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On S-prime Elements and their Generalizations in Multiplicative Lattices

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Abstract: A multiplicative lattice is a complete lattice endowed with a commutative, associative, join distributive multiplication in which 1 acts as a multiplicative identity, providing an abstract framework for the study of the ideal theory of commutative rings. In this paper, the concept of an $\mathcal{S}$-prime element is introduced, generalizing the notion of an $\mathcal{S}$-prime ideal of a commutative ring with unity to multiplicative lattices. Let $\mathcal{L}$ be a multiplicative lattice with k distinct prime elements and $\mathcal{S}$ be the multiplicative closed subset of $\mathcal{L}$. Initially, the fundamental characterizations of $\mathcal{S}$-prime elements are investigated. Furthermore, it is shown that every element of a reduced lattice is an idempotent and radical element, respectively. In particular, every proper element of a reduced lattice is an $\mathcal{S}$-prime element. The number of $\mathcal{S}$-prime elements in a given multiplicative is determined and $\mathcal{S}$ is the upset of $k-1$ prime elements in $\mathcal{L}$. Moreover, the sets of non $\mathcal{S}$-prime elements and nilpotent elements have the same cardinality in $\mathcal{L}$. It is further shown that the radical of every nilpotent element is an $\mathcal{S}$-prime element. The uniqueness of an $\mathcal{S}$-prime element in a local lattice is proved and consequently, every zero divisor is a nilpotent element. Finally, several illustrative examples are presented to demonstrate the structural behavior of $\mathcal{S}$-prime elements in $\mathcal{L}$.

Keywords: Multiplicative Lattice, Reduced Lattice, Prime Element, Nilpotent Element, Zero-divisor

1. Introduction

Ideal theory in commutative ring with unity has received significant attention in the mathematical literature. Various classes of ideals including prime ideal, primary ideal and maximal ideals has been extensively investigated in the ring theory. The theory of multiplicative lattices was first introduced by Ward and Dilworth [15] in 1939 to generalize the work of ideal theory of commutative ring with unity. This Multiplicative lattice structure inspired researchers to extend the ideal-theoretic concepts to the associated elements of multiplicative lattices. Analogously, the study of $\mathcal{S}$-prime ideals in a commutative ring with unity is extended to the study of $\mathcal{S}$-prime elements in multiplicative lattices. The concept of prime ideal admits various generalizations. One such generalization is given by the definition of $\mathcal{S}$-prime ideal in a commutative ring. The $\mathcal{S}$-prime ideal were first studied by Ahmed Hamed and Achraf Malek in 2019 [1]. Let $\mathcal{R}$ be a commuutative ring, $\mathcal{S} \subseteq \mathcal{R}$ a multiplicative set and $\mathscr{B}$ an ideal of $\mathcal{R}$ disjoint with $\mathcal{S}$. Then $\mathscr{B}$ is $\mathcal{S}$-prime if there exist an $s \in \mathcal{S}$ such that for all $g, h \in \mathcal{R}$ with $gh \in \mathscr{B}$. This implies either $sg \in \mathscr{B}$ or $sh \in \mathscr{B}$.

In 2021, Fuad Ali Ahmahdi, El Mehdi Bouba and Mohammed Tamekkante [7] developed the concept of weakly $\mathcal{S}$-prime ideal of commutative rings. Let $\mathscr{B}$ be an ideal of a ring $\mathcal{R}$. An ideal $\mathscr{B}$ is said to be weakly $\mathcal{S}$-prime ideal if there exist an $s \in \mathcal{S}$ such that for all $g, h \in \mathcal{R}$ if $0 \neq gh \in \mathscr{B}$ then either $sg \in \mathscr{B}$ or $sh \in \mathscr{B}$. This notion generalizes the concept of weakly prime ideals. Recently, Kalamani and Mythily [11] in 2023 examined the $\mathcal{S}$-prime ideal as a nilpotent ideal, discussing several properties of $\mathcal{S}$-prime ideal in commutative ring $\mathcal{R}$ and demonstrating that the nilradical of $\mathcal{R}$ constitutes the $\mathcal{S}$-prime ideal.

This notions of $\mathcal{S}$-prime ideals has been extended to lattice theory. In 2024, Shahabaddin Ebrahimi Atani [14] introduced the concept of $\mathcal{S}$-prime element of a bounded distributive lattice L. Let q be an element of L with $s \wedge q = 0$. Then q is an $\mathcal{S}$-prime element of L if there is an element $s \in \mathcal{S}$ such that for all $g, h \in L$ if $q \leq g \vee h$, then $q \leq g \vee s$ or $q \leq h \vee s$. In 2024, Kalamani and Mythily [10] encounter the $\mathcal{S}$-prime ideal concepts to the semilattice, in which the $\mathcal{S}$-meet Semilattice is introduced and considered the set $L_{\mathcal{S}}$ be the collection of all $\mathcal{S}$-prime ideal of $\mathcal{R}$ which is a poset under inclusion. The prime ideals and maximal ideals are defined in $\mathcal{S}$-meet semilattice and also defined the down-set of $\mathcal{S}$-meet semilattice.

In 2015, [9] C. Jayaram, Unsal Tekir and Ece Yetkin developed the 2-absorbing elements and weakly 2-absorbing elements in multiplicative lattices by means of 2-absorbing and weakly 2-absorbing ideals of a commutative ring with unity. They also characterized join principally generated C-lattices whose proper element are weakly 2-absorbing and every non-zero proper element is 2-absorbing element. Futher In 2015, [6] Fethi Callialp and Ece Yetkin and Unsal Tekir extended the 2-absorbing primary and weakly 2-absorbing primary ideals of a commutative ring with unity to their relevent 2-absorbing primary and weakly 2-absorbing primary elements in multiplicative lattices. In this a characterization of principal element domains in terms of 2-absorbing primary elements is obtained.

Recently, [13] in 2022 Sachin Sarode and Vinayak Joshi introduced a new class of X-element in multiplicative lattices with an $\mathcal{M}$-closed set X and replaced this subset by the set of zero-divisors $Z(L)$, Jacobson radical $J(L)$ and the nilradical $N(L)$ thereby generalizing r-ideals, j- ideals and n-ideals in a commutative ring with unity. Motivated by this work and the studies in [1, 7, 10, 11] on $\mathcal{S}$-prime ideal in commutative ring with unity, the notion of $\mathcal{S}$-prime elements is introduced in the multiplicative lattices.

Let $\mathcal{R}$ be a commutative ring with unity of and $\mathcal{L}(\mathcal{R})$ is the set of ideals of $\mathcal{R}$ endowed with a binary operation " . " as the ideal multiplication on $\mathcal{L}(\mathcal{R})$. Then $(\mathcal{L}(\mathcal{R}), \leq, 0, 1)$ is a multiplicative lattice where the partial order is defined by inclusion. Throughout the paper, $\mathcal{A}$ denotes a finite commutative ring with unity, $\mathcal{L}(\mathcal{A})$ denotes compactly generated multiplicative lattice with greatest compact element 1 and $\mathcal{S}$ be the multiplicative closed subset of $\mathcal{L}(\mathcal{A})$. This paper generalizes the number of $\mathcal{S}$-prime elements in a multiplicative lattice $\mathcal{L}(\mathcal{A})$ and extends some fundamental ring theoretic properties to $\mathcal{S}$-prime elements in multiplicative lattices.

2. Preliminaries

In this part, the essential definitions and known multiplicative lattice $\mathcal{L}$ properties that are necessary for this work are presented.

Definition 2.1. A complete lattice is a lattice $\mathcal{L}$ in which for every subset $\mathcal{E} \subseteq \mathcal{L}$, both $\vee\mathcal{E}$ and $\wedge\mathcal{E}$ exist in $\mathcal{L}$.

Definition 2.2. A complete lattice $\mathcal{L}$ is said to be a multiplicative lattice if there is defined a binary operation " . " called multiplication on $\mathcal{L}$ which satisfies,

1) $g.h = h.g$ for all $g, h \in \mathcal{L}$.
2) $g.(h.i) = (g.h).i$ for all $g, h, i \in \mathcal{L}$.
3) $g. \bigvee_\gamma h_\gamma = \bigvee_\gamma (g.h_\gamma)$ for all $g, h_\gamma \in \mathcal{L}$, $\gamma \in \Lambda$ (an index set).
4) $1.g = g.1$ for all $g \in \mathcal{L}$.

Definition 2.3. An element $g \in \mathcal{L}$ is called compact if for $\mathcal{X} \subseteq \mathcal{L}$, $g \leq \vee\mathcal{X}$ implies the existence of finitely many elements $g_1, g_2, ..., g_n$ in $\mathcal{X}$ such that $g \leq g_1 \vee g_2 \vee ... \vee g_n$. The set of compact elements of $\mathcal{L}$ will be denoted by $\mathcal{L}_*$.

Definition 2.4. A multiplicative lattice is said to be compactly generated if every element of $\mathcal{L}$ can be written as a join of compact elements.

Definition 2.5. An element $j \in \mathcal{L}$ is called a proper element if $j < 1$.

Definition 2.6. A proper element $j \in \mathcal{L}$ is called a prime element if $gh \leq j$ then either $g \leq j$ or $h \leq j$, where $g, h \in \mathcal{L}$.

Definition 2.7. A proper element m in $\mathcal{L}$ is said to be maximal if $m \nleq j$ for any other proper element j in $\mathcal{L}$.

Definition 2.8. A proper element $j \in \mathcal{L}$ is primary if $gh \leq j$ implies either $g \leq j$ or $h^r \leq j$ for some $r \in \mathbb{Z}^+$ and for every $g, h \in \mathcal{L}_*$.

Definition 2.9. A nilpotent element j of $\mathcal{L}$ is an element such that $j^r = 0$ for some $r \in \mathbb{Z}^+$. A lattice $\mathcal{L}$ is said to be a reduced whenever the zero element is the only nilpotent of $\mathcal{L}$ and $Nil(\mathcal{L})$ denotes the collection of nilpotent element of $\mathcal{L}$.

Definition 2.10. An element j of $\mathcal{L}$ is called idempotent if $j = j^2$.

Definition 2.11. An element $j \in \mathcal{L}$ is said to be zero divisor if $ji = 0$ for some $i \neq 0$ in $\mathcal{L}$.

Definition 2.12. The radical of $j \in \mathcal{L}$ is defined as $\sqrt{j} = \vee\{u \in \mathcal{L} \mid u^r \leq j \; for\ some \; r \in \mathbb{Z}^+\}$. If $j = \sqrt{j}$ then j is called a radical element.

Definition 2.13. A multiplicative lattice $\mathcal{L}$ is said to be a local lattice if $\mathcal{L}$ has the unique maximal element m and it is denoted by $(\mathcal{L} : m)$.

Definition 2.14. Let g and h be an element of $\mathcal{L}$. Then the residual of g by h is given by $(g : h) = \vee\{u \in \mathcal{L} \mid u.h \leq g\}$.

Definition 2.15. A subset $T \neq 0$ of a multiplicative lattice $\mathcal{L}$ is said to be multiplicatively closed if $1 \in T$ and $gh \in T$ for each $g, h \in T$.

Definition 2.16. A subset $\mathcal{C}$ of a lattice $\mathcal{L}$ is called a chain if any two elements of $\mathcal{C}$ are comparable.

3. Main Results

In this section, the definition of $\mathcal{S}$-prime element of multiplicative lattice $\mathcal{L}(\mathcal{A})$ is given and some characterization of $\mathcal{S}$-prime elements are studied with suitable examples. Let us take the ideals of a finite commutative ring $\mathcal{A}$ with unity of order n as the elements of $\mathcal{L}(\mathcal{A})$. Let $a \in \mathcal{A}$. Then (a) denotes the principal ideal generated by a and the notation $\mathfrak{a}$ is used in place of (a) where $\mathfrak{a} \in \mathcal{L}(\mathcal{A})$.

Definition 3.1. Let $\mathcal{L}(\mathcal{A})$ be a multiplicative lattice, $\mathcal{S} \subseteq \mathcal{L}(\mathcal{A})$ a multiplicative closed subset and $\mathfrak{p}$ be an element of $\mathcal{L}(\mathcal{A})$ with $\mathfrak{s} \nleq \mathfrak{p}$ for all $\mathfrak{s} \in \mathcal{S}$. Then $\mathfrak{p}$ is said to be an $\mathcal{S}$-prime element if there exist an $\mathfrak{s} \in \mathcal{S}$ such that for all $\mathfrak{g}, \mathfrak{h} \in \mathcal{L}(\mathcal{A})$, $\mathfrak{g}\mathfrak{h} \leq \mathfrak{p}$ implies either $\mathfrak{s}\mathfrak{g} \leq \mathfrak{p}$ or $\mathfrak{s}\mathfrak{h} \leq \mathfrak{p}$.

Example 3.1. Let $\mathcal{A} = \mathbb{Z}_{24}$, $\mathcal{L}(\mathcal{A}) = \{0, 1, 2, 3, 4, 6, 8, 12\}$ and its lattice structure is given in Figure 1. The element $\mathfrak{p} = 6$

is an $\mathcal{S}$-prime element and its corresponding $\mathcal{S} = \{1, 2, 4, 8\}$. Consider $\mathfrak{p} = 4$, $\mathcal{S} = \{1, 3\}$ here $2.6 \leq 4$ but $2, 6 \nleq 4$, by taking $\mathfrak{s} = 3$, $3.2 \nleq 4$ and $3.6 \nleq 4$. Hence $\mathfrak{p} = 4$ is not an $\mathcal{S}$-prime element.

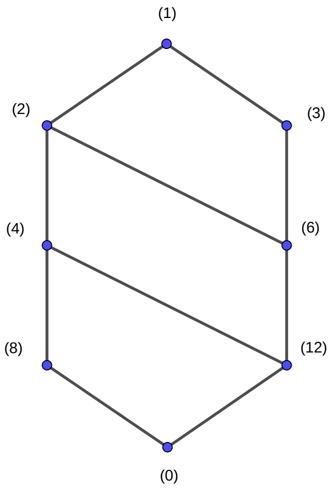

Figure 1. *Multiplicative Lattice* $\mathcal{L}(\mathbb{Z}_{24})$.

Theorem 3.1. If $\mathfrak{p}$ is an $\mathcal{S}$-prime element of $\mathcal{L}(\mathcal{A})$ then $\mathfrak{p}$ is a proper element.

Proof. Assume that $\mathfrak{p}$ be an $\mathcal{S}$-prime element of $\mathcal{L}(\mathcal{A})$ and $\mathcal{S} \subseteq \mathcal{L}(\mathcal{A})$ with $\mathfrak{s} \nleq \mathfrak{p}$ for all $\mathfrak{s} \in \mathcal{S}$. Suppose that, $\mathfrak{p}$ is not a proper element of $\mathcal{L}(\mathcal{A})$ i.e., $\mathfrak{p} = 1$. Since $\mathfrak{p}$ is the greatest element of $\mathcal{L}(\mathcal{A})$ and $\mathcal{S} \subseteq \mathcal{L}(\mathcal{A})$, $\mathfrak{s} \leq \mathfrak{p}$ for all $\mathfrak{s} \in \mathcal{S}$. This contradicts that $\mathfrak{s} \nleq \mathfrak{p}$ for all $\mathfrak{s} \in \mathcal{S}$. Hence $\mathfrak{p}$ is the proper element of $\mathcal{L}(\mathcal{A})$. Therefore, every $\mathcal{S}$-prime element of $\mathcal{L}(\mathcal{A})$ is a proper element.

Theorem 3.2. If $\mathfrak{p}$ is a prime element of $\mathcal{L}(\mathcal{A})$ then $\mathfrak{p}$ is an $\mathcal{S}$-prime element.

Proof. Let $\mathfrak{p}$ be a prime element and there exist an $\mathcal{S} \subseteq \mathcal{L}(\mathcal{A})$ such that $\mathfrak{s} \nleq \mathfrak{p}$ for all $\mathfrak{s} \in \mathcal{S}$. Consider $\mathfrak{g}\mathfrak{h} \leq \mathfrak{p}$ for every $\mathfrak{g}, \mathfrak{h} \in \mathcal{L}(\mathcal{A})$. Since $\mathfrak{p}$ is a prime element, $\mathfrak{g} \leq \mathfrak{p}$ or $\mathfrak{h} \leq \mathfrak{p}$. Taking $\mathfrak{s} = 1$, the element $\mathfrak{p}$ becomes the $\mathcal{S}$-prime element of $\mathcal{L}(\mathcal{A})$. Hence every prime element of $\mathcal{L}(\mathcal{A})$ is an $\mathcal{S}$-prime element. Conversely, an $\mathcal{S}$-prime element is not necessarily a prime element.

Some results on $\mathcal{S}$-prime element in the reduced lattice $\mathcal{L}(\mathcal{A})$ are generalized in the subsequent theorems.

Theorem 3.3. Every element of a reduced lattice $\mathcal{L}(\mathcal{A})$ is an idempotent element.

Proof. Obviously, 0 and 1 are idempotent elements. Let $\mathfrak{g} \neq 0$ be any element of $\mathcal{L}(\mathcal{A})$. Let $\mathcal{C}_{\mathfrak{g}}$ be the maximal chain containing $\mathfrak{g}$. If $0 \leqslant \mathfrak{g}$ then $\mathfrak{g}^2 \neq 0$, as $\mathcal{L}(\mathcal{A})$ is reduced. It is known that $\mathfrak{g}^2 \leqslant \mathfrak{g}$ i.e., $0 \leqslant \mathfrak{g}^2 \leqslant \mathfrak{g}$. This implies that $\mathfrak{g}^2 = \mathfrak{g}$. Therefore $\mathfrak{g}$ is an idempotent element. Suppose that $0 \leqslant \mathfrak{h} \leqslant \mathfrak{g}$ in the chain $\mathcal{C}_{\mathfrak{g}}$ where $\mathfrak{h} \neq 0 \in \mathcal{L}(\mathcal{A})$. As above, $\mathfrak{h}$ is the idempotent element. If $\mathfrak{h} \leqslant \mathfrak{g}$ then $\mathfrak{g}^2 = \mathfrak{h}$. Since $\mathfrak{h}$ is idempotent element, $\mathfrak{g}^2 = \mathfrak{h}^2$ and hence $\mathfrak{g} = \mathfrak{h}$. Therefore $\mathfrak{g}^2 = \mathfrak{g}$. Thus $\mathfrak{g}$ and $\mathfrak{h}$ are idempotent elements.

In the same way, it can be proved that all the elements in the chain $\mathcal{C}_{\mathfrak{g}}$ are idempotent elements. This is true for all chains in $\mathcal{L}(\mathcal{A})$. Hence every element of $\mathcal{L}(\mathcal{A})$ is an idempotent element.

Theorem 3.4. Every element of a reduced lattice $\mathcal{L}(\mathcal{A})$, is a radical element.

Proof. Clearly, 0 and 1 are radical elements. Let $\mathfrak{g}$ be any non-zero element of $\mathcal{L}(\mathcal{A})$. Consider the set $\mathcal{D} = \{\mathfrak{u} \in \mathcal{L}(\mathcal{A}) \mid \mathfrak{u}^r \leqslant \mathfrak{g}, for\ some\ r\ \in \mathbb{Z}^+\}$. As $0 \leqslant \mathfrak{g}$ and $\mathfrak{g} \leqslant \mathfrak{g}$, 0 and $\mathfrak{g}$ are in $\mathcal{D}$ for all $\mathfrak{g} \in \mathcal{D}$. If 0 and $\mathfrak{g}$ are the only elements in $\mathcal{D}$ then Sup $\mathcal{D}$ = $\mathfrak{g}$ i.e.,$\sqrt{\mathfrak{g}} = \mathfrak{g}$.

Suppose if there exist $\mathfrak{u} \neq 0$ and $\mathfrak{u} \neq \mathfrak{g}$ in $\mathcal{L}(\mathcal{A})$ such that $\mathfrak{u} \in \mathcal{D}$. Then either $\mathfrak{u} \leqslant \mathfrak{g}$ or $\mathfrak{g} \leqslant \mathfrak{u}$. If $\mathfrak{u} \leqslant \mathfrak{g}$ then Sup $\mathcal{D}$ = $\mathfrak{g}$. Hence $\sqrt{\mathfrak{g}} = \mathfrak{g}$.

If $\mathfrak{u} \nleqslant \mathfrak{g}$ then $\mathfrak{g} \leqslant \mathfrak{u}$. Since $\mathfrak{g} \neq 0$, $\mathfrak{g}^2 \leqslant \mathfrak{g}\mathfrak{u}$, as $\mathfrak{g}$ is the idempotent element. This implies that $1 \leq \mathfrak{u}$. This gives $\mathfrak{u} = 1$. Now $\mathfrak{g} \leqslant 1$, $\mathfrak{g}$ becomes the maximal element of $\mathcal{L}(\mathcal{A})$ and $1 \notin \mathcal{D}$. Since $\mathfrak{g}$ is the maximal element of $\mathcal{L}(\mathcal{A})$, Sup $\mathcal{D} = \mathfrak{g}$. Therefore $\sqrt{\mathfrak{g}} = \mathfrak{g}$. Hence $\mathfrak{g}$ is a radical element of $\mathcal{L}(\mathcal{A})$.

Recall that, for a chain $\mathcal{C}$, $a, b \in \mathcal{C}$ the half-open interval $(a, b] = \{x \mid a < x \leq b\}$ [8]. Then $\mathcal{C}(0, 1]$ of a reduced lattice $\mathcal{L}(\mathcal{A})$ is a multiplicative closed subset.

Theorem 3.5. Every chain $\mathcal{C}(0, 1]$ of a reduced lattice $\mathcal{L}(\mathcal{A})$ is a multiplicative closed subset $\mathcal{S}$.

Proof. Let $\mathcal{C}(0, 1]$ be a chain in $\mathcal{L}(\mathcal{A})$ and it is a subset of $\mathcal{L}(\mathcal{A})$. Clearly $1 \in \mathcal{C}(0, 1]$. Let $\mathfrak{g}, \mathfrak{h} \in \mathcal{C}(0, 1]$. Then either $\mathfrak{g} \leq \mathfrak{h}$ or $\mathfrak{h} \leq \mathfrak{g}$. Since $\mathcal{L}(\mathcal{A})$ is reduced, $\mathfrak{g}^r \neq 0$ and $\mathfrak{h}^r \neq 0$ for all $r \in \mathbb{Z}^+$. Assume that $\mathfrak{g} \leq \mathfrak{h}$. Then $\mathfrak{g}\mathfrak{h} = \mathfrak{g} \wedge \mathfrak{h} = \mathfrak{g} \in \mathcal{C}(0, 1]$ [2]. Hence every chain $\mathcal{C}(0, 1]$ of a reduced lattice is a multiplicative closed subset $\mathcal{S}$.

Theorem 3.6. If $\mathfrak{p}$ be a proper element of a reduced lattice $\mathcal{L}(\mathcal{A})$ then $\mathfrak{p}$ is an $\mathcal{S}$-prime element.

Proof. Let $\mathfrak{p}$ be an arbitrary proper element and $\mathcal{S}_{\mathfrak{t}}$ be the chain $\mathcal{C}(0, 1]$ of $\mathcal{L}(\mathcal{A})$ containing an atom $\mathfrak{t} \nleq \mathfrak{p}$. By theorem (3.5), the chain $\mathcal{S}_{\mathfrak{t}} \subseteq \mathcal{L}(\mathcal{A})$ is a multiplicative closed subset. Consider $\mathfrak{g}\mathfrak{h} \leq \mathfrak{p}$ where $\mathfrak{g}, \mathfrak{h} \in \mathcal{L}(\mathcal{A})$.

Case 1: If $\mathfrak{p}$ is a prime element of $\mathcal{L}(\mathcal{A})$ then $\mathfrak{g} \leq \mathfrak{p}$ or $\mathfrak{h} \leq \mathfrak{p}$. As the element $1 \in \mathcal{S}_{\mathfrak{t}}$, $\mathfrak{p}$ is an $\mathcal{S}$-prime element of $\mathcal{L}(\mathcal{A})$.

Case 2: Let $\mathfrak{p}$ be a non-prime element of $\mathcal{L}(\mathcal{A})$.

Assume that $\mathfrak{g} \leq \mathfrak{p}$ or $\mathfrak{h} \leq \mathfrak{p}$. If $\mathfrak{g} \leq \mathfrak{p}$ then $\mathfrak{s}\mathfrak{g} \leq \mathfrak{p}$ for $\mathfrak{s} = 1 \in \mathcal{S}_{\mathfrak{t}}$. The similar result is true for $\mathfrak{h} \leq \mathfrak{p}$. Suppose that $\mathfrak{g} \nleq \mathfrak{p}$ and $\mathfrak{h} \nleq \mathfrak{p}$. If both $\mathfrak{g} \in \mathcal{S}_{\mathfrak{t}}$ and $\mathfrak{h} \in \mathcal{S}_{\mathfrak{t}}$, then $\mathfrak{g}\mathfrak{h} \in \mathcal{S}_{\mathfrak{t}}$ i.e., $\mathfrak{g}\mathfrak{h} \nleq \mathfrak{p}$ which is a contradiction. Therefore either $\mathfrak{g} \notin \mathcal{S}_{\mathfrak{t}}$ or $\mathfrak{h} \notin \mathcal{S}_{\mathfrak{t}}$.

Assume that $\mathfrak{g} \in \mathcal{S}_{\mathfrak{t}}$ and $\mathfrak{h} \notin \mathcal{S}_{\mathfrak{t}}$. If $\mathfrak{g} \in \mathcal{S}_{\mathfrak{t}}$ then $\mathfrak{s} = \mathfrak{g}$. Thus $\mathfrak{s}\mathfrak{h} \leq \mathfrak{p}$. Consider $\mathfrak{g}, \mathfrak{h} \notin \mathcal{S}_{\mathfrak{t}}$ where $\mathfrak{g}$ or $\mathfrak{h}$ is an atom of $\mathcal{L}(\mathcal{A})$. If $\mathfrak{g}$ is an atom then $\mathfrak{t}\mathfrak{g} = 0 \leq \mathfrak{p}$. As the multiplication of any two distinct atom is the least element 0. By taking $\mathfrak{s} = \mathfrak{t}$, $\mathfrak{s}\mathfrak{g} \leq \mathfrak{p}$. Suppose that $\mathfrak{g}$ and $\mathfrak{h}$ are not atoms of $\mathcal{L}(\mathcal{A})$. If $\mathfrak{t} \leq \mathfrak{g}$ and $\mathfrak{t} \leq \mathfrak{h}$, then $\mathfrak{t} \leq \mathfrak{g}\mathfrak{h}$. Since $\mathfrak{g}\mathfrak{h} \leq \mathfrak{p}$, $\mathfrak{t} \leq \mathfrak{p}$. This is a contradiction. Therefore either $\mathfrak{t} \nleq \mathfrak{g}$ or $\mathfrak{t} \nleq \mathfrak{h}$. Assume that $\mathfrak{t} \nleq \mathfrak{g}$. Then $\mathfrak{t}\mathfrak{g} = 0 \leq \mathfrak{p}$ when $\mathfrak{s} = \mathfrak{t}$, $\mathfrak{s}\mathfrak{g} \leq \mathfrak{p}$.

Hence the proper element $\mathfrak{p}$ of a reduced lattice $\mathcal{L}(\mathcal{A})$ is an $\mathcal{S}$-prime element.

Example 3.2. Consider the reduced lattice $\mathcal{L}(\mathcal{A}) = \{0, 1, 2, 3, 5, 6, 7, 10, 14, 15, 21, 30, 35, 42, 70, 105\}$ and its lattice structure is given in Figure 2 where $\mathcal{A} = \mathbb{Z}_{210}$. Let us take the proper element $\mathfrak{p} = 70$, $\mathcal{S}_\mathfrak{t} = \{1, 3, 21, 105\}$ the atom element is $\mathfrak{t} = 105$. Take $\mathfrak{g} = 10$ and $\mathfrak{h} = 42$ such that $\mathfrak{gh} \leq \mathfrak{p}$ where $\mathfrak{g}, \mathfrak{h} \nleq \mathfrak{p}$. Since $\mathfrak{t} \nleq \mathfrak{g}$ and $\mathfrak{t} \nleq \mathfrak{h}$, choose $\mathfrak{s} = \mathfrak{t} \in \mathcal{S}_\mathfrak{t}$ such that $105.10 \leq \mathfrak{p}$ and $105.42 \leq \mathfrak{p}$. Thus $\mathfrak{p} = 70$ is an $\mathcal{S}$-prime element.

Similarly, for each proper element in $\mathcal{L}(\mathcal{A})$ there exist a corresponding $\mathcal{S}_\mathfrak{t} \subseteq \mathcal{L}(\mathcal{A})$ with $\mathfrak{s} \nleq \mathfrak{p}\ \forall \mathfrak{s} \in \mathcal{S}$. Thus every proper element in a reduced lattice is an $\mathcal{S}$-prime element.

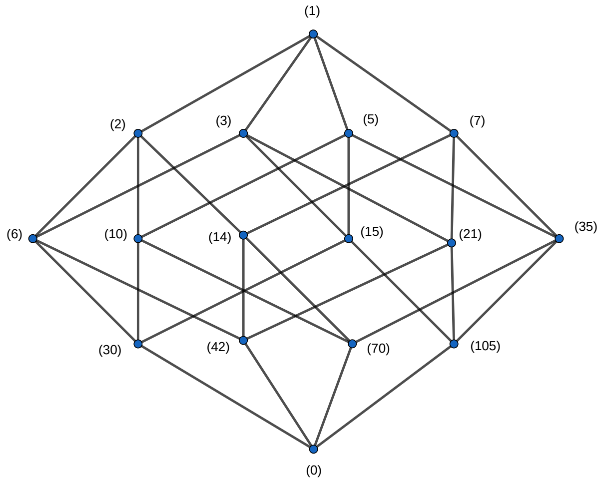

Figure 2. *Multiplicative Lattice* $\mathcal{L}(\mathbb{Z}_{210})$.

The generalization of $\mathcal{S}$-prime elements in multiplicative lattice $\mathcal{L}(\mathcal{A})$ is investigated in the next theorem.

Theorem 3.7. Let $\mathcal{L}(\mathcal{A})$ be the multiplicative lattice of a commutative ring $\mathcal{A}$ with unity of order n. Then the number of $\mathcal{S}$-prime elements of $\mathcal{L}(\mathcal{A})$ is $|\mathcal{L}(\mathcal{A})| - |\mathcal{S}_{\mathfrak{p}'}|$, where $\mathcal{S}_{\mathfrak{p}'}$ is the set of non $\mathcal{S}$-prime elements of $\mathcal{L}(\mathcal{A})$.

Proof. Let $n = \prod_{i=1}^{k} \mathfrak{p}_i^{\alpha_i}$ be the prime factorization of n with $\alpha_i \geq 1$ for all i. If $\mathfrak{p} \in \mathcal{L}(\mathcal{A})$ then $\mathfrak{p} = \mathfrak{p}_1^{\beta_1}\mathfrak{p}_2^{\beta_2}...\mathfrak{p}_k^{\beta_k}$ where $0 \leq \beta_i \leq \alpha_i$ be an element of $\mathcal{L}(\mathcal{A})$. Let $\mathcal{S}$ be the upset of $\mathfrak{p}_2^{\alpha_2}\mathfrak{p}_3^{\alpha_3}....\mathfrak{p}_k^{\alpha_k}$ where $\alpha_1 \leq \alpha_i$, $i = 2, 3, ..., k$. Then $\mathcal{S} \subseteq \mathcal{L}(\mathcal{A})$ with $\mathfrak{s} \nleq \mathfrak{p}$ for all $\mathfrak{s} \in \mathcal{S}$. Consider $\mathfrak{gh} \leq \mathfrak{p}$ for every $\mathfrak{g}, \mathfrak{h} \in \mathcal{L}(\mathcal{A})$.

Case 1: $\beta_i = 1$ for some i. Let $\beta_1 = 1$, which implies that $\mathfrak{p} = \mathfrak{p}_1\mathfrak{p}_2^{\beta_2}\mathfrak{p}_3^{\beta_3}..\mathfrak{p}_k^{\beta_k}$. Assume that either $\mathfrak{g} \leq \mathfrak{p}$ or $\mathfrak{h} \leq \mathfrak{p}$. If $\mathfrak{g} \leq \mathfrak{p}$ then $\mathfrak{sg} \leq \mathfrak{p}$ for $\mathfrak{s} = 1 \in \mathcal{S}$. Suppose that $\mathfrak{g} \nleq \mathfrak{p}$ and $\mathfrak{h} \nleq \mathfrak{p}$. This implies that either $\mathfrak{g} \notin \mathcal{S}$ or $\mathfrak{h} \notin \mathcal{S}$. Let us take $\mathfrak{g} \notin \mathcal{S}$. Then $\mathfrak{g} \leq \mathfrak{p}_1$. If for every $\mathfrak{s} \in \downarrow \mathfrak{p}_2^{\beta_2}\mathfrak{p}_3^{\beta_3}..\mathfrak{p}_k^{\beta_k} \subseteq \mathcal{S}$ then $\mathfrak{sg} \leq \mathfrak{p}$. Thus $\mathfrak{p}$ is an $\mathcal{S}$-prime element of $\mathcal{L}(\mathcal{A})$.

Case 2: $\beta_1 \neq 1$ for all i. Assume that $\mathfrak{g}, \mathfrak{h} \nleq \mathfrak{p}$. This implies that either $\mathfrak{g} \notin \mathcal{S}$ or $\mathfrak{h} \notin \mathcal{S}$. Then $\mathfrak{g} \leq \mathfrak{p}_1$. Let $\mathfrak{z}$ be an element of $\mathcal{L}(\mathcal{A})$ such that $\mathfrak{z} = \mathfrak{p}_1^{\beta_1-1}\mathfrak{p}_2^{\beta_2-1}....\mathfrak{p}_k^{\beta_k-1}$ then there exist a chain $\mathcal{C}(\mathfrak{z}, \mathfrak{p}_1)$ which is disjoint with $\mathcal{S}$. Thus for all $\mathfrak{g} \in \mathcal{C}(\mathfrak{z}, \mathfrak{p}_1)$ and for any $\mathfrak{s} \in \mathcal{S}$, $\mathfrak{sg} \nleq \mathfrak{p}$. Hence $\mathfrak{p}$ is not an $\mathcal{S}$-prime element of $\mathcal{L}(\mathcal{A})$.

Define $\mathcal{S}_{\mathfrak{p}'} = \{\mathfrak{p} \in \mathcal{L}(\mathcal{A}) \mid \mathfrak{sg} \nleq \mathfrak{p}\ for\ any\ \mathfrak{s} \in \mathcal{S}\ and\ for\ all\ \mathfrak{g} \in \mathcal{C}(\mathfrak{z}, \mathfrak{p}_1)\}$ is the set of non $\mathcal{S}$-prime elements $\mathcal{L}(\mathcal{A})$. Then the total number of $\mathcal{S}$-prime elements of $\mathcal{L}(\mathcal{A}) = \tau(n) - |\mathcal{S}_{\mathfrak{p}'}|$ where $\tau(n)$ is the number of divisors of n.

Hence the number of $\mathcal{S}$-prime elements of $\mathcal{L}(\mathcal{A}) = |\mathcal{L}(\mathcal{A})| - |\mathcal{S}_{\mathfrak{p}'}|$.

Corollary 3.1. $|\mathcal{S}_{\mathfrak{p}'}| = \prod_{\alpha_i}$.

Proof. If $\mathfrak{p} \in \mathcal{S}_{\mathfrak{p}'}$ then $\mathfrak{p} = \mathfrak{p}_1^{\beta_1}\mathfrak{p}_2^{\beta_2}....\mathfrak{p}_k^{\beta_k}$ where $\beta_i \neq 1$ for all i. If $n = \mathfrak{p}_1^{\alpha_1}\mathfrak{p}_2^{\alpha_2}....\mathfrak{p}_k^{\alpha_k}$, $\alpha_i \geq 0$ then $|\mathcal{L}(\mathcal{A})| = \tau(n) = (\alpha_1 + 1)...(\alpha_k + 1)$. Since $0 \leq \beta_i \leq \alpha_i$ where $\beta_i \neq 1$, there are $\alpha_1\alpha_2....\alpha_k$ elements in $\mathcal{S}_{\mathfrak{p}'}$.

Hence $|\mathcal{S}_{\mathfrak{p}'}| = \prod_{i=1}^{k} \alpha_i$.

Example 3.3. Consider $\mathcal{L}(\mathbb{Z}_{60})$= $\{0, 1, 2, 3, 4, 5, 6, 10, 12, 15, 30\}$ where $n = 60 = 2^2.3.5$ and $|\mathcal{L}(\mathbb{Z}_{60})| = \tau(n) = 12$ its lattice structure is given in figure 3. For a proper element $\mathfrak{p} = 0$ its corresponding $\mathcal{S}$ is the upset of $(2^2.3)$ i.e., $\mathcal{S} = \{1, 2, 3, 4, 6, 12\}$. Here $20.30 \leq 0$. Taking $\mathfrak{s} \in \downarrow (2^2.3)$ $=12 \in \mathcal{S}$, $12.20 \leq 0$ and $12.30 \leq 0$. Hence 0 is an $\mathcal{S}$-prime element. Now for $\mathfrak{p} = 4$, $\mathcal{S}$ is the upset of (3.5) which is $\mathcal{S} = \{1, 3, 5, 15\}$. For $10.30 = 0 \leq 4$ by taking any $\mathfrak{s} \in \mathcal{S}$, $\mathfrak{p} = 4$ is not an $\mathcal{S}$-prime element. Therefore the non $\mathcal{S}$-prime elements of $\mathcal{L}(\mathbb{Z}_{60})$ are 4 and 1. Thus $|\mathcal{S}_{\mathfrak{p}'}| = \prod_{\alpha_i}$, as $\prod_{\alpha_i} = 2$. Then the number of $\mathcal{S}$-prime elements in $\mathcal{L}(\mathbb{Z}_{60})$ $= 12 - 2 = 10$. Hence there are 10 $\mathcal{S}$-prime elements in $\mathcal{L}(\mathbb{Z}_{60})$.

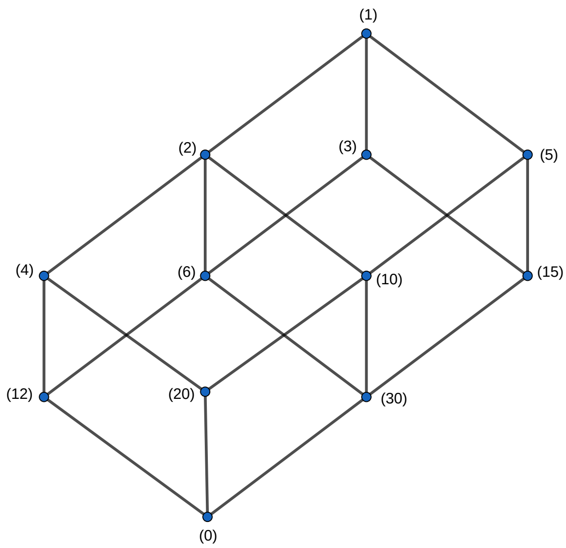

Figure 3. *Multiplicative Lattice* $\mathcal{L}(\mathbb{Z}_{60})$.

The next theorem describes a relation between radical of a nilpotent element and an $\mathcal{S}$-prime element.

Theorem 3.8. The radical of the nilpotent element of $\mathcal{L}(\mathcal{A})$ is an $\mathcal{S}$-prime element of $\mathcal{L}(\mathcal{A})$.

Proof. Assumes that $\mathfrak{p}$ is a nilpotent element of $\mathcal{L}(\mathcal{A})$ and $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \dots, \mathfrak{p}_k$ be the distinct prime elements of $\mathcal{L}(\mathcal{A})$. Consider the set $\mathcal{D} = \{\mathfrak{u} \in \mathcal{L}(\mathcal{A}) \mid \mathfrak{u}^r \leqslant \mathfrak{g}$, for some $r \in \mathbb{Z}^+\}$. Suppose if there exist $\mathfrak{u} \neq 0$ and $\mathfrak{u} \neq \mathfrak{p}$ in $\mathcal{L}(\mathcal{A})$ such that $\mathfrak{u} \in \mathcal{D}$. Then either $\mathfrak{u} \leq \mathfrak{p}$ or $\mathfrak{p} \leq \mathfrak{u}$. If $\mathfrak{u} \leq \mathfrak{p}$ then $\mathfrak{u} \leq \mathfrak{p} \leq \mathfrak{p}_1 \wedge \mathfrak{p}_2 \wedge \mathfrak{p}_3 \wedge \cdots \wedge \mathfrak{p}_k$. Thus $\mathfrak{u}$ is a nilpotent and $\sup \mathcal{D} = \mathfrak{p}$. Hence $\sqrt{\mathfrak{p}} = \mathfrak{p}$. If $\mathfrak{u} \nleq \mathfrak{p}$ then $\mathfrak{p} \leq \mathfrak{u}$. Assume that $\mathfrak{u}$ is nilpotent. Then $\mathfrak{u}^r = 0 \leq \mathfrak{p}$ for some $r \in \mathbb{Z}^+$. Thus $\mathfrak{u} \in \mathcal{D}$. Suppose that $\mathfrak{u}$ is not nilpotent. As $\mathfrak{u}^r \leq \mathfrak{u}$ and $\mathfrak{p}$ is nilpotent, $\mathfrak{u}^r \nleq \mathfrak{p}$. Thus the set $\mathcal{D}$ contains only nilpotent elements i.e., $\mathcal{D} = \mathrm{Nil}\,\mathcal{L}(\mathcal{A})$. Since $\mathfrak{p} \leq \mathfrak{p}_1 \wedge \mathfrak{p}_2 \wedge \mathfrak{p}_3 \wedge \cdots \wedge \mathfrak{p}_k$, the meet of prime elements is same as the join of nilpotent elements. Thus $\sup \mathcal{D} = \mathfrak{p}_1 \wedge \mathfrak{p}_2 \wedge \mathfrak{p}_3 \wedge \cdots \wedge \mathfrak{p}_k$. Therefore $\sqrt{\mathfrak{p}} = \mathfrak{p}_1 \wedge \mathfrak{p}_2 \wedge \mathfrak{p}_3 \wedge \cdots \wedge \mathfrak{p}_k$. Now consider $\mathfrak{g}\mathfrak{h} \leq \mathfrak{p}_1 \wedge \mathfrak{p}_2 \wedge \mathfrak{p}_3 \wedge \cdots \wedge \mathfrak{p}_k$ and $\mathcal{S}$ is the upset of $\mathfrak{p}_2\mathfrak{p}_3 \dots \mathfrak{p}_k$. The proof now follows as in theorem 3.7, case 1. For $\mathfrak{g}, \mathfrak{h} \nleq \mathfrak{p}_1 \wedge \mathfrak{p}_2 \wedge \mathfrak{p}_3 \wedge \cdots \wedge \mathfrak{p}_k$. Then either $\mathfrak{g} \notin \mathcal{S}$ or $\mathfrak{h} \notin \mathcal{S}$. If $\mathfrak{g} \notin \mathcal{S}$ then $\mathfrak{g} \leq \mathfrak{p}_1$. By taking $\mathfrak{s} = \mathfrak{p}_2\mathfrak{p}_3 \dots \mathfrak{p}_k \in \mathcal{S}$, $\mathfrak{s}\mathfrak{g} \leq (\mathfrak{p}_2\mathfrak{p}_3 \dots \mathfrak{p}_k)\mathfrak{p}_1 \leq \mathfrak{p}_1 \wedge \mathfrak{p}_2 \wedge \mathfrak{p}_3 \wedge \cdots \wedge \mathfrak{p}_k$. This implies that $\mathfrak{s}\mathfrak{g} \leq \mathfrak{p}_1 \wedge \mathfrak{p}_2 \wedge \mathfrak{p}_3 \wedge \cdots \wedge \mathfrak{p}_k$. Thus $\mathfrak{p}_1 \wedge \mathfrak{p}_2 \wedge \mathfrak{p}_3 \wedge \cdots \wedge \mathfrak{p}_k$ is an $\mathcal{S}$-prime element. Hence $\sqrt{\mathfrak{p}}$ is an $\mathcal{S}$-prime element of $\mathcal{L}(\mathcal{A})$.

Corollary 3.2. The radical of the nilpotent element of a local lattice $\mathcal{L}(\mathcal{A})$ is a prime element.

As a consequence of theorem 3.7 and corollary 3.1 the following corollary is obtained.

Corollary 3.3. $|Nil\mathcal{L}(\mathcal{A})| = |\mathcal{S}_{\mathfrak{p}'}|$.

Proof. Let $n = \mathfrak{p}_1^{\alpha_1}\mathfrak{p}_2^{\alpha_2}...\mathfrak{p}_k^{\alpha_k}$, $\alpha_i \geq 0$ and $\mathfrak{p}$ be a nilpotent element $\mathcal{L}(\mathcal{A})$, i.e., $\mathfrak{p} \leq \mathfrak{p}_1 \wedge \mathfrak{p}_2 \wedge .... \wedge \mathfrak{p}_k$. Then $\mathfrak{p} = \mathfrak{p}_1^{\beta_1}\mathfrak{p}_2^{\beta_2}....\mathfrak{p}_k^{\beta_k}$ where $1 \leq \beta_i \leq \alpha_i$ for each i. Since $1 \leq \beta_i \leq \alpha_i$, there are α_i choices for each β_i. Thus $|Nil\mathcal{L}(\mathcal{A})| = \prod_{i=1}^{k} \alpha_i$.

Hence $|Nil\mathcal{L}(\mathcal{A})| = |\mathcal{S}_{\mathfrak{p}'}|$.

The following theorem provides a characterization of an $\mathcal{S}$-prime element and their properties in a local lattice $\mathcal{L}(\mathcal{A})$ by means of equivalent conditions.

Theorem 3.9. If $\mathcal{L}(\mathcal{A})$ is a multiplicative lattice the following conditions are equivalent

1. $\mathcal{L}(\mathcal{A})$ is a local lattice
2. $\mathcal{L}(\mathcal{A})$ has a unique $\mathcal{S}$-prime element.
3. Every zero-divisor element is nilpotent element.

Proof. (1) $\implies$ (2)

Let $\mathcal{L}(\mathcal{A})$ be a local lattice. Then it has a unique maximal element $\mathfrak{m}$. Since every maximal element is prime say $\mathfrak{p}$, it is obvious that $\mathfrak{p}$ is an $\mathcal{S}$-prime element. Suppose that there exist another $\mathcal{S}$-prime element say $\mathfrak{p}'$. Since $\mathcal{L}(\mathcal{A})$ is local, every element except 1 is a nilpotent element. Thus the multipliative closed subset $\mathcal{S}$ of $\mathcal{L}(\mathcal{A})$ has the greatest element 1. Consider $\mathfrak{g}\mathfrak{h} \leq \mathfrak{p}'$ where $\mathfrak{g}, \mathfrak{h} \in \mathcal{L}(\mathcal{A})$. Then either $\mathfrak{s}\mathfrak{g} \leq \mathfrak{p}'$ or $\mathfrak{s}\mathfrak{h} \leq \mathfrak{p}'$ for some $\mathfrak{s} \in \mathcal{S}$. Since $\mathfrak{s} = 1$, $\mathfrak{g} \leq \mathfrak{p}'$ or $\mathfrak{h} \leq \mathfrak{p}'$. This implies that $\mathfrak{p}'$ is prime element of $\mathcal{L}(\mathcal{A})$. Thus $\mathfrak{p} = \mathfrak{p}'$.

Hence $\mathcal{L}(\mathcal{A})$ has a unique $\mathcal{S}$-prime element. (1) $\implies$ (3)

Assume that $\mathcal{L}(\mathcal{A})$ has a unique maximal element say $\mathfrak{m}$. Suppose that $\mathfrak{e}$ be a zero divisor $\mathcal{L}(\mathcal{A})$. If $\mathfrak{e} = 1$ then there exist no $\mathfrak{f}$ in $\mathcal{L}(\mathcal{A})$ such that $\mathfrak{e}\mathfrak{f} = 0$. Thus $\mathfrak{e} \neq 1$. Since $\mathfrak{e} \leq \mathfrak{m}$, $\mathfrak{e} \leq$ the prime element $\mathfrak{p}$. Then $\mathfrak{e}$ is a nilpotent element.

Hence every zero divisor is a nilpotent element.

(2) $\implies$ (3) is obvious

(3) $\implies$ (1)

It is clear that the maximal element exist in $\mathcal{L}(\mathcal{A})$. Suppose that $\mathcal{L}(\mathcal{A})$ has maximal elements $\mathfrak{m}_1$ and $\mathfrak{m}_2$. Let $\mathfrak{e} \in \mathcal{L}(\mathcal{A})$ be a zero divisor. Then $\mathfrak{e} \in Nil\mathcal{L}(\mathcal{A})$ such that $\mathfrak{e} \leq \mathfrak{m}_1 \wedge \mathfrak{m}_2$ i.e., $\mathfrak{e} \leq \mathfrak{p}_1 \wedge \mathfrak{p}_2$. Since every maximal element is a zero divisor, it follows that $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are zero divisors. Thus $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are nilpotent elements, i.e., $\mathfrak{m}_1 \leq \mathfrak{m}_1 \wedge \mathfrak{m}_2$ and $\mathfrak{m}_2 \leq \mathfrak{m}_1 \wedge \mathfrak{m}_2$ which is not possible as $\mathfrak{m}_1 \nleq \mathfrak{m}_2$. Therefore $\mathfrak{m}_1 = \mathfrak{m}_2$. Thus $\mathcal{L}(\mathcal{A})$ has a unique maximal element

Hence $\mathcal{L}(\mathcal{A})$ is a local lattice.

4. Conclusion

In this work, the definition of an $\mathcal{S}$-prime element is introduced within the framework of multiplicative lattices. Futher, every proper element of a reduced lattice constitutes an $\mathcal{S}$-prime element and the associated multiplicative closed subset is $\mathcal{C}(0, 1]$ is illustrated through an example. Moreover, the number of $\mathcal{S}$-prime elements in a multiplicative lattice $\mathcal{L}(\mathcal{A})$ is determined in relation to a commutative ring of order n. Finally, the behaviour of $\mathcal{S}$-prime element is characterized with respect to nilpotent elements, with particular emphasis on the case of local lattice $\mathcal{L}(\mathcal{A})$.

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Abbreviations

$\mathcal{L}$	Multiplicative Lattices
$\mathcal{R}$	Commutative Ring with Unity
$\mathcal{A}$	Finite Commutative Ring with Unity of Order n
$\mathcal{L}(\mathcal{A})$	Compactly Generated Multiplicative Lattice with Greatest Compact Element 1
n	Order of the Ring $\mathcal{A}$
$\tau(n)$	Number of Divisors of n
$\mathcal{S}$	Multiplicatively Closed Subset of $\mathcal{L}(\mathcal{A})$
$\mathcal{S}_{\mathfrak{t}}$	Multiplicatively Closed Subset of $\mathcal{L}(\mathcal{A})$ Chain $\mathcal{C}(0, 1]$
$\mathcal{C}_{\mathfrak{g}}$	Maximal Element of $\mathcal{L}(\mathcal{A})$
$\mathfrak{p}$	Proper Element of $\mathcal{L}(\mathcal{A})$
$\mathfrak{m}$	Maximal Element of $\mathcal{L}(\mathcal{A})$
$\mathfrak{t}$	Atom Element of Reduced Lattice $\mathcal{L}(\mathcal{A})$
$\mathcal{S}_{\mathfrak{p}'}$	Collection of Non $\mathcal{S}$-prime Element of $\mathcal{L}(\mathcal{A})$
Nil$\mathcal{L}(\mathcal{A})$	Set of Nilpotent Elements of $\mathcal{L}(\mathcal{A})$

Author Contributions

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Conflicts of Interest

The authors declare no conflicts of interest.

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