h:\ v_h(0,t) = -\sum_{k=1}^{N}\varphi_k(t)s_k(0),\ v_h(L,t) = -\sum_{k=1}^{N}\varphi_k(t)s_k(L) \right\}$, solve the problem: Find $w_{h\tau} \in V_{1h}$ such that

$$\begin{array}{ll} \dfrac{\partial w_{h\tau}}{\partial t} - c^2\dfrac{\partial^2 w_{h\tau}}{\partial x^2} = f - \displaystyle\sum_{k=1}^{N}\varphi^p_{k\tau}s_k + c^2\sum_{k=1}^{N}\varphi_{k\tau}\frac{d^2 s_k}{dx^2} & \\ \text{in } (0,L)\times(0,T] & \\ w_{h\tau}(0,t) = -\displaystyle\sum_{k=1}^{N}\varphi_{k\tau}(t)s_k(0) & t \in [0,T] \\ w_{h\tau}(L,t) = -\displaystyle\sum_{k=1}^{N}\varphi_{k\tau}(t)s_k(L) & t \in [0,T] \\ w_{h\tau}(x,0) = g_0(x) - \displaystyle\sum_{k=1}^{N}\varphi_{k\tau}(0)s_k(x) & x \in [0,L] \end{array}$$

d) Recompute

$$u_{h\tau N}(x,t) = w_{h\tau}(x,t) + \sum_{k=1}^{N}\varphi_{k\tau}(t)s_k(x).$$

Remark 3.1.

The formulas of the φ_k and $\dfrac{d\varphi_k}{dt}$ are exact and do not depend on the solution $u_{h\tau}$, but can be very expensive in the computational sense so they are replaced in the Predictor-corrector Algorithm by piecewise interpolations $\varphi_{k\tau}$ of φ_k and $\varphi^p_{k\tau}$ of $\dfrac{d\varphi_k}{dt}$ at the points $t_i,\ i = 1,\cdots,N_t$ of the triangulation of $[0,T]$ by polynomials of degree 2. These approximations by interpolation converge at the order $\mathcal{O}(\tau^2)$.

3.3. Error Estimates

Let u be the solution of Problem (1) and $u_{h\tau N}$ the approximate solution of u using the Predictor-corrector algorithm 3.2. Suppose that

$u_{h\tau N}(x,t) = w_{h\tau}(x,t) + \sum_{k=1}^{N}\varphi_{k\tau}(t)s_k(x)$ with $N \geq (h^2+\tau^2)^{-2}$, one has the following lemma.

Lemma 3.1.

If the solution u of (1) is such that $u \in \mathcal{C}([0,T], L^2(0,L))$, $t \mapsto \dfrac{\partial u}{\partial t}(\cdot,t)$ is piecewise continuous and $u(x,t) = w(x,t) + \sum_{k=1}^{\infty}\varphi_k(t)s_k(x)$ with $w \in C^3([0,T], H^2(0,L))$ and the φ_k defined as in (6) or (7). If $u_{h\tau N}$ is an approximation of u using the Predictor-corrector FEM with the decomposition $u_{h\tau N}(x,t) = w_{h\tau}(x,t) + \sum_{k=1}^{N}\varphi_{k\tau}(t)s_k(x)$, $N \in \mathbb{N}$, $N > 1$, then $\int_0^t \left\| \dfrac{\partial w}{\partial t}(\cdot,s)\right\|_2 ds < \infty$ and

$$\left\| g_0 - \sum_{k=1}^{N}\varphi_{k\tau}(0)s_k \right\|_2 < \infty.$$

Proof. Let $s \in [0,T]$, suppose w has the Fourier decomposition (6) or (7). $w(x,t) = \sum_{k=1}^{\infty} w_k(t)s_k(x)$, one has

$$\begin{aligned} \left\|\frac{\partial w}{\partial t}(\cdot,s)\right\|_2^2 &= \int_0^L \left|\frac{\partial w}{\partial t}(x,s)\right|^2 + \left|\frac{\partial^2 w}{\partial x\partial t}(x,s)\right|^2 dx \\ &\quad + \int_0^L \left|\frac{\partial^3 w}{\partial x^2\partial t}(x,s)\right|^2 dx \\ &= \int_0^L \sum_{k=1}^{\infty}\left(\left|\frac{dw_k}{dt}(s)\right|^2 |s_k(x)|^2 \right. \\ &\quad + \left|\frac{dw_k}{dt}(s)\right|^2 \left|\frac{ds_k}{dx}(x)\right|^2 \\ &\quad \left. + \left|\frac{dw_k}{dt}(s)\right|^2 \left|\frac{d^2 s_k}{dx^2}(x)\right|^2\right) dx \\ &\leq \frac{L}{2}\sum_{k=1}^{\infty}\left(1 + \left(\frac{k\pi}{L}\right)^2 + \left(\frac{k\pi}{L}\right)^4\right)\left|\frac{dw_k}{dt}(s)\right|^2 \end{aligned}$$

But from [24] page 23 and 24, $\left|\dfrac{dw_k}{dt}(s)\right| \leq \dfrac{C}{k^3}$ for any $s \in [0,T]$ since $w \in C^3([0,T], H^2(0,L))$. This implies

$$\left\|\frac{\partial w}{\partial t}(\cdot,s)\right\|_2^2 \leq C\sum_{k=1}^{\infty}\left(\frac{1}{k^6} + \frac{1}{k^4} + \frac{1}{k^2}\right) \leq C_1 < \infty$$

where $C > 0$ and $C_1 > 0$. One can deduce that

$$\int_0^t \left\|\frac{\partial w}{\partial t}(\cdot,s)\right\|_2 ds \leq \int_0^T C_1 ds < \infty.$$

$$\begin{aligned}\left\|g_0-\sum_{k=1}^{N}\varphi_{k\tau}(0)s_k\right\|_2 &= \left\|\sum_{k=1}^{\infty}(\varphi_k(0)+w_k(0))s_k\right.\\ &\quad \left.-\sum_{k=1}^{N}\varphi_{k\tau}(0)s_k\right\|_2\\ &\leq \sum_{k=1}^{N}|\varphi_k(0)-\varphi_{k\tau}(0)|||s_k||_2\\ &\quad +\sum_{k=N+1}^{\infty}|\varphi_k(0)|||s_k||_2\\ &\quad +\sum_{k=1}^{\infty}|w_k(0)|||s_k||_2.\end{aligned}$$

From [24] page 23 and 24, $|\varphi_k(0)| \leq \dfrac{C}{k^2}$ and $|w_k(0)| \leq \dfrac{C}{k^2}$, then $\displaystyle\sum_{k=N+1}^{\infty} |\varphi_k(0)|||s_k||_2 \leq C\frac{L}{2}\sum_{k=N+1}^{\infty}\frac{1}{k^2}$ $\displaystyle\leq C\int_{N+1}^{\infty}\frac{1}{x^2}dx \leq \frac{C}{N}$ and $\displaystyle\sum_{k=1}^{\infty}|w_k(0)|\,||s_k||_2 \leq C.$

From Remark 3.1 $|\varphi_k(0)-\varphi_{k\tau}(0)| \leq C\tau^2$, one can conclude that

$$\left\|g_0-\sum_{k=1}^{N}\varphi_{k\tau}(0)s_k\right\|_2 \leq C(\tau^2+1/N+1) < \infty.$$

The following lemma from [11], Theorem 1.2, page 8 gives us some error estimates for regular solutions.

Lemma 3.2. [[11, p. 8 (1.2)]]

Let u_h be the solution of (9), u be the solution of (1) and assume that $g_0(0) = g_0(L) = 0$. Then

$$\begin{aligned}||u_h(\cdot,t)-u(\cdot,t)||_0 &\leq ||u(\cdot,0)-g_0(\cdot)||_0\\ &+ Ch^2\left(||g_0||_2+\int_0^t\left\|\frac{\partial u}{\partial t}\right\|_2 ds\right),\ t\geq 0.\end{aligned}$$

Here we require that the solution of the continuous problem has the regularity implicitly assumed by the presence of the norms on the right.

Theorem 3.1.

Let u be the solution of (1) such that $u \in \mathcal{C}([0,T],L^2(0,L))$, $t \mapsto \dfrac{\partial u}{\partial t}(\cdot,t)$ is piecewise continuous and $u(x,t) = w(x,t) + \displaystyle\sum_{k=1}^{\infty}\varphi_k(t)s_k(x)$ with $w \in C^3([0,T],H^2(0,L))$ and the φ_k defined as in (6) or (7). Let $u_{h\tau N}$ be an approximation of u using the Predictor-corrector FEM with the decomposition

$u_{h\tau N}(x,t) = w_{h\tau}(x,t) + \displaystyle\sum_{k=1}^{N}\varphi_{k\tau}(t)s_k(x)$, $N \in \mathbb{N}$, $N \geq (h^2+\tau^2)^{-2}$. Then there exists $C>0$ such that $||u-u_{h\tau N}||_0 \leq C(h^2+\tau^2)$.

Proof. In this proof C is an arbitrary positive constant. Let $t\in[0,T]$, assume that $w(x,t)=\displaystyle\sum_{k=1}^{\infty}w_k(t)s_k(x)$ and

$w_{h\tau}(x,t) = \displaystyle\sum_{k=1}^{\infty}w_{h\tau k}(t)s_k(x)$ where w_k and $w_{h\tau k}$ are defined by (6) or (7).

$$\begin{aligned}||u(\cdot,t)-u_{h\tau N}(\cdot,t)||_0^2 &= \int_0^L\left|\sum_{k=1}^{N}(\varphi_k(t)+w_k(t)-\varphi_{k\tau}(t)\right.\\ &\quad \left.-w_{h\tau k}(t))s_k(x)+\sum_{k=N+1}^{\infty}(\varphi_k(t)+w_k(t)-w_{h\tau k}(t))s_k(x)\right|^2\\ &= \sum_{k=1}^{N}\int_0^L|\varphi_k(t)+w_k(t)-\varphi_{k\tau}(t)-w_{h\tau k}(t)|^2|s_k(x)|^2dx\\ &\quad +\sum_{k=N+1}^{\infty}\int_0^L|\varphi_k(t)+w_k(t)-w_{h\tau k}(t)|^2|s_k(x)|^2dx\\ &= \sum_{k=1}^{N}\int_0^L(|\varphi_k(t)-\varphi_{k\tau}(t)|^2\\ &\quad +2|\varphi_k(t)-\varphi_{k\tau}(t)||w_k(t)-w_{h\tau k}(t)|\\ &\quad +|w_k(t)-w_{h\tau k}(t)|^2)|s_k(x)|^2dx+\sum_{k=N+1}^{\infty}\int_0^L(|\varphi_k(t)|^2\\ &\quad +2|\varphi_k(t)||w_k(t)-w_{h\tau k}(t)|\\ &\quad +|w_k(t)-w_{h\tau k}(t)|^2)|s_k(x)|^2dx\\ &= \sum_{k=1}^{N}\left(|\varphi_k(t)-\varphi_{k\tau}(t)|^2||s_k||_0^2\right.\\ &\quad +2|\varphi_k(t)-\varphi_{k\tau}(t)|\int_0^L|w_k(t)-w_{h\tau k}(t)||s_k(x)|^2dx\\ &\quad \left.+\int_0^L|w_k(t)-w_{h\tau k}(t)|^2|s_k(x)|^2dx\right)\\ &\quad +\sum_{k=N+1}^{\infty}\left(|\varphi_k(t)|^2||s_k||_0^2\right.\\ &\quad +2|\varphi_k(t)|\int_0^L|w_k(t)-w_{h\tau k}(t)||s_k(x)|^2dx\\ &\quad \left.+\int_0^L|w_k(t)-w_{h\tau k}(t)|^2|s_k(x)|^2dx\right)\\ &\leq C\left(\tau^4+2\tau^2||w(\cdot,t)-w_{h\tau}(\cdot,t)||_0+||w(\cdot,t)-w_{h\tau}(\cdot,t)||_0^2\right.\\ &\quad \left.+\frac{2}{N^2}||w(\cdot,t)-w_{h\tau}(\cdot,t)||_0+\frac{1}{N}\right)\\ &\leq C\left(\tau^4+1/N+(\tau^2+1/N^2)||w(\cdot,t)-w_{h\tau}(\cdot,t)||_0\right.\\ &\quad \left.+||w(\cdot,t)-w_{h\tau}(\cdot,t)||_0^2\right)\end{aligned}$$

But from Lemma 3.2 and Lemma 3.1,

$$
\begin{aligned}
&||w(\cdot,t)-w_{h\tau}(\cdot,t)||_0 \le ||w(\cdot,0)-w_{h\tau}(\cdot,0)||_0 \\
&+Ch^2\left(||w_{h\tau}(\cdot,0)||_2+\int_0^t\left|\left|\frac{\partial w}{\partial t}(\cdot,s)\right|\right|_2 ds\right)\\
\le\ & \left|\left|g_0-\sum_{k=1}^{\infty}\varphi_k(0)s_k-g_0+\sum_{k=1}^{N}\varphi_{k\tau}(0)s_k\right|\right|_0\\
&+Ch^2\left(\left|\left|g_0-\sum_{k=1}^{N}\varphi_{k\tau}(0)s_k\right|\right|_2+\int_0^t\left|\left|\frac{\partial w}{\partial t}(\cdot,s)\right|\right|_2 ds\right)\\
\le\ & \sum_{k=1}^{N}|\varphi_k(0)-\varphi_{k\tau}(0)|||s_k||_0+\sum_{k=N+1}^{\infty}|\varphi_k(0)|||s_k||_0+Ch^2\\
\le\ & C(\tau^2+1/N+h^2)\\
\le\ & C(h^2+\tau^2)\quad \text{since } N=\mathcal{O}((h^2+\tau^2)^{-2})
\end{aligned}
$$

Then

$$||u(\cdot,t)-u_{h\tau N}(\cdot,t)||_0^2 \le C(\tau^4+(\tau^2+\frac{1}{N^2})(\tau^2+h^2)+(\tau^2+h^2)^2)$$

and

$||u(\cdot,t)-u_{h\tau N}(\cdot,t)||_0 \le C(h^2+\tau^2)$ for any $t\in[0,T]$.

Hence

$$||u_{h\tau N}-u||_0=\int_0^T ||u_{h\tau N}(\cdot,t)-u(\cdot,t)||_0^2 dt \le C(h^2+\tau^2).$$

4. Numerical Experiments

The main prupose of this section is to present some computational results carried out with the Predictor-corrector algorithm 3.2. Even though the examples considered may not represent any real world situation, the difficulties due to singularities in the solution are well reflected. We consider three examples with a known solutions. In Examples 4.1, 4.2 and 4.3 the error is computed in the discrete L^2-norm defined by the formulas

$$||u(\cdot,t)-u_h(\cdot,t)||_d=\left(\frac{1}{N_x+1}\sum_{i=0}^{N_x}|u(x_i,t)-u_h(x_i,t)|^2\right)^{1/2}$$

for t fixed and for $N_{x,t}=(Nx+1)(N_t+1)$

$$||u-u_{h\tau}||_d=\left(\frac{1}{N_{x,t}}\sum_{j=0}^{N_t}\sum_{i=0}^{N_x}|u(x_i,t_j)-u_{h\tau}(x_i,t_j)|^2\right)^{1/2}.$$

It is worth noting that the discrete L^2-norm $||\cdot||_{d,0}$ and the L^2-norm are equivalent in the sense of convergence and order of convergence. All computations are carried out using a code developped in python and executed in a laptop computer so very large computations cannot be done. However the results obtained are sufficient to make definite conclusions about the efficacy of the FEM. The resulting linear systems are solved iteratively with a generalized minimal residual method (GMRES) and all the integrals are computed using a 10-point Legendre Gauss formula on the stifness, mass matrices and load vectors, and a 100-points Legendre Gauss formula for integrals used to compute the Fourier coefficients. It is shown that more quadrature points will not significantly change the numerical results.

In all the computations , starting from a triangulation $\mathcal{T}_{h_0,\tau_0}$, the refinement are perfomed by dividing the concerned lentgh by two ($h_{i+1}=h_i/2$ and/or $\tau_{i+1}=\tau_i/2$). If the exact solution u is known then the order or rate of convergence ν is computed using the formula

$$\nu=\begin{cases}\log_2\left|\dfrac{||u-u_{h_i\tau_j}||_d-||u-u_{h_{i-1}\tau_j}||_d}{||u-u_{h_{i+1}\tau_j}||_d-||u-u_{h_i\tau_j}||_d}\right| & \text{if } j \text{ is fixed}\\[2ex] \log_2\left|\dfrac{||u-u_{h_i\tau_j}||_d-||u-u_{h_i\tau_{j-1}}||_d}{||u-u_{h_i\tau_{j+1}}||_d-||u-u_{h_i\tau_j}||_d}\right| & \text{if } i \text{ is fixed}\\[2ex] \log_2\left|\dfrac{||u-u_{h_i\tau_j}||_d}{||u-u_{h_{i+1}\tau_{j+1}}||_d}\right| & \text{if } h_k=\tau_k\ \ \forall k\end{cases}.$$

In all the following examples ν_{exp} is the expected order of convergence.

4.1. *Example 1: A Smooth Solution*

This example solves Problem (9) with the exact solution $u(x,t)=\sqrt{2}t^2\sin\left(\frac{3\pi}{L}x\right)$ and $f=\dfrac{\partial u}{\partial t}-\dfrac{\partial^2 u}{\partial x^2}$, $g_0(x)=0, T=1, L=1$. Set $e_{h\tau}(t)=||u(\cdot,t)-u_{h\tau}(\cdot,t)||_d$, one has:
For $\tau=1/4$

Table 1. *Error estimates in the discrete L^2-norm and $h=\tau$ without adaptation (N = 0).*

h	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}	ν_{exp}
$e_{h\tau}(1/2)$	$3.00e^{-3}$	$1.13e^{-3}$	$3.08e^{-4}$	$7.92e^{-5}$	$2.00e^{-5}$	$5.03e^{-6}$	
Order	$--$	$--$	1.18	1.84	1.95	1.98	2.00
$e_{h\tau}(3/4)$	$4.44e^{-3}$	$1.67e^{-3}$	$4.55e^{-4}$	$1.17e^{-4}$	$2.95e^{-5}$	$7.42e^{-6}$	
Order	$--$	$--$	1.19	1.85	1.95	1.98	2.00
$e_{h\tau}(1)$	$6.01e^{-3}$	$2.26e^{-3}$	$6.17e^{-4}$	$1.59e^{-4}$	$4.01e^{-5}$	$1.01e^{-5}$	
Order	$--$	$--$	1.18	1.84	1.96	1.98	2.00

Table 2. *Error estimates in the discrete L^2-norm at $t = 1/2$, $t = 1$ and $h = \tau$.*

h	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}	ν_{exp}
$e_{h\tau}(1/2)$	$3.00e^{-3}$	$1.12e^{-3}$	$3.03e^{-4}$	$7.80e^{-5}$	$1.97e^{-5}$	$4.95e^{-6}$	
Order	$--$	1.42	1.88	1.96	1.98	1.99	2.00
$e_{h\tau}(1)$	$6.01e^{-3}$	$2.25e^{-3}$	$6.14e^{-4}$	$1.58e^{-4}$	$3.99e^{-5}$	$1.00e^{-5}$	
Order	$--$	1.41	1.88	1.96	1.98	1.99	2.00
$\|\|u - u_{h\tau}\|\|_d$	$3.66e^{-3}$	$1.33e^{-3}$	$3.58e^{-4}$	$9.13e^{-5}$	$2.30e^{-5}$	$5.76e^{-6}$	
Order	$--$	1.47	1.89	1.97	1.99	2.00	2.00

In this example the solution is smooth, the optimal convergence rates are observed, it is also showing the efficacy of the finite element code.

4.2. Example 2: A singular Solution with a Finite Fourier Series Decomposition

This example solves Problem (9) with the exact solution
$u(x,t) = t\sin\left(\frac{\pi}{L}x\right) - \frac{1}{2}t^{3/2}\sin\left(\frac{2\pi}{L}x\right) + \sqrt{2}t^2\sin\left(\frac{3\pi}{L}x\right)$ and $f = \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2}$, $g_0(x) = 0, T = 1, L = 1.$

Table 3. *Error estimates in the discrete L^2-norm and $h = \tau$ without adaptation (N = 0).*

$\tau = h$	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	ν_{exp}
$\|\|u - u_{h\tau}\|\|_d$	$8.14e^{-2}$	$1.56e^{-2}$	$8.75e^{-3}$	$4.65e^{-3}$	$2.40e^{-3}$	$1.22e^{-3}$	
Order	$--$	2.38	0.84	0.91	0.95	0.98	1.00

Table 4. *Error estimates in the discrete L^2-norm and $h = \tau$ with $N = 1$ and $\alpha = 1$.*

$\tau = h$	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	ν_{exp}
$\|\|u - u_{h\tau 1}\|\|_d$	$6.86e^{-2}$	$4.03e^{-3}$	$1.44e^{-3}$	$4.00e^{-4}$	$1.09e^{-4}$	$3.12e^{-5}$	
Order	$--$	4.09	1.48	1.85	1.87	1.81	1.5

Table 5. *Error estimates in the discrete L^2-norm and $h = \tau$ with $N = 2$ and $\alpha = 1$.*

$\tau = h$	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	ν_{exp}
$\|\|u - u_{h\tau 2}\|\|_d$	$6.86e^{-2}$	$3.66e^{-3}$	$1.33e^{-3}$	$3.58e^{-4}$	$9.13e^{-5}$	$2.30e^{-5}$	
Order	$--$	4.23	1.45	1.90	1.97	1.99	2.00

Table 6. *Error estimates in the discrete L^2-norm and $h = \tau$ with $N = 3$ and $\alpha = 1$*

$\tau = h$	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	ν_{exp}
$\|\|u - u_{h\tau 3}\|\|_d$	$8.68e^{-14}$	$1.64e^{-13}$	$5.37e^{-12}$	$1.72e^{-11}$	$2.05e^{-11}$	$2.08e^{-11}$	

In Table 3 the convergence rate is reduced to 1, proof of the existence of a time singularity. In fact the solution has a line singularity at $t = 0$. The predictor-corrector algorithm is applied for the number of functions in the Fourier decomposition to be $N = 1,\ 2,\ 3$ and the singular exponent $\alpha = 1$ (the convergence rate). In Table 4 the convergence rate is ameliorated, proof that a singular part has been removed, but the convergence is still not optimal, one needs to increase the value of N. In Table 5 the optimal convergence rate is recovered. In Table 6 the error is almost zero, this is because the regular part approximated is 0 and is very well approximated, only the errors due to the computations of integrals appear.

4.3. Example 3: A Singular Solution with an Infinite Fourier Series Decomposition and a Line Singularity

This example solves Problem (9) with the exact solution $u(x,t) = tx(x-1)$ and $f = \dfrac{\partial u}{\partial t} - 10^{-2}\dfrac{\partial^2 u}{\partial x^2}$, $g_0(x) = 0$, $T = 1, L = 1.$

Table 7. *Error estimates in the discrete L^2-norm and $h = \tau$ without adaptation.*

$\tau = h$	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	ν_{exp}
$\|\|u - u_{h\tau}\|\|_d$	$1.57e^{-2}$	$1.38e^{-2}$	$8.67e^{-3}$	$4.85e^{-3}$	$2.57e^{-3}$	$1.32e^{-3}$	
Order	$--$	0.19	0.66	0.83	0.92	0.96	1.00

Table 8. *Error estimates in the discrete L^2-norm and $h = \tau$ with $N = 40$ and $\alpha = 1$.*

$\tau = h$	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	ν_{exp}
$\|\|u - u_{h\tau N}\|\|_d$	$1.19e^{-5}$	$2.05e^{-6}$	$6.27e^{-7}$	$2.45e^{-7}$	$6.38e^{-8}$	$1.60e^{-8}$	
Order	$--$	2.54	1.71	1.36	1.94	1.99	2.00

Table 7 shows that a singularity occurs and affects the convergence. In fact the solution has a line singularity at $t = 0$. With the application of the predictor-corrector algorithm, the optimal convergence rate is recovered for $N = 40$ as Table 8 illustrates.

4.4. *Example 4: A Solution with an Infinite Fourier Series Decomposition and with a Line Singularity*

This example solves Problem (9) with the exact solution $u(x,t) = t^{1/2}\cos(\frac{\pi x}{2})(1 - e^{-x})$ and $f = \dfrac{\partial u}{\partial t} - c^2\dfrac{\partial^2 u}{\partial x^2}$, $c = 1/\pi^4$, $g_0(x) = 0$, $T = 1$, $L = 1$.

Table 9 presents the finite element approximation with $\tau = h$ without adaptation and it is shown that the optimal convergence rate is not obtained. Table 10 presents the Predictor-corrector solution with the singular exponent $\alpha = 1/2$. The number N of Fourier coefficients to compute is defined as a function of the diameter h of each triangulation to be $N = 1 + \lfloor h^{-1} \rfloor$, where $\lfloor h^{-1} \rfloor$ is the integer part of h^{-1}.

Table 9. *Error estimates in the continuous L^2-norm, $h = \tau$, without adaptation.*

$\tau = h$	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	ν_{exp}
$\|\|u - u_{h\tau}\|\|_0$	$7.57e^{-2}$	$5.23e^{-2}$	$3.68e^{-2}$	$2.59e^{-2}$	$1.83e^{-2}$	$1.29e^{-2}$	
Order	$--$	0.53	0.51	0.50	0.50	0.50	0.50

Table 10. *Error estimates in the continuous L^2-norm, $h = \tau$, without adaptation with $N = 1 + \lfloor h^{-1} \rfloor$.*

$\tau = h$	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	ν_{exp}
$\|\|u - u_{h\tau}\|\|_0$	$--$	$1.96e^{-4}$	$5.31e^{-5}$	$1.16e^{-5}$	$2.31e^{-6}$	$5.47e^{-7}$	
Order	$--$	$--$	1.89	2.19	2.33	2.08	2.00
N	3	5	9	17	33	65	

4.5. *Example 5: A solution with an Infinite Fourier Series and with a Line Ingularity*

This example solves Problem (9) with the exact solution $u(x,t) = (1\sqrt{1-t})(1 + \cos(\pi x))\ln(x+1)$ and $f = \dfrac{\partial u}{\partial t} - c^2\dfrac{\partial^2 u}{\partial x^2}$, $c = 10^{-2}$, $g_0(x) = 0$, $T = 1$, $L = 1$.

The number N of Fourier coefficients is defined to be $N = 1 + \lfloor h^{-1} \rfloor$.

In Table 11 a convergence is observed and the convergence rate that is 1 is better than the expected convergence rate 0.5. however the convergence rate is still not optimal and the FEM requires an adptation. Table 12 shows that the optimal convergence rate is obtained with the Predictor-corrector FEM.

Table 11. *Error estimates in the continuous L^2-norm, $h = \tau$, without adaptation.*

$\tau = h$	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	ν_{exp}
$\|\|u - u_{h\tau}\|\|_0$	$9.06e^{-2}$	$3.86e^{-2}$	$1.88e^{-2}$	$9.38e^{-3}$	$4.69e^{-3}$	$2.34e^{-3}$	
Order	$--$	1.23	1.03	1.01	1.00	1.00	1.00

Table 12. *Error estimates in the continuous L^2-norm, $h = \tau$, without adaptation with $N = 1 + \lfloor h^{-1} \rfloor$.*

$\tau = h$	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	ν_{exp}
$\|\|u - u_{h\tau}\|\|_0$	$4.34e^{-3}$	$1.07e^{-3}$	$2.25e^{-4}$	$4.29e^{-5}$	$7.83e^{-6}$	$1.41e^{-6}$	
Order	$--$	2.03	2.25	2.39	2.45	2.48	2.00
N	3	5	9	17	33	65	

5. Conclusion

The Predictor-corrector FEM developped in this paper is based on the Fourier decomposition of the solution and an approximation of the time singular solution. The FEM is proved to be efficient to recover the optimal convergence rate of the Crank-Nicolson scheme. In this paper and the various examples the solution is assumed to have only one time singularity, the case with more singularities can be treated locally by adding a time-dependent cut-off function or by segmenting the time domain $(0, T)$ into subdomains where the singular ponts are the endpoints of the subdomains, a sufficient appoximation at each interior endpoint of a subdomain being considered as an initial condition in the next subdomain. The case of Neumann and mixed boundary conditions in space can also be easiy adapted as the basis in $L^2(0, L)$ for the Fourier decomposition can be found.

ORCID

0000-0003-0492-8247 (Jake Leonard Nkeck)

Abbreviations

FEM Finite Element Method

Authors Contributions

Jake Leonard Nkeck: Conceptualization, Formal Analysis, Methodology, Resources, Software, Validation, Writing – original draft, Writing – review & editing

Conflicts of Interest

The author declares no conflicts of interest.

References

[1] Chapman, A. (1974). Heat Transfer (Fourth ed.). Macmillan Publishing Company, New York.

[2] Grisvard, P. (1992).Singularities in Boundary Value Problems. Recherches en mathématiques appliquées, Masson, Paris. https://doi.org/10.1007/978-3-0348-8625-3

[3] Grisvard, P. (1985). Elliptic Problems in Nonsmooth Domains. Pintman Advanced Publishing Program, Boston. https://doi.org/10.1137/1.9781611972030

[4] Koslov, V. A., Maz'ya, V. G., Rosmann, J. (2001). Spectral Problems Associated with Corner Singularities of Solutions to Elliptic Equations. AMS Mathematical Surveys and Monographs Rhode Island. https://doi.org/10.1090/surv/085

[5] Shu-Yu, H. (2006). Removable Singularities of Semilinear Parabolic Equations. Advances in Differential Equations 15, 137–158. https://doi.org/10.57262/ade/1355854766

[6] Lesnic, D., Elliott, L., Ingham, B. D. (1995). Treatment of singularities in time-dependent problems using boundary element methods. Engineering Analysis with Boundary Elements 16, 65–70. https://doi.org/10.1016/j.jsv.2003.09.059

[7] Kan, T., Takahashi, J.(2016). Time-dependent singularities in semilinear parabolic equations: Behavior at the singularities. J. Differential Equations, 260, 7278–7319. https://doi.org/10.1016/j.jde.2016.01.026

[8] Brenner, S. C., Scott, L. R.(2008). The Mathematical Theory of Finite Element Methods. Springer Science+Business Media, New York. https://doi.org/10.1007/978-0-387-75934-0

[9] Ciarlet, P. (2002). The Finite Element Method for Elliptic Problems. Society for Industrial and Applied Mathematics, Amsterdam. https://doi.org/10.1137/1.9780898719208

[10] Steinbach, O.(2015). Space-time finite element methods for parabolic problems. Computational Methods in Applied Mathematics 15(4), 551–566. https://doi.org/10.1515/cmam-2015-0026

[11] Thomée, V. (2006). Galerkin Finite Element Methods for Parabolic Problems. Springer-Verlag, Heidelberg. https://doi.org/10.1137/1028090

[12] Jung, M., Langer, U. (2013). Methode der finiten Elemente für Ingenieure. Springer Science+Business Media. https://doi.org/10.1007/978-3-658-01101-7

[13] Kondratiev, V. (1967). Boundary Values problems for elliptic equations in domains with conical or angular points. Translated. Moscow Mathematical Society 16, 227–313.

[14] Nkemzi, B., Tanekou, S. (2018). Predictor-corrector p - and hp -versions of the finite element method for Poisson's equation in polygonal domains. Computer Methods in Applied Mechanics and Engineering 333, 74–93. https://doi.org/10.1016/j.cma.2018.01.027

[15] Nkemzi, B., Nkeck, J. (2020). A predictor-corrector finite element method for Maxwell's equations in polygonal domains. Mathematical Problems in Engineering, 1–13. https://doi.org/10.1155/2020/3502513

[16] Nkemzi, B. (2006).On the solution of Maxwell's equations in polygonal domains. Mathematical Methods in Applied Science 29, 1053–1080 https://doi.org/10.1002/mma.717

[17] Renardy, M., and Rogers, R. C. (2004). An Introduction to Partial Differential Equations. (Second ed.) Springer, New York. https://doi.org/10.1007/b97427

[18] Adams, R. A. (1975). Sobolev Spaces. Academic Press, San Francisco.

[19] Mclean, W. (2000). Strongly Elliptic Systems and Boundary Integral Equations. Cambridge University Press, Cambridge. https://doi.org/10.2307/3621632

[20] Dautray, R., and Lions, J-L. (1999). Mathematical Analysis and Numerical Methods for Science and Technology: Evolution Problems II, Volume 6., Springer-Verlag, Berlin. https://doi.org/10.1007/978-3-642-61529-0

[21] Howell, K. B. (2001). Principles of Fourier Analysis. Chapman & Hall/CRC, New York. https://doi.org/10.1201/9781420036909

[22] Körner, T. W. (1988). Fourier Analysis. Cambridge University Press, Cambridge. https://doi.org/10.1017/CBO9781107049949

[23] Nkemzi, B., and Jung, M. (2013). A Postprocessing Finite Element Strategy for Poisson's Equation in Polygonal Domains: Computing the Stress Intensity Factors. Lecture Notes in Applied and Computational Mechanics 66, 153–173 https://doi.org/10.1007/978-3-642-30316-6_7

[24] Jackson, D. (1930). The Theory of Approximation. American Mathematical Society Colloquium Publications, Volume 11.