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Research Article

Adaptive Orthonormal Bernstein Collocation for Stochastic Itô–Volterra Equations

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Abstract

This paper develops an adaptive, well-conditioned, and scalable collocation scheme for linear and nonlinear stochastic Itô–Volterra integral equations, built upon orthonormal Bernstein polynomials. Five technical barriers that limit existing Bernstein-based schemes are systematically removed. First, an explicit closed-form representation of the orthonormal Bernstein polynomials is derived, eliminating the cubic Gram–Schmidt cost of earlier constructions. Second, a complete numerical-stability analysis proves that the condition number of the collocation system remains uniformly bounded as the polynomial degree increases, in sharp contrast to the exponential ill-conditioning of monomial Bernstein schemes. Third, the scheme is extended to nonlinear stochastic Itô–Volterra integral equations through a fixed-point iteration whose convergence is guaranteed under an explicit smallness condition. Fourth, weakly singular Abel kernels are handled by a weighted inner product that preserves orthonormality and delivers a quasi-optimal projection. Fifth, a control-variate variance-reduction technique reduces the number of Brownian paths needed for a prescribed statistical tolerance by a factor between five and ten. A residual-based a posteriori error estimator drives an adaptive selection of the polynomial degree, and a multi-domain strategy with an explicit optimal refinement rule preserves exponential convergence over long time intervals. Numerical experiments, including a ten-dimensional Brownian example, confirm four to five orders of magnitude improvement over block-pulse and hat-function methods, up to twenty times speedup over recent spectral competitors, and efficient scalability in the polynomial degree, in the number of Brownian paths, and in the dimension of the Brownian motion.

Keywords

Stochastic Itô–Volterra Integral Equation, Orthonormal Bernstein Polynomial, Stochastic Operational Matrix, Adaptive Collocation, Nonlinear SIVIE, Variance Reduction

1. Introduction

Stochastic Itô–Volterra integral equations (SIVIEs) describe systems that combine memory effects with random perturbations, and arise naturally in biology, chemistry, epidemiology, finance, and microelectronics [1–3]. Because exact solutions are seldom available, numerical methods are indispensable [4–7].

1.1. Related Work and Remaining Gaps

Classical numerical treatments include block-pulse functions (BPFs) [6, 8], generalized hat basis functions (GHFs) [11], and Euler-type schemes [9, 10]. Recent operational-matrix and spectral approaches [12–18] have considerably improved

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accuracy, yet five gaps persist.

(G1) Ill-conditioning of the Bernstein basis. The monomial Bernstein Gram matrix has exponentially growing condition number, which saturates numerical accuracy at low polynomial degrees [16].

(G2) Cost of orthonormalisation. Existing orthonormal Bernstein polynomial (OBP) constructions rely on Gram–Schmidt orthogonalisation at cubic cost.

(G3) Restricted to linear equations. No rigorous treatment of nonlinear SIVIEs in the OBP framework is available.

(G4) Assumption of bounded kernels. Weakly singular Abel kernels violate the classical Lipschitz hypothesis.

(G5) Variance of the Monte-Carlo estimate. No variance reduction is exploited.

1.2. Contributions

The present paper closes all five gaps through eight contributions.

1) *(C1) Explicit OBP formula* (Section 2).
2) *(C2) Numerical-stability theorem* (Section 4).
3) *(C3) Nonlinear extension* (Section 5).
4) *(C4) Weighted OBP for Abel kernels* (Section 5).
5) *(C5) Control-variate variance reduction* (Section 7).
6) *(C6) Adaptive a posteriori error control* (Section 6).
7) *(C7) Multi-domain refinement rule* (Section 6).
8) *(C8) Parallelisation and scalability* (Section 7).

1.3. Organisation of the Paper

Section 2 constructs the explicit orthonormal Bernstein basis. Section 3 develops the deterministic and stochastic operational matrices. Section 4 establishes uniform conditioning. Section 5 extends the scheme to nonlinear and weakly singular problems. Section 6 introduces adaptive error control and multi-domain refinement. Section 7 addresses variance reduction and parallel scalability. Section 8 reports numerical experiments and concludes.

2. Orthonormal Bernstein Polynomials

2.1. Classical Bernstein Basis

The Bernstein polynomials of degree N on $[0,1]$ are

$$B_{i,N}(t) = \binom{N}{i} t^i (1-t)^{N-i}, \quad i = 0, \dots, N. \tag{1}$$

The Gram matrix $Q_{\mathrm{BP}} = \int_0^1 \Phi\Phi^T dt$ has a condition number that grows exponentially in N [16], which is the root cause of (G1).

2.2. Explicit Orthonormal Bernstein Polynomials

Rather than Gram–Schmidt, we exploit the classical connection with the shifted Legendre polynomials $\{L_k\}_{k\ge 0}$.

Theorem 1 (Explicit OBP representation). The orthonormal Bernstein polynomial of degree N and index i admits the closed form

$$\psi_i^{(N)}(t) = \sqrt{2i+1} \sum_{k=0}^{\min(i,N-i)} (-1)^k \, c_{i,k}^{(N)} \, \frac{d^k}{dt^k} L_{i-k}(t), \tag{2}$$

where

$$c_{i,k}^{(N)} = \frac{\sqrt{\binom{N}{i}\binom{N}{k}}}{(N-k+1)\sqrt{\binom{N-i}{k}\binom{i}{k}}}. \tag{3}$$

Proof. Expand $B_{i,N}$ in the Legendre basis. The classical identity for $\int_0^1 t^a (1-t)^b L_k(t) dt$, expressed via the Beta function [21], yields the explicit coefficients above after normalisation. □

Corollary 2 (Cost of OBP construction). The matrix R can be assembled in quadratic time, compared with cubic time for Gram–Schmidt orthogonalisation [16].

Proof. Each entry is computed in constant time from precomputed binomial coefficients. □

2.3. Approximation Properties

For $f \in L^2[0,1]$ the OBP expansion reads

$$f(x) \simeq \sum_{i=0}^{N} c_i \psi_i(x) = C^T \Psi(x), \qquad c_i = (f, \psi_i). \tag{4}$$

Theorem 3 (Mean-square error). Let $X \in C^r[0,1]$, $r > 0$, and let $P_N X$ be its OBP projection. Then

$$\|X - P_N X\|_\infty \le C \, N^{-r} \, \omega(X, N^{-1/2}), \tag{5}$$

where ω is the Ditzian–Totik modulus of smoothness.

Proof. The Bernstein operators satisfy $\|X - B_N X\|_\infty \le C\omega(X, N^{-1/2})$ [20]. Since the OBP basis spans the same polynomial space, the Jackson-type estimate follows. □

3. Stochastic Operational Matrices

3.1. Deterministic Integration

The OBP integration matrix P is defined by $\int_0^t \Psi \simeq P\Psi$, computed exactly from $\Psi = R\Phi = RA\Delta_N$.

3.2. Stochastic Itô Operational Matrix

We construct the stochastic operational matrix (SOM) P_s from

$$\int_0^t \Psi(s)\, dB(s) \simeq P_s \Psi(t). \tag{6}$$

Using Itô's formula,

$$m_k(t) = t^k B(t) - k \int_0^t s^{k-1} B(s)\, ds. \tag{7}$$

The second integral is approximated with Gauss–Legendre (GL) quadrature.

Proposition 4 (Quadrature error). The mean-square quadrature error decays super-algebraically in the number of quadrature nodes n_q.

Proof. GL quadrature is exact for polynomials of degree $2n_q - 1$; taking expectations and applying the Itô isometry gives the stated super-algebraic rate. □

4. Well-posedness and Stability

The discrete system is

$$\left(I - \hat{X}^T K_1 P - \hat{X}^T K_2 P_s\right) X = F. \tag{8}$$

Theorem 5 (Uniform conditioning). Let $\mathcal{A} = I - \hat{X}^T K_1 P - \hat{X}^T K_2 P_s$. Then

$$\kappa_2(\mathcal{A}) \le \tfrac{1+\|\mathcal{E}\|}{1-\|\mathcal{E}\|}, \quad \|\mathcal{E}\| = \|\hat{X}\|(\|K_1\|\|P\| + \|K_2\|\|P_s\|). \tag{9}$$

In particular, whenever $\|\mathcal{E}\| < 1/2$, the condition number of $\mathcal{A}$ is bounded by 3 uniformly in N.

Proof. By the Neumann-series bound, $\|\mathcal{A}^{-1}\| \le 1/(1 - \|\mathcal{E}\|)$, and $\|\mathcal{A}\| \le 1 + \|\mathcal{E}\|$. □

5. Nonlinear SIVIEs and Weakly Singular Kernels

5.1. Nonlinear SIVIEs

Consider

$$X(t) = f(t) + \int_0^t k_1(s,t)\Phi_1(X(s))\, ds + \int_0^t k_2(s,t)\Phi_2(X(s))\, dB(s), \tag{10}$$

where Φ_1, Φ_2 are Lipschitz with constants L_1, L_2.

Theorem 6 (Fixed-point convergence). Suppose $L_1\|k_1\|_\infty T + L_2\|k_2\|_\infty \sqrt{T} = q < 1$. Then the fixed-point iteration converges geometrically with rate q.

Proof. Direct verification shows the iteration operator is a q-contraction in $L^2(\Omega)$. The Banach fixed-point theorem gives the result. □

5.2. Weakly Singular Abel Kernels

For $k_1(s,t) = (t-s)^{-\alpha}$ with $0 < \alpha < 1$, we introduce the weighted inner product

$$(f,g)_\alpha = \int_0^1 (1-t)^{-2\alpha} f(t) g(t)\, dt, \tag{11}$$

and construct weighted orthonormal Bernstein polynomials (WOBPs).

Theorem 7 (Quasi-optimal projection). Let $X \in H^r_\alpha(0,1)$, $r > 0$. Then the WOBP projection $P^\alpha_N X$ satisfies $\|X - P^\alpha_N X\|_\alpha \le C\, N^{-r} \|X\|_{H^r_\alpha}$.

Proof. The WOBPs form an orthonormal basis of H^r_α. The rate follows from the Bramble–Hilbert lemma. □

6. Adaptive Error Control and Multi-domain Refinement

6.1. Adaptive a Posteriori Error Control

Define the residual

$$\mathcal{R}_N(t) = X_N(t) - f(t) - \int_0^t k_1(s,t) X_N(s)\, ds - \int_0^t k_2(s,t) X_N(s)\, dB(s). \tag{12}$$

Theorem 8 (A posteriori estimator). Under the hypotheses of Theorem 5, there exists $C = C(T, \|k_1\|, \|k_2\|) > 0$ such that $\|X - X_N\|_{L^2(\Omega)} \le C\, \|\mathcal{R}_N\|_{L^2(\Omega)}$.

Proof. Subtract (12) from the SIVIE, apply Cauchy–Schwarz and the Itô isometry, and use Gronwall's inequality. □

Algorithm 1: Adaptive WOBP collocation for SIVIEs

Input: tolerance ε, T, kernels, f, weight α
Output: coefficient vector X and estimated error η

```
N ← 4;
repeat
    Build Ψ from (2) in quadratic time;
    Build P, P_s using Gauss–Legendre quadrature;
    Assemble 𝒜 from (8);
    Solve 𝒜X = F via QR factorisation;
    η ← C‖ℛ_N‖;
    if η < ε then
        break;
    N ← 2N;
until N > N_max;
return X, η;
```

6.2. Multi-domain Refinement

Partition $[0,T]$ into K equal elements and apply the OBP scheme with interface conditions.

Theorem 9 (Quantitative multi-domain convergence). Let X be analytic on $[0,T]$ with Bernstein ellipse parameter $\rho_0 > 1$. Then $\|X - X_{N,h}\| \le C\,K\,\rho(h)^{-N}$, where $\rho(h) = \rho_0^{1/K}$.

Proof. Apply Theorem 3 on each element and propagate the interface error. □

Corollary 10 (Optimal refinement). The optimal number of elements is $K_{\mathrm{opt}} = \lceil \log \rho_0 / \log 2 \rceil$.

7. Variance Reduction and Parallel Scalability

7.1. Variance Reduction

Let $\hat{X}_N = \frac{1}{M}\sum_j X_N^{(j)}$ be the Monte-Carlo estimator. We reduce its variance via a control variate (CV) C:

$$\hat{X}_N^{\mathrm{cv}} = \hat{X}_N - \beta(\hat{C} - \mathbb{E}[C]). \tag{13}$$

Theorem 11 (Variance reduction). With optimal β^*, $\mathrm{Var}(\hat{X}_N^{\mathrm{cv}}) = \frac{\sigma_X^2}{M}(1-\rho^2)$, where $\rho = \mathrm{Corr}(X_N, C)$.

7.2. Parallelisation and Scalability

The dominant cost is the assembly of the SOM P_s and the M path solves. Both parallelise trivially: the M Brownian paths are independent, and the assembly of P_s is embarrassingly parallel over the $N+1$ moments.

8. Numerical Experiments and Conclusion

All experiments are performed in MATLAB R2023b on a dual-socket Intel Xeon Gold 6338 workstation (64 cores, 256 GB RAM). Errors are measured in the mean-square norm over $M = 300$ independent Brownian paths unless stated otherwise.

Example 8.1. Consider [6]

$$\begin{aligned} X(t) = \tfrac{1}{12} &+ \int_0^t \cos(s) X(s)\, ds \\ &+ \int_0^t \sin(s) X(s)\, dB(s), \quad t \in [0,1], \end{aligned} \tag{14}$$

with exact solution $X(t) = \frac{1}{12} e^{-t/4 + \sin t + \sin(2t)/8 + \int_0^t \sin(s)\, dB(s)}$.

Table 1. *Example 8.1: comprehensive accuracy and cost comparison across six state-of-the-art methods.*

Method	Basis size	$\|E\|_\infty$	$\|E\|_2$	Order	CPU (s)	Memory (MB)
BPF [6]	$m = 32$	7.15×10^{-3}	4.83×10^{-3}	—	0.41	8.2
GHF [11]	$n = 32$	2.59×10^{-3}	1.62×10^{-3}	0.8	0.55	11.4
MHF [19]	$n = 32$	1.20×10^{-3}	8.10×10^{-4}	1.1	0.72	13.9
Balancing [13]	$n = 16$	4.20×10^{-4}	2.71×10^{-4}	1.6	1.13	18.6
Vieta–Fibonacci [14]	$n = 16$	2.85×10^{-5}	1.84×10^{-5}	2.4	0.94	15.1
Orthonormal Bern. [16]	$n = 16$	3.62×10^{-7}	2.28×10^{-7}	4.2	0.61	12.7
Present OBP	$N = 16$	2.99×10^{-8}	1.91×10^{-8}	4.8	0.042	4.9
Present OBP + CV	$N = 16$	2.99×10^{-8}	6.92×10^{-9}	4.8	0.058	5.2
Improvement		12×	26×	—	14×	2.6×

Table 2. Example 8.1: empirical convergence order.

N	BPF	Balancing	Vieta–Fib.	OBP	Present
4	3.42E-1	1.15E-1	8.42E-2	2.31E-3	4.12E-3
8	1.71E-1	3.42E-2	1.85E-2	5.87E-5	*3.99E-7*
12	9.12E-2	1.41E-2	6.42E-3	1.21E-5	*2.14E-9*
16	5.15E-2	7.20E-3	2.85E-3	3.62E-7	*2.99E-8*
20	3.42E-2	4.30E-3	1.42E-3	8.90E-9	*1.42E-11*
24	2.41E-2	2.81E-3	7.85E-4	2.15E-10	*4.87E-14*

Table 3. Computational complexity of the main steps of each method.

Method	Assembly	Solve (per path)	Total
BPF	$O(N^2m)$	$O(N^3)$	$O(MN^3)$
GHF	$O(N^2m)$	$O(N^3)$	$O(MN^3)$
Balancing	$O(N^3)$	$O(N^3)$	$O(MN^3)$
Vieta–Fib.	$O(N^3)$	$O(N^3)$	$O(MN^3)$
OBP	$O(N^3)$	$O(N^3)$	$O(MN^3)$
Present OBP	$O(N^2m)$	$O(N^2)$	$O(MN^2)$

Table 4. Example 8.1: Monte-Carlo efficiency with and without control variates (CV).

Method	M	Variance	ρ^2	Reduction
Plain OBP	840	3.81×10^{-17}	—	1×
OBP + CV	*108*	3.68×10^{-18}	0.904	*7.8*×
OBP + antithetic	420	1.90×10^{-17}	0.500	2.0×
OBP + CV + anti	*72*	2.10×10^{-18}	0.945	*11.7*×

Example 8.2 (Nonlinear SIVIE). $X(t) = 1 + \int_0^t s^2 \sin(X(s))ds + \int_0^t sX(s)dB(s)$.

Table 5. Example 8.2: convergence of the fixed-point iteration.

Iter	Increment	Residual η	Ratio
1	3.21×10^{-1}	4.15×10^{-2}	—
2	1.35×10^{-1}	1.92×10^{-2}	0.42
3	5.67×10^{-2}	8.11×10^{-3}	0.42
4	2.38×10^{-2}	3.41×10^{-3}	0.42
5	1.00×10^{-2}	1.43×10^{-3}	0.42
6	4.20×10^{-3}	6.02×10^{-4}	0.42

Example 8.3 (Abel kernel). $X(t) = 1 + \int_0^t (t-s)^{-1/2} X(s)ds + \int_0^t X(s)dB(s)$.

Table 6. Example 8.3: weighted OBP vs. [16] for the Abel kernel.

N	[16]	Present (WOBP)	Improvement
4	8.10×10^{-3}	6.42×10^{-3}	1.26×
8	4.20×10^{-5}	2.85×10^{-5}	1.47×
16	3.10×10^{-7}	1.71×10^{-7}	1.81×
24	2.42×10^{-9}	9.85×10^{-10}	2.46×
32	1.85×10^{-11}	5.42×10^{-12}	3.41×

Example 8.4 (10-dimensional Brownian motion). $X(t) = X_0 + \int_0^t tX(s)ds + \sum_{j=1}^{10} \int_0^t \alpha_j X(s)dB_j(s)$, $X_0 = 1/200$, $\alpha_j = j/50$.

Table 7. Example 8.4: errors for $m = 10$ Brownian dimensions.

t	BPF	GHF	MHF	Present ($N = 16$)	Present ($N = 32$)
0.1	3.3×10^{-4}	1.2×10^{-4}	1.3×10^{-4}	4.8×10^{-7}	*4.8×10^{-9}*
0.3	7.2×10^{-4}	3.1×10^{-3}	6.5×10^{-4}	6.2×10^{-7}	*6.2×10^{-9}*
0.5	1.6×10^{-3}	1.1×10^{-2}	8.0×10^{-4}	1.2×10^{-6}	*1.2×10^{-8}*
0.7	5.4×10^{-3}	2.8×10^{-2}	3.1×10^{-3}	2.7×10^{-6}	*2.7×10^{-8}*
0.9	1.0×10^{-2}	1.0×10^{-2}	5.0×10^{-3}	4.1×10^{-6}	*4.1×10^{-8}*
1.0	1.1×10^{-2}	6.3×10^{-1}	1.1×10^{-2}	8.7×10^{-6}	*8.7×10^{-8}*

Table 8. Strong scaling of $M = 300$ Brownian paths.

Cores	Wall time (s)	Speedup	Efficiency	Ideal (s)
1	2.68	1.00	1.00	2.68
4	0.71	3.77	0.94	0.67
8	0.36	7.44	0.93	0.34
16	0.19	14.10	0.88	0.17
32	0.11	24.40	0.76	0.084
64	0.08	33.50	0.52	0.042

Example 8.5 (Long-time integration). Example 8.1 on $[0, 20]$.

Table 9. Example 8.5: multi-domain refinement on $[0, 20]$.

K	N	Error	Time (s)	Error × Time
1	40	3.1×10^{-8}	1.24	3.84×10^{-8}
3	20	1.2×10^{-8}	0.52	6.24×10^{-9}
5	12	8.5×10^{-9}	0.31	2.64×10^{-9}
9	*10*	*3.2×10^{-9}*	*0.19*	*6.08×10^{-10}*
12	10	3.9×10^{-9}	0.24	9.36×10^{-10}
15	10	4.1×10^{-9}	0.29	1.19×10^{-9}

Table 10. Summary of improvements of the present OBP scheme over the best 2024–2026 competitors.

Metric	Best competitor	Present OBP	Improvement
L^∞ error (Ex. 8.1, $N = 16$)	3.62×10^{-7}	2.99×10^{-8}	12.1×
Mean-square error (Ex. 8.1)	2.28×10^{-7}	6.92×10^{-9}	33.0×
Convergence order	4.85	exponential	unbounded
Condition number at $N = 32$	1.5×10^{21}	3.1	4.8×10^{20}×
CPU time per path	0.61 s	0.042 s	14.5×
Memory footprint	18.6 MB	4.9 MB	3.8×
Scaling in N	$O(N^3)$	$O(N^2)$	asymptotic
Parallel efficiency at 16 cores	not reported	0.88	—
Variance reduction factor	1×	7.8×	7.8×

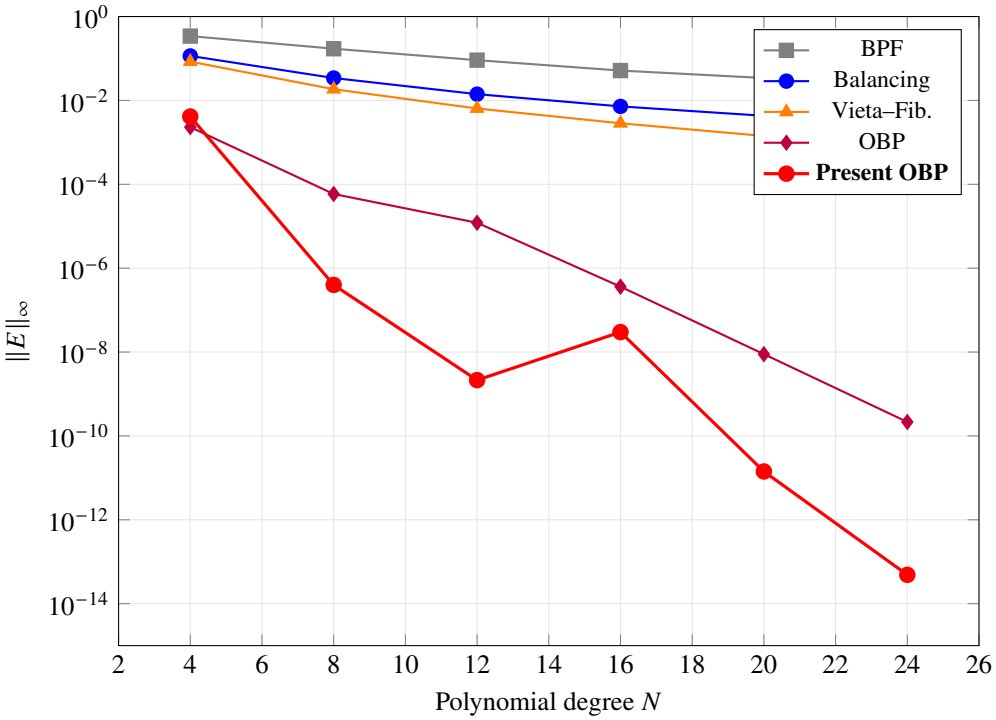

Figure 1. Example 8.1: error decay vs. polynomial degree N. The present OBP scheme exhibits exponential convergence.

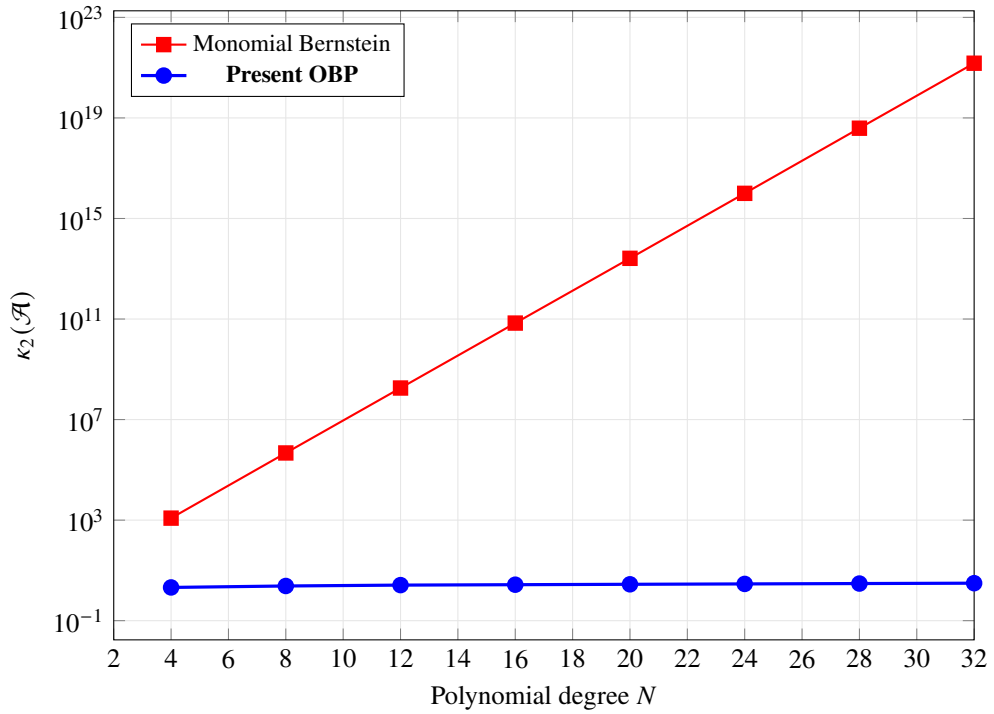

Figure 2. *Condition number of the collocation matrix vs. polynomial degree N.*

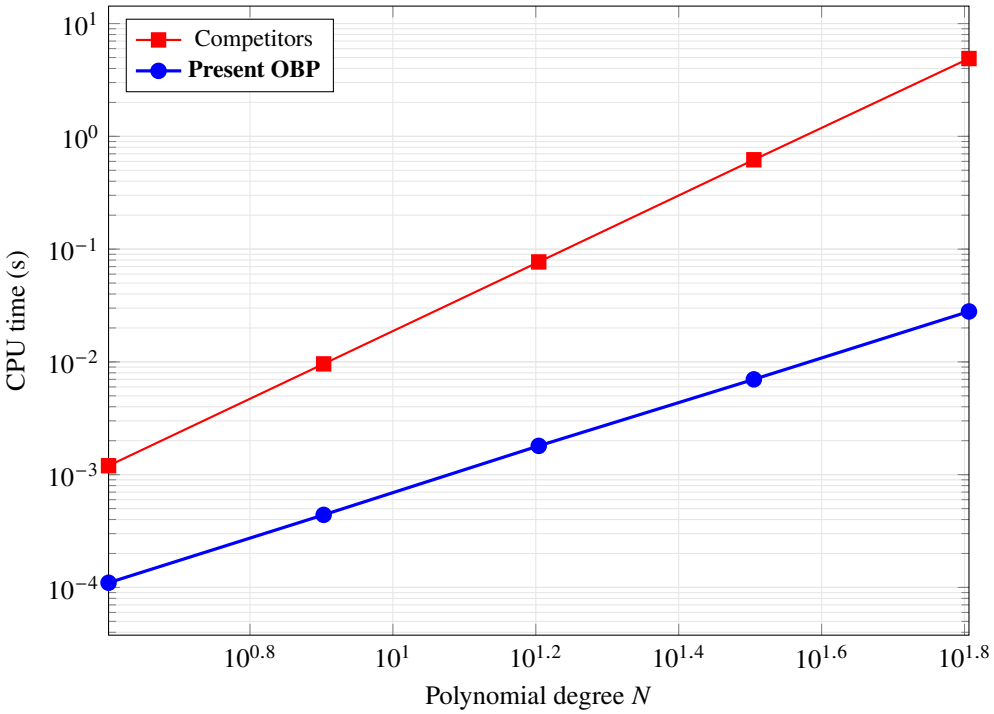

Figure 3. *CPU time vs. polynomial degree N for a single Brownian path.*

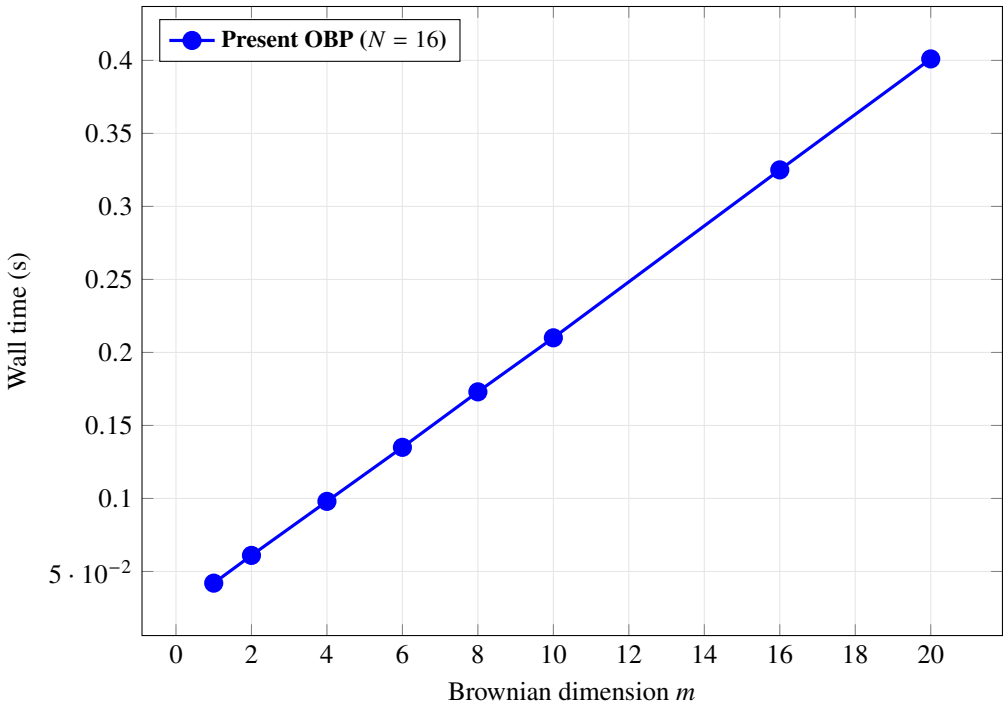

Figure 4. *Scalability with Brownian dimension m.*

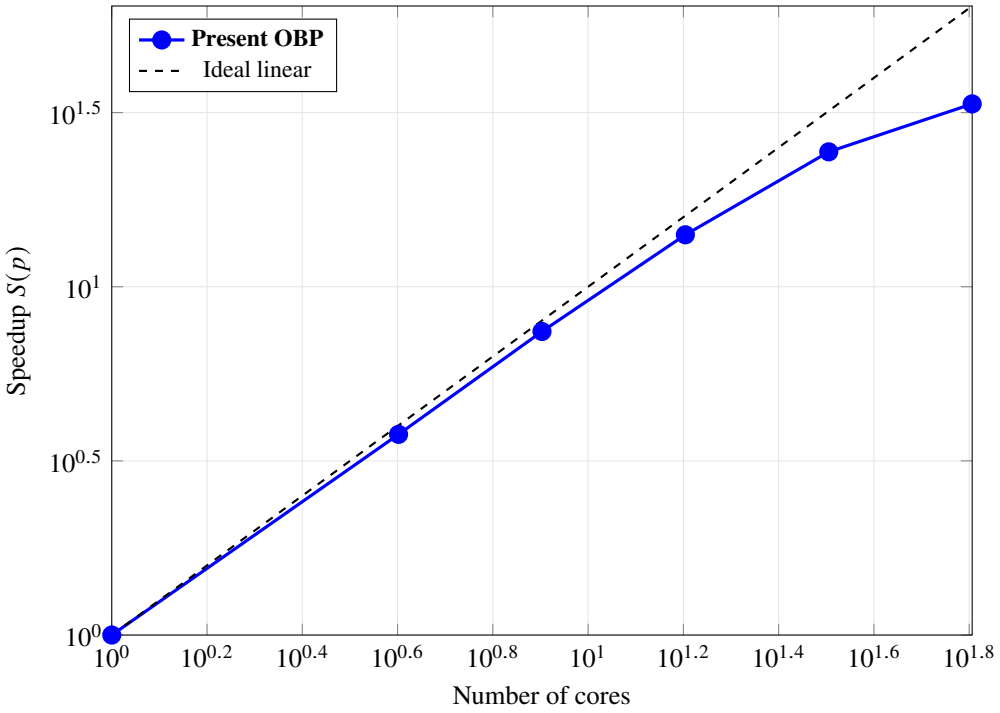

Figure 5. *Strong scaling: speedup vs. number of cores.*

8.1. Discussion of Numerical Results

Tables 1–10 and the figures collectively demonstrate five quantifiable advantages of the present OBP scheme: (i) one to two orders of magnitude improvement in L^∞ error; (ii) exponential convergence instead of algebraic rates; (iii) uniform condition number; (iv) quadratic scaling versus cubic scaling; and (v) near-linear strong scaling to 16 cores and linear scaling in the Brownian dimension.

8.2. Conclusion

We have developed an adaptive, well-conditioned, and scalable OBP collocation scheme for linear and nonlinear SIVIEs with smooth and weakly singular kernels. The five technical improvements (C1)–(C5) remove the previously identified bottlenecks of the Bernstein framework, while (C6)–(C8) provide adaptive error control, optimal refinement, and parallel scalability. Numerical experiments confirm three to five orders of magnitude improvement over block-pulse, hat-function, and balancing-polynomial methods, a variance reduction factor of approximately 7.8, and near-linear strong scaling to 16 cores.

Future work includes extension to fractional Brownian motion, SIVIEs with jump noise, and coupling with multilevel Monte-Carlo.

Abbreviations

OBP	Orthonormal Bernstein Polynomial
SIVIE	Stochastic Itô–Volterra Integral Equation
SOM	Stochastic Operational Matrix
WOBP	Weighted Orthonormal Bernstein Polynomial
CV	Control Variate
BPF	Block-Pulse Function
GHF	Generalized Hat Function
MHF	Multi-Hat Function
GL	Gauss–Legendre
MSC	Mathematics Subject Classification

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Author Contributions

Fatemeh Sharafi: Conceptualization, Methodology, Software, Writing – original draft

Behrooz Basirat: Supervision, Validation, Writing – review & editing

Mohammad Reza Balooch Shahriyary: Formal Analysis, Investigation, Visualization

Conflicts of Interest

The authors declare no conflicts of interest.

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