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Structures Associated with the Direct Product of Alternating and Dihedral Groups Acting on the Cartesian Product of Two Sets

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Abstract: Whenever a permutation group acts on a set, combinatorial and invariant properties, and mathematical structures that result from this group action are studied. Various mathematicians have studied these properties over time using different groups acting on both ordered and unordered sets. The combinatorial properties (transitivity and primitivity) and invariants (ranks and subdegrees) of the direct product between alternating and dihedral groups acting on the Cartesian product of two sets have already been studied and it was found out that the group action is transitive, imprimitive, the rank is 6, and subdegrees are obtained according to theorem 2.3. This research seeks to extend this by constructing and analyzing the properties (simple/multigraph, self-pairedness, connectedness, degree of the vertex, girth, and directedness) of these mathematical structures (suborbital graphs) that result from the group action. This research for $n \geqslant 3$, suborbital graphs can be classified into three categories; First, those constructed when only the first components of the vertex set are identical and second, those when only the second components of the vertex set are identical. The suborbital graphs of the first and second category are simple, self-paired, have $n-$ disconnected components, are regular with degree $n-1$ and girth is 3. The third category of suborbital graphs in which neither the first nor the second components of the vertex set are identical and they are; simple, self-paired, connected, regular with degree of vertex varying from graph to graph, and girth 3.

Keywords: Permutation, Group Action, Alternating Group, Dihedral Group, Suborbital, Suborbital Graph

1. Introduction

Suppose that (G, X) is transitive. Then the group action can be also expanded easily to when two permutation groups (A_n) and (D_n) are combined by direct product $(A_n \times D_n \,|(a_1, d_1)\, \forall a_1 \in A_n, d_1 \in D_n)$ to act on the Cartesian product of two sets $(X \times Y \,|(x, y)\, \forall x \in X, y \in Y)$ i.e. $(a_1, d_1)\,(x, y) = (a_1 x, d_1 y)$.

Let $G = A_n \times D_n$ act transitively on $K = X \times Y$ and $Stab_G\,(x_1, y_1)$ be the stabilizer of a point $(x_1, y_1) \in X \times Y$, the stabilizer partitions $X \times Y$ into orbits called suborbits $[\Delta_o = \{x_1, y_1\}, \Delta_1, \Delta_2, ..., \Delta_{r-1}]$'. Rank r is the number of these suborbits and subdegrees is the lengths of these suborbits $\{n_i = |\Delta_i|\,(i = 0, 1, 2, ..., r-1)\}$. From each suborbit we obtain suborbital, $O\,(x, y)$ which is used to construct a suborbital graph $\Gamma\,(x, y)$ whose vertices are the elements of $X \times Y$ and a directed line $(\delta \to \gamma)$ exists if $(\delta \to \gamma) \in O(x, y)$. Therefore, $O\,(y, x)$ is a suborbital also and it is either equal or disjoint from $O\,(x, y)$. Suppose $\Gamma\,(y, x) = \Gamma\,(x, y)$ then the suborbital graph has a pair of lines between vertices that are directed oppositely so they are replaced by one undirected line, and the suborbital graph becomes undirected and self-paired. If $\Gamma\,(y, x)\ and\ \Gamma\,(x, y)$ are disjoint then they are similar and called paired suborbital graphs. This paper seeks to construct suborbital graphs of the group action $A_n \times D_n$ on $X \times Y$ where $n \geq 3$.

2. Notation and Preliminary Results

Definition 2.1. A graph $G\,(V, E)$ is a mathematical structure that is composed of a set V whose elements are called vertices, points, or nodes and a set E of unordered pairs of the vertices called edges(arcs). Trivial graph has only one vertex. [1]

Definition 2.2. A group G is a set with a binary operation and satisfies the rules; identity, inverse, closure and associativity. [2]

Definition 2.3. A permutation of n elements of a set X is a $1-1$ mapping from X onto itself. The set that contains all permutations of X form a group G under composition of permutations called a Symmetric group (S_n). [3]

Definition 2.4. Alternating group, A_n is the group formed by the collection of set of all even permutations in a Symmetric group (S_n) its number of elements is $|A_n| = \frac{n!}{2}$. [4]

Definition 2.5. A Dihedral group, D_n is a group that is composed of symmetries and rigid motions of a regular polygon P_n which has n sides. The degree (number of sides) is n and order (number of elements) $2n$. [5]

Definition 2.6. Let H be a permutation group and Y a non-empty set. A group action of H on Y is the function $H \times Y \rightarrow Y$ satisfying:

1. $e \cdot h = h \,\forall\, h \in H$ and $e \in H$ - *Identity law*.
2. $(h \cdot k) \cdot y = h \cdot (k \cdot y) \;\forall\; h, k \in H$ and $y \in Y$ - *Associative law*. [6]

Definition 2.7. Let H be a permutation group that act on a set Y. The set that contains elements of Y fixed by $h \in H$ is the fixed point set of h, $Fix\,(h)$. Thus $Fix\,(h) = \{y \in Y \,|hy = x\}$. [7]

Definition 2.8. The stabilizer of an element y in set Y, $Stab_H\,(y)$, is the set composed of all elements in H that fix y i.e. $Stab_H\,(y) = \{h \in H \,|hy = y\}$. [8]

Definition 2.9. When a permutation group H acts on a set Y then, Y is divided into disjoint subsets called orbits. For each $y \in Y$ the orbit that contains y is denoted by $Orb_H\,(y) = \{hy \,|h \in H\}$ that contains all images of y under every h in H. [9]

Definition 2.10. A loop in a graph H is a line that joins a vertex to itself. [10]

Definition 2.11. Suppose that H be a graph, and $v \in V$. The degree of $v \in V$ is the number of edges incident onto it. A loop at v is counted twice when determining degree of v. A vertex with degree zero is known as an isolated vertex. [11]

Definition 2.12. A simple graph has no loop at any vertex and no multiple edges between any pair of vertices. When multiple edges are allowed between any pair of vertices, and loops are optional but accepted, the graph is called a multigraph. [12]

Definition 2.13. A walk in a graph H is a non-empty set of the list that contains alternating vertices and edges. A path is a walk with no vertex repeated (except end vertices). If the end vertices repeat in a path, it is called a cycle (closed path). The girth is the length of the shortest cycle (if any) in H. [13]

Definition 2.14. When any pair of vertices of a graph H are joined by some path, then H is said to be connected if not H is disconnected. [14]

Definition 2.15. A graph H is called a directed graph or diagraph if each edge is associated with a direction otherwise undirected. [15]

Definition 2.16. A self-complementary (or self-paired) graph is a graph H such that $H \cong \bar{H}$ i.e. H is isomorphic to its complement $\bar{H}$. If a self-complementary graph H has $n-$vertices, then $n = 0, 1\,(\mathrm{mod}4)$. Because the complement of each disconnected graph is connected, each self-complementary graph is connected. [16]

Theorem 2.1. (Wielandt Theorem) Let a group H act on a set Y. Then there exist at least one self-paired suborbit of H on Y iff $|G|$ is even. [17].

Theorem 2.2. (Cameron Theorem) Let a finite permutation group G act on a finite set X and $g \in G$, then the number of self-paired sub-orbits of G is given by: [18].

$$\frac{1}{|G|}\sum_{g \in G} |Fix(g)^2| \tag{1}$$

Theorem 2.3. Suppose that the group $A_n \times D_n$ action on $X \times Y$ is transitive, then the rank is 6 for $n \geqslant 3$ and the corresponding subdegrees are; [19]

$$|\Delta_i| = \begin{cases} 1, & \text{trivial orbit} \\ p, & \text{if } i \leqslant r \\ p-1, & \text{if } i > r \end{cases}$$

$\forall i = 1, 2, ..., 5$, $p = \left[\frac{|K|-1}{5}\right]$ and $r = (|K| - 1) \mod 5$.

3. Main Results

3.1. Suborbital Graphs of the Group $A_3 \times D_3$ Action on $X \times Y$

By theorem 2.3 when $n = 3$, the rank is 6 and the corresponding subdegrees are $1, 2, 2, 2, 1, 1$. The suborbits are

$|\Delta_0| = |Orb_G(x_1, y_1)| = |\{(x_1, y_1)\}| = 1$ (Trivial Orbit)
$|\Delta_1| = |Orb_G(x_1, y_2)| = |\{(x_1, y_2), (x_1, y_3)\}| = 2$
$|\Delta_2| = |Orb_G(x_2, y_2)| = |\{(x_1, y_2), (x_2, y_3)\}| = 2$
$|\Delta_3| = |Orb_G(x_3, y_2)| = |\{(x_3, y_2), (x_3, y_3)\}| = 2$
$|\Delta_4| = |Orb_G(x_2, y_1)| = |\{(x_2, y_1)\}| = 1$
$|\Delta_5| = |Orb_G(x_3, y_1)| = |\{(x_3, y_1)\}| = 1$

Given that;
$\Delta_0 = \{(x_1, y_1)\}$
$\Delta_1 = \{(x_1, y_2), (x_1, y_3)\}$
$Stab_G(x_1, y_1) = \{(e_a, e_d), (e_a, (d_2\; d_3))\}$

The suborbital O_1 that corresponds to suborbit Δ_1;

$O_1 = \{(e_a, e_d)\,(x_1, y_1)\,, (e_a, e_d)\,(x_1, y_2)\,, (e_a, e_d)\,(x_1, y_3)\} = \{(x_1, y_1)\,, (x_1, y_2)\,, (x_1, y_3)\}$

$O_1 = (e_a, (d_2, d_3))((x_1, y_1), (x_1, y_2), (x_1, y_3))$
$= \{(x_1, y_1), (x_1, y_3), (x_1, y_2)\}$

Using suborbital O_1, the suborbital graph Γ_1 is drawn by constructing a line from (x_1, y_1) to $\{(x_1, y_3), (x_1, y_2)\}$ so that only the first components are identical.

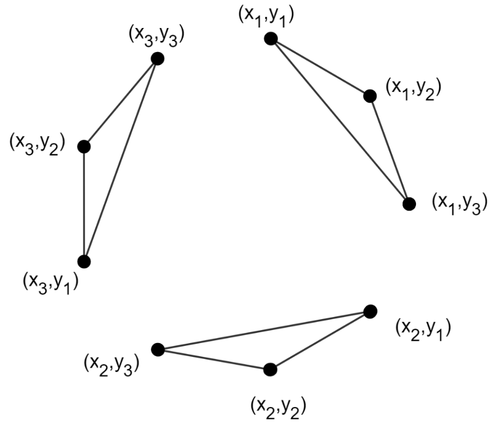

Figure 1. *Suborbital graph Γ_1 corresponding to suborbit Δ_1 of the group $A_3 \times D_3$ action on $X \times Y$*

The suborbital graph Γ_1 above is simple, self-paired, undirected, regular, degree 2 and is made up of 3 disconnected components and girth 3.

NOTE: The suborbital graph Γ_1 is similar to suborbital graph Γ_5 and Γ_4 but in Γ_4 it only the second components are identical.

Given that

$\Delta_0 = \{(x_1, y_1)\}$, $\Delta_2 = \{(x_2, y_2), (x_2, y_3)\}$ and $Stab_G(x_1, y_1) = \{(e_a, e_d), (e_a, (d_2\ d_3))\}$.

The suborbital O_2 that corresponds to the suborbit Δ_2;

$O_2 = \{(e_a, e_d)(x_1, y_1), (e_a, e_d)(x_2, y_2), (e_a, e_d)(x_2, y_3)\}$
$= \{(x_1, y_1), (x_2, y_2), (x_2, y_3)\}$

$O_2 = (e_a, (d_2, d_3))((x_1, y_1), (x_2, y_2), (x_2, y_3))$
$= \{(x_1, y_1), (x_2, y_3), (x_2, y_2)\}$

Using suborbital O_2, a suborbital graph Γ_2 is drawn by constructing a line from (x_1, y_1) to $\{(x_2, y_3), (x_2, y_2)\}$ so that neither first nor the second components are similar.

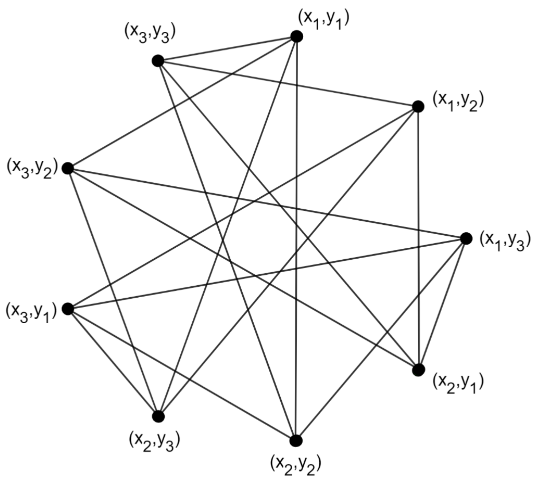

Figure 2. *Suborbital graph Γ_2 corresponding to suborbit Δ_2 of the group $A_3 \times D_3$ action on $X \times Y$.*

The suborbital graph Γ_2 is simple, self-paired, undirected, regular, has degree 4, is connected, and girth 3.

NOTE: The suborbital graph Γ_2 is similar to suborbital graph Γ_3.

3.2. Suborbital Graphs of the Group $A_4 \times D_4$ Action on $X \times Y$

By theorem 2.3 when $n = 4$, the rank is 6 and the corresponding subdegrees are $1, 3, 3, 3, 3, 3$. The suborbits are;

$|\Delta_0| = |Orb_G(x_1, y_1)| = |\{(x_1, y_1)\}| = 1$ (Trivial Orbit)

$|\Delta_1| = |Orb_G(x_1, y_2)| = |\{(x_1, y_2), (x_1, y_3), (x_1, y_4)\}| = 3$

$|\Delta_2| = |Orb_G(x_2, y_1)| = |\{(x_2, y_1), (x_3, y_1), (x_4, y_1)\}| = 3$

$|\Delta_3| = |Orb_G(x_2, y_2)| = |\{(x_2, y_2), (x_3, y_2), (x_4, y_2)\}| = 3$

$|\Delta_4| = |Orb_G(x_2, y_3)| = |\{(x_2, y_3), (x_3, y_3), (x_4, y_3)\}| = 3$

$|\Delta_5| = |Orb_G(x_2, y_4)| = |\{(x_2, y_4), (x_3, y_4), (x_4, y_4)\}| = 3$

Given that

$\Delta_0 = \{(x_1, y_1)\}$, $\Delta_1 = \{(x_1, y_2), (x_1, y_3), (x_1, y_4)\}$ and

$$Stab_G(x_1, y_1) = \{(e_a, e_d), (e_a, (d_2\ d_4)), ((a_2\ a_3\ a_4), e_d), ((a_2\ a_3\ a_4), (d_2\ d_4)), ((a_2\ a_4\ a_3), e_d), ((a_2\ a_4\ a_3), (d_2\ d_4))\}$$

The suborbital O_1 that corresponds to suborbit Δ_1;

$$O_1 = \{((e_a, e_d))(x_1, y_1), (x_1, y_2), (x_1, y_3), (x_1, y_4)\}$$
$$= \{(x_1, y_1), (x_1, y_2), (x_1, y_3), (x_1, y_4)\}$$
$$O_1 = \{((e_a, (d_2\ d_4)))(x_1, y_1), (x_1, y_2), (x_1, y_3),)(x_1, y_4)\}$$
$$= \{(x_1, y_1), (x_1, y_2), (x_1, y_3), (x_1, y_4)\}$$
$$\vdots$$
$$O_1 = \{(((a_2\ a_4\ a_3), (d_2\ d_4)))(x_1, y_1), (x_1, y_2), (x_1, y_3), ((a_2\ a_4\ a_3), (d_2\ d_4))(x_1, y_4)\}$$
$$= \{(x_1, y_1), (x_1, y_2), (x_1, y_3), (x_1, y_4)\}$$

Using suborbital O_1, suborbital graph Γ_1 is drawn by constructing a line from (x_1, y_1) to $\{(x_1, y_2), (x_1, y_3), (x_1, y_4)\}$ such that only the first components are identical.

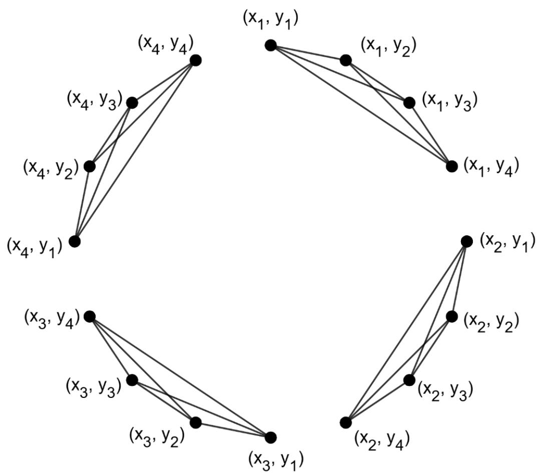

Figure 3. *Suborbital graph Γ_1 corresponding to suborbit Δ_1 of the group $A_4 \times D_4$ action on $X \times Y$.*

The suborbital graph Γ_1 is simple, self-paired, undirected, regular with degree 3, composed of four disconnected components and girth 3.

Given that;

$\Delta_0 = \{(x_1, y_1)\}$, $\Delta_2 = \{(x_2, y_1), (x_3, y_1), (x_4, y_1)\}$ and

$Stab_G(x_1, y_1) = \{(e_a, e_d), (e_a, (d_2\ d_4)), ((a_2\ a_3\ a_4), e_d), ((a_2\ a_3\ a_4), (d_2\ d_4)), ((a_2\ a_4\ a_3), e_d), ((a_2\ a_4\ a_3), (d_2\ d_4))\}$

The suborbital O_2 that correspond to suborbit Δ_2;

$$O_2 = \{(e_a, e_d)((x_1, y_1), (x_2, y_1), (x_3, y_1), (x_4, y_1))\} = \{(x_1, y_1), (x_2, y_1), (x_3, y_1), (x_4, y_1)\}$$

$$O_2 = \{(e_a, (d_2\ d_4))((x_1, y_1), (x_2, y_1), (x_3, y_1), (x_4, y_1))\} = \{(x_1, y_1), (x_2, y_1), (x_3, y_1), (x_4, y_1)\}$$

$$\vdots$$

$$O_2 = \{((a_2\ a_4\ a_3), (d_2\ d_4))((x_1, y_1), (x_2, y_1), (x_3, y_1), (x_4, y_1))\} = \{(x_1, y_1), (x_4, y_1), (x_2, y_1), (x_3, y_1)\}$$

Using suborbital O_2, a suborbital graph Γ_2 is drawn by constructing a line from (x_1, y_1) to $\{(x_4, y_1), (x_2, y_1), (x_3, y_1)\}$ such that only the second components are similar.

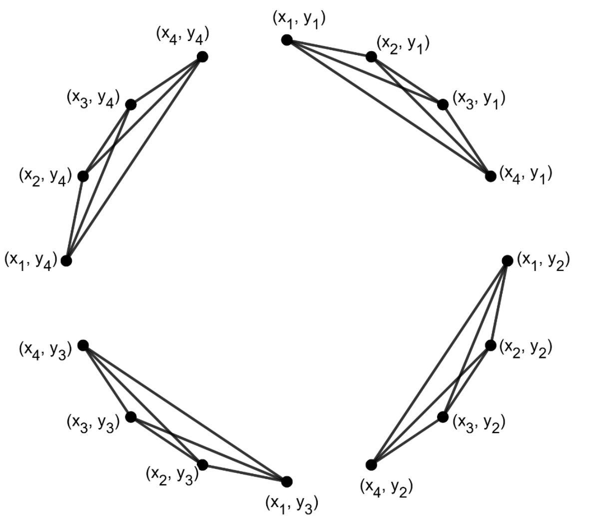

Figure 4. *Suborbital graph Γ_2 corresponding to suborbit Δ_2 of the group $A_4 \times D_4$ action on $X \times Y$.*

The suborbital graph Γ_2 is simple, undirected, regular with degree 3, composed of four disconnected components and the girth 3.

NOTE: Suborbital graph Γ_1 is identical to Γ_2 in terms of properties but in Γ_2 the only the second components are identical.

Given that

$\Delta_0 = \{(x_1, y_1)\}$, $\Delta_3 = \{(x_2, y_2), (x_3, y_2), (x_4, y_2)\}$ and

$Stab_G(x_1, y_1) = \{(e_a, e_d), (e_a, (d_2\ d_4)), ((a_2\ a_3\ a_4), e_d), ((a_2\ a_3\ a_4), (d_2\ d_4)), ((a_2\ a_4\ a_3), e_d), ((a_2\ a_4\ a_3), (d_2\ d_4))\}$

The suborbital O_3 that correspond to suborbit Δ_3;

$$O_3 = \{(e_a, e_d)((x_1, y_1), (x_2, y_2), (x_3, y_2), (x_4, y_2))\} = \{(x_1, y_1), (x_2, y_2), (x_3, y_2), (x_4, y_2)\}$$

$$O_3 = \{(e_a, (d_2\ d_4))((x_1, y_1), (x_2, y_2), (x_3, y_2), (x_4, y_2))\} = \{(x_1, y_1), (x_2, y_4), (x_3, y_4), (x_4, y_4)\}$$

$$\vdots$$

$$O_3 = \{((a_2\ a_4\ a_3), (d_2\ d_4))((x_1, y_1), (x_2, y_2), (x_3, y_2), (x_4, y_2))\} = \{(x_1, y_1), (x_4, y_4), (x_2, y_4), (x_3, y_4)\}$$

Using suborbital O_3, the suborbital graph Γ_3 is drawn by constructing a line from (x_1, y_1) to $\{(x_4, y_2), (x_2, y_2), (x_3, y_2)\}$ such that none of the components are identical.

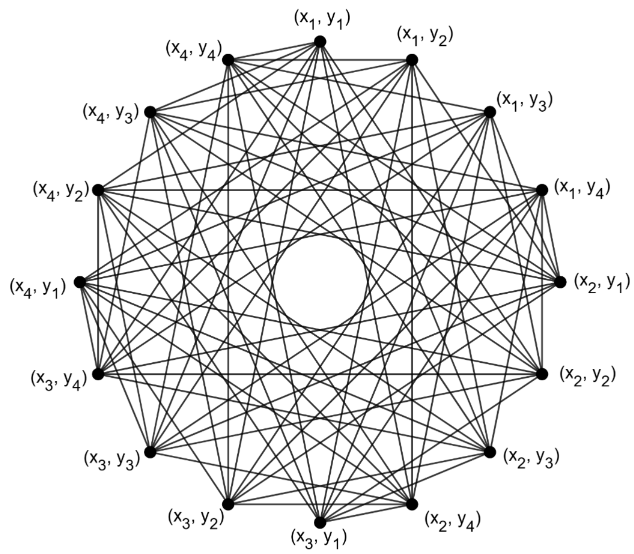

Figure 5. *Suborbital graph Γ_3 corresponding to suborbit Δ_3 of the group $A_4 \times D_4$ acting on $X \times Y$.*

The suborbital graph Γ_3 above is simple, undirected, regular with degree 9, connected and girth 3.

NOTE: Suborbital graphs Γ_3, Γ_4 and Γ_5 are identical hence have the same structure and properties.

3.3. Suborbital Graphs of the Group $A_5 \times D_5$ Action on $X \times Y$

By theorem 2.3 when $n = 5$, the rank is 6 and the corresponding subdegrees are $1, 5, 5, 5, 5, 4$. The suborbits are;

$|\Delta_0| = |Orb_G(x_1, y_1)| = |\{(x_1, y_1)\}| = 1$ (Trivial Orbit)

$|\Delta_1| = |Orb_G(x_2, y_1)| = |\{(x_2, y_1), (x_2, y_2), (x_2, y_3), (x_2, y_4), (x_2, y_5)\}| = 5$

$|\Delta_2| = |Orb_G(x_3, y_1)| = |\{(x_3, y_1), (x_3, y_2), (x_3, y_3), (x_3, y_4), (x_3, y_5)\}| = 5$

$|\Delta_3| = |Orb_G(x_4, y_1)| = |\{(x_4, y_1), (x_4, y_2), (x_4, y_3), (x_4, y_4), (x_4, y_5)\}| = 5$

$|\Delta_4| = |Orb_G(x_5, y_1)| = |\{(x_5, y_1), (x_5, y_2), (x_5, y_3), (x_5, y_4), (x_5, y_5)\}| = 5$

$|\Delta_5| = |Orb_G(x_1, y_2)| = |\{(x_1, y_2), (x_1, y_3), (x_1, y_4), (x_1, y_5)\}| = 4$

Given that;

$\Delta_0 = \{(x_1, y_1)\}$
$\Delta_1 = \{(x_2, y_1), (x_2, y_2), (x_2, y_3), (x_2, y_4), (x_2, y_5)\}$,
$Stab_G(x_1, y_1) = \{(e_a, e_d), (e_a, (d_2\ d_5)(d_3\ d_4),$
$((a_2\ a_3)(a_4\ a_5), e_d), ((a_3\ a_4\ a_5), e_d),$
$((a_3\ a_4\ a_5), (d_2\ d_5)(d_3\ d_4)), ((a_3\ a_5\ a_4), e_d),$
$((a_3\ a_5\ a_4), (d_2\ d_5)(d_3\ d_4)),$
$((a_2\ a_3)(a_4\ a_5), (d_2\ d_5)(d_3\ d_4)),$
$((a_2\ a_3\ a_4), e_d), ((a_2\ a_3\ a_4), (d_2\ d_5)(d_3\ d_4)),$
$((a_2\ a_3\ a_5), e_d)), ((a_2\ a_3\ a_5), (d_2\ d_5)(d_3\ d_4)),$
$((a_2\ a_4\ a_3), e_d), ((a_2\ a_4\ a_3), (d_2\ d_5)(d_3\ d_4)),$
$((a_2\ a_4\ a_5), e_d), ((a_2\ a_4\ a_5), (d_2\ d_5)(d_3\ d_4)),$
$((a_2\ a_4)(a_3\ a_5), e_d),$
$((a_2\ a_4)(a_3\ a_5), (d_2\ d_5)(d_3\ d_4)),$
$(a_2\ a_5\ a_3), e_d), ((a_2\ a_5\ a_3), (d_2\ d_5)(d_3\ d_4)),$
$(a_2\ a_5\ a_4), e_d), ((a_2\ a_5\ a_4), (d_2\ d_5)(d_3\ d_4)),$
$((a_2\ a_5)(a_3\ a_4), e_d),$
$((a_2\ a_5)(a_3\ a_4), (d_2\ d_5)(d_3\ d_4))\}$

The suborbital O_1 that correspond to suborbit Δ_1;
$O_1 = \{(e_a, e_d)((x_1, y_1), (x_2, y_1), (x_2, y_2), (x_2, y_3),$
$(x_2, y_4), (x_2, y_5))$
$= (x_1, y_1), (x_2, y_1), (x_2, y_2), (x_2, y_3), (x_2, y_4), (x_2, y_5)\}$
$O_1 = \{(e_a, (d_2\ d_5)(d_3\ d_4)((x_1, y_1), (x_2, y_1), (x_2, y_2),$
$(x_2, y_3), (x_2, y_4), (x_2, y_5)$
$= (x_1, y_1), (x_2, y_1), (x_2, y_5), (x_2, y_4), (x_2, y_3), (x_2, y_2)\}$

$\vdots$

$O_1 = \{((a_2\ a_5)(a_3\ a_4), (d_2\ d_5)(d_3\ d_4))(x_1, y_1), (x_2, y_1),$
$(x_2, y_2), (x_2, y_3), (x_2, y_4), (x_2, y_5)$
$= (x_1, y_1), (x_5, y_1), (x_5, y_5), (x_5, y_4), (x_5, y_3), (x_5, y_2)\}$

Using suborbital O_1, the suborbital graph Γ_1 is drawn by constructing a line from (x_1, y_1) to
$\{(x_2, y_1), (x_2, y_2), (x_2, y_3), (x_2, y_4), (x_2, y_5)\}$
such that neither of the components are identical.

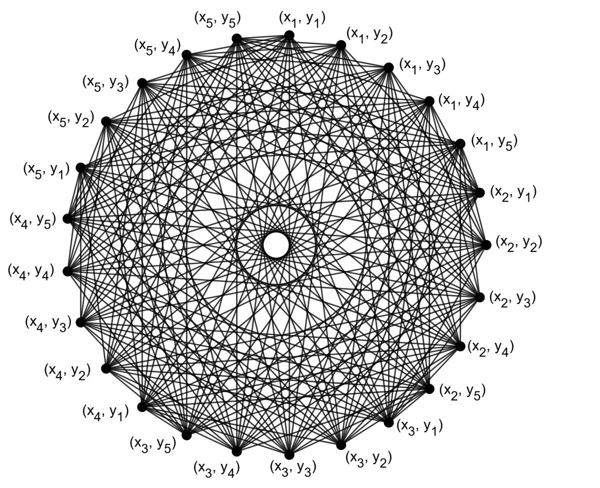

Figure 6. *Suborbital graph Γ_1 corresponding to suborbit Δ_1 of the group $A_5 \times D_5$ action on $X \times Y$.*

The suborbital graph Γ_1 above is simple, undirected, regular with degree 16, connected and the girth 3.

NOTE: Suborbital graphs, Γ_1, Γ_2, Γ_3 and Γ_4 are identical hence they have the same structure and properties.

The suborbital O_5 that correspond to suborbit Δ_5;
$O_5 = \{(e_a, e_d)((x_1, y_1), (x_1, y_2), (x_1, y_3), (x_1, y_4), (x_1, y_5)),$
$= (x_1, y_1), (x_1, y_2), (x_1, y_3), (x_1, y_4), (x_1, y_5)\}$
$O_5 = \{(e_a, (d_2\ d_5)(d_3\ d_4)((x_1, y_1), (x_1, y_2), (x_1, y_3),$
$(x_1, y_4), (x_1, y_5))$
$= (x_1, y_1), (x_1, y_5), (x_1, y_4), (x_1, y_3), (x_1, y_2)\}$

$\vdots$

$O_4 = \{((a_2\ a_5)(a_3\ a_4), (d_2\ d_5)(d_3\ d_4))((x_1, y_1), (x_1, y_2),$
$(x_1, y_3), (x_1, y_4), (x_1, y_5))$
$= (x_1, y_1), (x_1, y_5), (x_1, y_4), (x_1, y_3), (x_1, y_2)\}$

Using suborbital O_5, the suborbital graph Γ_5 is drawn by constructing a line from (x_1, y_1) to
$\{(x_1, y_5), (x_1, y_4), (x_1, y_3), (x_1, y_2)\}$ such that only the first components are identical.

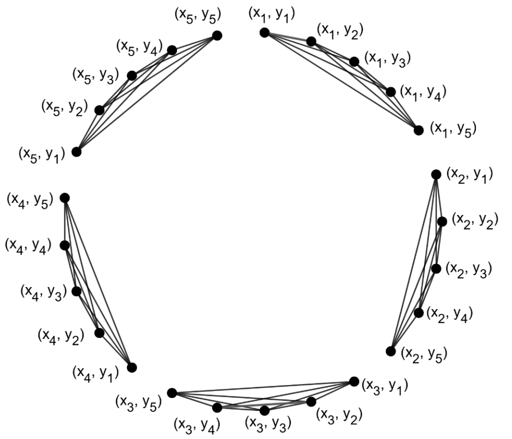

Figure 7. *Suborbital graph Γ_5 corresponding to suborbit Δ_5 of the group $A_5 \times D_5$ action on $X \times Y$.*

The suborbital graph Γ_5 above is simple, undirected, regular with degree 4, has five disconnected components, and the girth 3.

3.4. *Suborbital Graphs of the Group $A_n \times D_n$ Action on $X \times Y$*

Suborbital graphs constructed when the group $A_n \times D_n$ acts on the set $X \times Y$ fall into either of the following three categories;

(a) These are suborbital graphs drawn by constructing a line from (x_1, y_1) to $\{(x_1, y_2), (x_1, y_3), \ldots, (x_1, y_n)\}$, such that only the first components of the pairs are identical. These suborbital graphs have n disconnected components, are simple, self-paired, regular with degree $n - 1$, and have girth 3.

(b) These are suborbital graphs drawn by constructing a line from (x_1, y_1) to $\{(x_2, y_1), (x_3, y_1), ..., (x_n, y_1)\}$ such that only the second components of the sets are identical. These suborbital graphs; have n disconnected components, are simple, self-paired, regular with degree $(n-1)$ and girth 3.

(c) These are suborbital graphs drawn by constructing a line from (x_1, y_1) to $\{(x_2, y_2), (x_3, y_3), ..., (x_n, y_n)\}$ such that none of the components is identical. These suborbital graphs; are simple, connected, self-paired, are regular with degree that varies form graph to graph and girth 3.

4. Conclusion

All suborbital graphs of the group $(A_n \times D_n)$ action on set $(X \times Y)$ for $n \geqslant 3$ are simple, self-complementary (self-paired), undirected, regular degree, and have a girth 3. However, some graphs are connected while others disconnected this further proves that the group action is imprimitive.

Symbols

A_n	Alternating group (degree n and order is $\frac{n!}{2}$)
D_n	Dihedral group of order $2n$
$X \times Y$	Cartesian product between set X and set Y
$\text{Stab}_G(x)$	Stabilizer of a point x in G
$\text{Fix}_G(x)$	A point x fixed an element in G
$\text{Orb}_G(x)$	The orbit of a point x in G
Δ	Suborbit of G on set K
$\|G\|$	Order of G
$\forall$	for all
$\emptyset$	An empty set
Γ	Suborbital graph
O	Suborbital
$G = A_n \times D_n$	Direct product between Alternating group and Dihedral group

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Conflicts of Interest

The authors declare no conflicts of interest.

References

[1] Shanker, G. R. (2009). Discrete Mathematical Structures. *New Age International (p) Limited Publishers*, 2, 218.

[2] Kurzweil, H. and Stellmacher, B. (2004). Permutation Groups. *The Theory of FiniteGroups* (pp. 77-97). New York: Springer New York. https://doi.org/10.1007/0-387-21768-1_4

[3] Burnside William. (1911). Theory of Groups of finite Order. *Cambridge University Press*. 2. 1.

[4] Thomas, W. J. (2010). Abstract Algebra (Theory and Applications). *Stephen F. Austin University*.

[5] John, B. F. (2003). A First Course in Abstract Algebra. *Pearson Education Inc*. 7th Edition.

[6] Gachogu, R., Kamuti, I. N., Gachuki, M. N. (2017). Properties of Suborbitals of Dihedral group acting on ordered subset.*Advances in Pure Mathematic*s, 7, 375-382. https://doi.org/10.4236/apm.2017.78024

[7] Nyaga, L. N., Kamuti, I. N., Mwathi, C. W., Akanga, J. R. (2011). Ranks and Subdegrees of the Symmetric Group S_n Acting on Unordered r-element Subsets. http://ir.jkuat.ac.ke/handle/123456789/918

[8] Gachimu, R., kamuti, I., Nyaga, L., Rimberia, J., Kamaku, P. (2016). Properties invariants Associated with the Action of the Alternating Group on Unordered subsets. *International Journal of Pure and Applied Mathematics*, 106(1), 333-346. https://doi.org/10.12732/ijpam.v106i1.27

[9] Njagi, L. (2016). Ranks and Subdegrees of Suborbital Graphs of Symmetric Group Acting on Ordered Pairs. *Journal of Advanced Research in Applied Science*, 3(2). https://doi.org/10.53555/nnas.v3i2.664

[10] Edgar, G. G., Michael, M. P. (2002). Discrete Mathematics with Group Theory. *Prentice-Hall Inc*, 2, 286.

[11] Douglas, E. E., Winston, J. C. (2006). Discrete Mathematics. *John Wiley and Sons Inc*, 1, 510.

[12] Seymour Lipschutz, Marc Lars Lipson. (2007). Discrete Mathematics. *Mc Graw-Hill companies*, 3, 157.

[13] Oscar Levin. (2021). Discrete Mathematics. 3, 243. https://discrete.openmathbooks.org/dmoi3.html

[14] Richard Johnsonbaugh. (2018). Discrete Mathematics. *Pearson*, 8th Edition, 385.

[15] Liu, C. L.(2001). Elements of Discrete Mathematics. *Tata McGraw-Hill Publishing Company*, 2, 139.

[16] Richard, A. Gibbs. (1974). Self-Complementary Groups. Journal of Combinatorial Theory (B), 16, 106-123. https://doi.org/10.1016/0095-8956(74)90053-7

[17] Helmut Wielandth. (1964). Multiply Transitive Groups. Finite Permutation Groups, 19-43. https://doi.org/10.1016/B978-0-12-749650-4.50007-2

[18] Cameron, P. J. (1975). Suborbits in transitive permutation groups. Proceedings of the NATO Advanced Study Institute on Combinatorics, 16: 419-450. https://doi.org/10.1007/978-94-010-1826-5_20

[19] Victor, J. M., Namu, N. L., Muriuki, G. D. (2024). Combinatorial Properties and Invariants Associated with the Direct Product of Alternating and Dihedral Groups Acting on the Cartesian Product of Two Sets. American Journal of Applied Mathematics, 12(6), 258-265. https://doi.org/10.11648/j.ajam.20241206.15