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The Symmetry of Solutions for a Class of KIRCHHOFF Equations on the Unit Ball and in the Entire Space

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Abstract: This paper is mainly concerned with the symmetry and monotonicity of solutions to a fractional parabolic Kirchhoff equation. We first establishes the asymptotic narrow region principle, the asymptotic maximum principle near infinity, the asymptotic strong maximum principle and the Hopf principle for antisymmetric functions in bounded and unbounded domains. By the method of moving plane, it then derives the symmetry of positive solutions on the unit sphere and in the entire space. Next, we point out how to apply these tools and methods to obtain asymptotic radial symmetry and monotonicity of positive solutions in a unit ball and on the whole space. By some researches, we find that no matter how we set the initial value, it will not affect the property of the solution approaching a radially symmetric function as t approaches infinity. Throughout the paper, establishing the maximum principle plays a central role in exploring and studying the fractional parabolic Kirchhoff equation. After establishing different maximum principles, one can study the properties of a solution to the parabolic equation under different conditions. Finally, the novelty of this article is that it is the first time to apply method of moving plane to fractional parabolic Kirchhoff problems and the ideas and methods presented in this article are applicable to studying different non local parabolic problems, various operators and the symmetry of solutions in different regions.

Keywords: Fractional Laplace Operator, Maximum Principle, Moving Plane Method, Parabolic Equation

1. Introduction

This article studies the symmetry of the positive solutions of the fractional parabolic Kirchhoff equation

$$\begin{aligned}&\tfrac{\partial u}{\partial t}(x,t)\\&+\left(a+b\int_{\mathbb{R}^n}\left|(-\Delta)^{\frac{s}{2}}u(x,t)\right|^2dx\right)(-\Delta)^s u(x,t)\\&=f(t,u(x,t)),\quad u>0\end{aligned} \tag{1}$$

on the unit sphere and the entire space where $a \geq 0, b > 0, s \in (0,1)$ is a constant and

$$\begin{aligned}&\int_{\mathbb{R}^n}\left|-\Delta^{\frac{s}{2}}u(x,t)\right|^2dx\\&=\int\int_{\mathbb{R}^{2n}}\tfrac{|u(x,t)-u(y,t)|^2}{|x-y|^{n+2s}}dydx\end{aligned} \tag{2}$$

holds for every fixed $t > 0$. In this article, for each fixed $t > 0$, the fractional Laplacian operator is defined as

$$(-\Delta)^s u(x,t) = C_{n,s}PV\int_{\mathbb{R}^n}\frac{u(x,t)-u(y,t)}{|x-y|^{n+2s}}dy \tag{3}$$

where $C_{n,s}$ is the standardized constant and PV represents the Cauchy principal value.

The Kirchhoff equation originated from Krichhoff-type problems. Krichhoff-type problems have been developed for many years in various physical and biological models. In order to extend the vibration of elastic ropes for this classic D'Alembert's wave equation, Kirchhoff [1] studied the following wave equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2\right)\frac{\partial^2 u}{\partial x^2}=0 \tag{4}$$

Where $u = u(x,t)$ is the lateral displacement of coordinates x and time t, ρ is the mass density, P_0 is the initial axial tensile, h is the cross-sectional area, E is the Young's modulus, L is the length. Moreover, Pohozaev [2] also studied the Kirchhoff equation mentioned above. Afterwards, Lions [3] studied an abstract functional model of the following equation.

$$u_{tt} + \left(a + b\int_{\Omega} |\nabla u|^2\right) \Delta u = f(x, u). \tag{5}$$

In the following years, Alves [4], Spagnolo [5] and Li [6] have studied a lot of research in this area. Pucci and Saldi [7] established the existence and multiplicity of non trivial and non negative global solutions for a class of Kirchhoff eigenvalue problems involving critical nonlinearity and fractional Laplacian operators. For more fractional Kirchhoff problems, we can refer to He [8] and Zhang [9]. This paper investigates the radial symmetry of positive solutions to a class of fractional order Kirchhoff equations for parabolic systems.

The study of non locality of fractional Laplacian operators is quite difficult. Caffarelli and Silvestre [10] use the extension method to transform non local problems into local ones in high dimensions. This extension method can be effectively used to study equations involving fractional Laplacian operators. In paper of Chen, Li and Ou [11], the author proved that if $u \in H^{\frac{\alpha}{2}}(\mathbb{R}^n)$ is the positive weak solution of

$$(-\Delta)^{\frac{\alpha}{2}} u = u^p(x),\ x \in \mathbb{R}^n, \tag{6}$$

then it satisfies the integral equation

$$u(x) = C\int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} u^p(y)dy. \tag{7}$$

Thus, by using the integral form of the moving plane method, the radial symmetry of the positive solution under critical conditions and the non existence of the positive solution under subcritical conditions can be obtained. Both extension methods and integral equations need additional conditions to be added to the solution. But this is not necessary for directly studying pseudo differential equations such as equation.

$$C_{n,s} \lim_{\epsilon \to 0} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{a(x-z)(u(x)-u(z))}{|x-z|^{n+\alpha}} dz = f(x, u) \tag{8}$$

which involves consistent non local elliptical operators where $0 < c_0 \leq a(y) \leq C_1$ and equation

$$F_\alpha(u(x)) \equiv C_{n,\alpha} \lim_{\epsilon \to 0} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{G(u(x)-u(z))}{|x-z|^{n+\alpha}} dz = f(x, u) \tag{9}$$

which involves completely nonlinear non local operators. Few extension methods and integral equation studies are used for these operators such that it is necessary for applying direct methods necessary about studying typical non local operators. Jarohs and Weth [12] studied direct methods for non local operators. They introduced the maximum principle of antisymmetric functions and proved the radial symmetry of positive solutions through the moving plane method. However, they only focused on the maximum principle of bounded domains and only studied weak solutions defined by the inner product of $H^{\frac{\alpha}{2}}(\Omega)$.

Due to the work of Chen and Li [13, 14], they have developed a series of methods whether bounded or unbounded for using the moving plane method on non local problems. For example, we know fractional Laplace and p-Laplace which no longer use extension methods and integral equations to obtain the symmetry, monotonicity and non existence of positive solutions for various semilinear equations involving non local operators. Inspired by the use of the moving plane method in [14], [15] and [16] to prove the symmetry and monotonicity of the positive solutions of fractional parabolic Kirchhoff equations containing non local operators and non local term $\int_{\mathbb{R}^n} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 dxdt$, this paper studies the asymptotic symmetry of solutions to fractional parabolic Kirchhoff equations which is a trend that the initial solution of the parabolic equation has no symmetry and becomes symmetric as time approaches infinity.

Here are some symbols that will be used in the following of this article.

The ω-limits set of u:

$$\omega(u) := \left\{\varpi |\ \varpi = \lim_{t_k \to \infty} u(\cdot, t_k)\right\},$$

$$T_\kappa = \{x \in \mathbb{R}^n | \text{for some } \kappa \in \mathbb{R}\ ,\ x_1 = \kappa\}$$

are the moving planes,

$$\Sigma_\kappa = \{x \in \mathbb{R}^n | x_1 > \kappa\}$$

is the right area of plane T_κ and

$$x^\kappa = (2\kappa - x_1, x_2, \ldots, x_n)$$

is the reflection area of T_κ for x.

To make (2) and (3) meaningful, it is assumed that

$$u \in H^s(\mathbb{R}^n) \bigcap L_{2s} \bigcap C^{1,1}_{loc}(\mathbb{R}^n),$$

where

$$H^s(\mathbb{R}^n)=\left\{u(\cdot,t)\in L^2(\mathbb{R}^n):\frac{|u(x,t)-u(y,t)|}{|x-y|^{\frac{n}{2}+s}}\in L^2(\mathbb{R}^n\times\mathbb{R}^n)\text{for every fixed t}\right\},$$

$$L_{2s}=\left\{u(\cdot,t)\in L^1_{loc}(\mathbb{R}^n)|\int_{\mathbb{R}^n}\frac{|u(x,t)|}{1+|x|^{n+2s}}dx<+\infty\right\}$$

and $C^{1,1}_{loc}(\mathbb{R}^n)$ is the usual Hölder space on each $V\subset\subset\mathbb{R}^n$. For any function u that satisfies (1.1), let

$$I(u(x,t))=a+b\int_{\mathbb{R}^n}\left|(-\Delta)^{\frac{s}{2}}u(x,t)\right|^2dx,$$

$$u_\kappa(x)=u(x^\kappa),$$

$$w_\kappa(x)=u_\kappa(x)-u(x),$$

$$w_\kappa(x)=-w_\kappa(x^\kappa).$$

The Dirichlet problem on the unit ball is as follows：

$$\begin{cases}\frac{\partial u(x,t)}{\partial t}+I(u(x,t))(-\Delta)^su(x,t)=f(t,u(x,t)), & (x,t)\in B_1(0)\times(0,+\infty),\\ u(x,t)=0, & (x,t)\in B_1^c(0)\times(0,+\infty).\end{cases}\tag{10}$$

Theorem 1.1. (Radial symmetry of solutions on a unit sphere)

Assuming that function $u(x,t)\in\left(C^{1,1}_{loc}(B_1(0))\cap C(\overline{B_1(0)})\times C^1(0,+\infty)\right)$ is a positive uniformly bounded solution of (1.1) where $B_1(0)$ is a open ball with center 0 and radius 1, $\overline{B_1(0)}$ is the closure of $B_1(0)$, $C(\overline{B_1(0)})$ is the space composed of continuous functions in $\overline{B_1(0)}$ and $C^1(0,+\infty)$ is the space composed of 1-times continuously differentiable functions in $(0,+\infty)$.

Let $\alpha\in(0,1)$ and $\frac{\alpha}{2s}\in(0,1)$. Assume $f(t,u)\in L^\infty(\mathbb{R}^+\times\mathbb{R})$ is $C^{\frac{\alpha}{2s}}_{loc}$ for any t and $L^\infty(\mathbb{R}^+\times\mathbb{R})$ where $L^\infty(\mathbb{R}^+\times\mathbb{R})$ is the space composed of Lebesgue measurable functions in $\mathbb{R}^+\times\mathbb{R}$ with ess $\sup_{\mathbb{R}^+\times\mathbb{R}}|f(t,u)|<+\infty$. For t, f is Lipschitz continuous with respect to u and

$$f(x,t,0)\geq 0,\quad \forall t\geq 0.$$

For any $\varpi(x)\in w(u)$, either

$$\varpi(x)\equiv 0,$$

or $\varpi(x)\in w(u)$ is radially symmetrical and strictly decreasing about the origin.

The following section introduces the asymptotic symmetry of the solution of equation KIRCHHOFF.

$$\frac{\partial u(x,t)}{\partial t}+I(u(x,t))(-\Delta)^su(x,t)=f(t,u(x,t)),\quad (x,t)\in\mathbb{R}^n\times(0,+\infty)\tag{11}$$

in the entire space. Firstly, the following assumptions are made for the nonlinear term：

(G): Let $\alpha\in(0,1),\ \frac{\alpha}{2s}\in(0,1),\ f(t,u)$ belongs to $C^{\frac{\alpha}{2s}}_{loc}$ for time t and it is uniformly Lipschitz continuous with respect to t for u. we also need

$$f(t,0)=0,\ f_u(t,0)<-\delta,\quad t>0$$

where $\delta>0$ is a constant. But $f_u\equiv\frac{\partial f}{\partial u}$ is continuous near $u=0$.

Theorem 1.2. (Radial symmetry of solutions in $\mathbb{R}^n$)

Let $u(x,t)\in\left(C^{1,1}_{loc}(\mathbb{R}^n)\cap L_{2s}\cap H^s(\mathbb{R}^n)\times C^1(0,+\infty)\right)$ be the positive uniformly bounded solution of (11). It satisfies

$$\lim_{|x|\to+\infty}u(x,t)=0,\tag{12}$$

for sufficiently large $t$where f satisfies (G). Then either for all $\varpi(x)\in w(u)$, it is always equal to 0, either $\varpi(x)\in w(u)$ is radially symmetrical and decreases for some points in $\mathbb{R}^n$ which means there exists $\tilde{x}\in\mathbb{R}^n$ and $x\in\mathbb{R}^n$ such that

$$\varpi\left(x-\tilde{x}\right)=\varpi\left(|x-\tilde{x}|\right).$$

2. Some Maximum Principles

This section mainly establishes some maximum principles related to fractional parabolic Kirchhoff operators involving antisymmetic functions. These play a key role in applying the moving plane method to solve the radial symmetry of the positive solution of equations.

Theorem 2.1. (Asymptotic narrow region principle)

Let Ω be a subset contained within a narrow area

$$\{x|\kappa - l < x_1 < \kappa\} \subset \Sigma_\kappa$$

where l is sufficiently small. For a sufficiently large $\bar{t}$, Assume for any x, $w_\kappa(x,t) \in \left(C^{1,1}_{loc}(\Omega) \cap L_{2s}\right) \times C^1([\bar{t}, +\infty])$ is uniformly bounded and semicontinous over region $\bar{\Omega}$. It also satisfies

$$\begin{cases} \frac{\partial w_\kappa}{\partial t}(x,t) + I(u(x,t))(-\Delta)^s w_\kappa(x,t) = c_\kappa(x,t) w_\kappa(x,t), & (x,t) \in \Omega \times [\bar{t}, +\infty), \\ w_\kappa(x,t) \geq 0, & (x,t) \in (\Sigma_\kappa \setminus \Omega) \times [\bar{t}, +\infty), \\ w_\kappa(x,t) = -w_\kappa(x^\kappa, t), & (x,t) \in \Omega \times [\bar{t}, +\infty), \end{cases} \tag{13}$$

where $c_\kappa(x,t)$ is bounded. The following conclusion holds:

(i)If Ω is bounded, then for sufficiently small l,

$$\varliminf_{t \to \infty} w_\kappa(x,t) \geq 0, \ \forall x \in \Omega.$$

(ii)If Ω is unbounded, then for $t \geq \bar{t}$,

$$\varliminf_{|x| \to \infty} w_\kappa(x,t) \geq 0, \ \forall x \in \Omega.$$

Proof: Let m be a constant to be selected. Let

$$\tilde{w}_\kappa(x,t) = e^{mt} w_\kappa(x,t),$$

then

$$\frac{\partial \tilde{w}_\kappa}{\partial t} = m e^{mt} w_\kappa + e^{mt} \partial \frac{w_\kappa}{\partial t}$$

and

$$\begin{aligned} I(u)(-\triangle)^s \tilde{w}_\kappa(x,t) &= I(u)(-\triangle)^s (e^{mt} w_\kappa(x,t)) \\ &= I(u) e^{mt} C_{n,s} P.V \int_{\mathbb{R}^n} \frac{w_\kappa(x,t) - w_\kappa(y,t)}{|x-y|^{n+2s}} dy \\ &= I(u) e^{mt} (-\triangle)^s w_\kappa(x,t). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\partial \tilde{w}_\kappa}{\partial t} + I(u)(-\triangle)^s \tilde{w}_\kappa &= e^{mt} \left(m w_\kappa + \frac{\partial w_\kappa}{\partial t} + I(u)(-\triangle)^s w_\kappa \right) \\ &= e^{mt} (m w_{\kappa} + c_\kappa w_\kappa) \\ &= \tilde{w}_\kappa (m + c_\kappa). \end{aligned}$$

The following will prove $\lim_{t \to +\infty} w_\kappa(x,t) \geq 0$ by proving for any $\forall T > \bar{t}$, $(x,t) \in \Omega \times [\bar{t}, T]$,

$$\tilde{w}_\kappa(x,t) \geq \min\{0, \inf_\Omega \tilde{w}_\kappa(x, \bar{t})\}. \tag{14}$$

If (14) is false, By observing the lower semicontinuity of (13) and w_κ on $\bar{\Omega} \times [\bar{t}, T]$, it can be concluded that the existence of $(x_0, t_0) \in \Omega \times (\bar{t}, T)$ such that

$$\tilde{w}_\kappa(x_0, t_0) = \min_{\Sigma_\kappa \times (\bar{t}, T]} \tilde{w}_\kappa(x,t) < \min\{0, \underset{\Omega}{inf}\, \tilde{w}_\kappa(x, \bar{t})\}.$$

Because (x_0, t_0) may reach its extremum inside the cylinder. Thus,

$$\frac{\partial \tilde{w}_\kappa}{\partial t}(x_0, t_0) \leq 0$$

and

$$\begin{aligned}
& I(u)(-\triangle)^s \tilde{w}_\kappa(x_0, t_0) \\
&= I(u) C_{n,s} P.V \int_{\mathbb{R}^n} \frac{w_\kappa(x_0, t_0) - w_\kappa(y, t_0)}{|x_0 - y|^{n+2s}} dy \\
&= I(u) C_{n,s} P.V \int_{\Sigma_\kappa} \frac{w_\kappa(x_0, t_0) - w_\kappa(y)}{|x_0 - y|^{n+2s}} dy + I(u) C_{n,s} P.V \int_{\mathbb{R}^n \setminus \Sigma_\kappa} \frac{w_\kappa(x_0, t_0) - w_\kappa(y)}{|x_0 - y|^{n+2s}} dy \\
&= I(u) C_{n,s} P.V \int_{\Sigma_\kappa} \frac{w_\kappa(x_0, t_0) - w_\kappa(y)}{|x_0 - y|^{n+2s}} dy + I(u) C_{n,s} P.V \Sigma_\kappa \frac{w_\kappa(x_0, t_0) - w_\kappa(y^\kappa)}{|x_0 - y^\kappa|^{n+2s}} dy \\
&\leq I(u) C_{n,s} P.V \int_{\Sigma_\kappa} \frac{w_\kappa(x_0, t_0) - w_\kappa(y)}{|x_0 - y^\kappa|^{n+2s}} dy + I(u) C_{n,s} P.V \int_{\Sigma_\kappa} \frac{w_\kappa(x_0, t_0) + w_\kappa(y)}{|x_0 - y^\kappa|^{n+2s}} dy \\
&= 2 I(u) C_{n,s} w_\kappa(x_0, t_0) \int_{\Sigma_\kappa} \frac{1}{|x_0 - y^\kappa|^{n+2s}} dy \\
&\leq \frac{c}{l^{2s}} \tilde{w}_\kappa(x_0, t_0) I(u)
\end{aligned}$$

where the proof of the last inequality is shown in the proof of Theorem 2.1 in [15]. we can conclude that

$$\begin{aligned}
\frac{\partial \tilde{w}_\kappa}{\partial t}(x_0, t_0) &= -I(u)(-\triangle)^s \tilde{w}_\kappa(x_0, t_0) + (m + c_\kappa(x_0, t_0)) \tilde{w}_\kappa(x_0, t_0) \\
&\geq \left(-\frac{c}{l^{2s}} I(u) + m + c_\kappa(x_0, t_0) \right) \tilde{w}_\kappa(x_0, t_0).
\end{aligned}$$

Since $c_\kappa(x, t)$ is bounded for all (x, t), we can choose sufficiently small l such that

$$-\frac{c}{l^{2s}} I(u) + C_\kappa(x_0, t_0) < -\frac{c}{2l^{2s}}.$$

After this, we choose $m = \frac{C}{2l^{2s}} > 0$ so that the right-hand side of the above equation is strictly greater than 0. Since $\tilde{w}_\kappa(x_0, t_0) < 0$, it is a contradiction. Thus, using the boundedness of w_κ, there is a $c_1 > 0$ such that

$$\tilde{w}_\kappa(x, t) \geq \min \left\{ 0, \inf_\Omega \tilde{w}_\kappa(x, \bar{t}) \right\} \geq -C_1, \quad (x, t) \in \Omega \times [\bar{t}, T], \quad \forall T > \bar{t}.$$

We have

$$w_\kappa(x, t) \geq e^{-mt}(-C_1), \quad \forall t > \bar{t}.$$

Let $t \to +\infty$, it can be derived that

$$\lim_{t \to +\infty} w_\kappa(x, t) \geq 0, \quad x \in \Omega.$$

As Ω is unbounded, the same conclusion can be drawn.

Theorem 2.2. (Asymptotic maximum of antisymmetric functions)

Let Ω be a bounded domain in Σ_κ. Assume $v(x, t) \in \left(C^{1,1}_{loc}(\Omega) \cap L_{2s} \cap H^s \right) \times C^1([0, +\infty])$ is semicontinuous on $\bar{\Omega}$ with respect to x and satisfies

$$\begin{cases}
\frac{\partial v}{\partial t}(x, t) + I(v)(-\Delta)^s v(x, t) \geq c_\kappa(x, t) v(x, t), & (x, t) \in \Omega \times [0, +\infty), \\
v(x^\kappa, t) = -v(x, t), & (x, t) \in (\Sigma_\kappa \times [0, +\infty), \\
v(x, t) \geq 0, & (x, t) \in (\Sigma_\kappa \setminus \Omega) \times [0, +\infty), \\
v(x, 0) \geq 0, & x \in \Omega.
\end{cases} \tag{15}$$

If $c_\kappa(x,t)$ is bounded, then

$$v(x,t) \geq 0, \quad (x,t) \in \Omega \times [0,T], \quad \forall\, T > 0.$$

Proof: Because $c_\kappa(x,t)$ is bounded, we can select $m < 0$ such that $m + c_\kappa(x,t) < 0$. Let

$$\tilde{v}(x,t) = e^{mt} v(x,t),$$

then $\tilde{v}(x,t)$ satisfies

$$\frac{\partial \tilde{v}}{\partial t} + I(v)(-\Delta)^s \tilde{v} \geq (m + c_\kappa(x,t)) v(\tilde{x},t), \quad (x,t) \in \Omega \times [0,+\infty).$$

The following statement states that

$$\tilde{v}(x,t) \geq \inf_{\Omega} \tilde{v}(x,0) \geq 0$$

is established in region $\Omega \times [0,T]$. If it is false, there exists $(x_0,t_0) \in \Omega \times (0,T)$ such that

$$\tilde{v}(x_0,t_0) = \inf_{R^n \times [0,T]} \tilde{v}(x,t) < 0.$$

Then

$$(m + c(x_0,t_0))\tilde{v}(x_0,t_0) > 0.$$

However

$$\frac{\partial v}{\partial t}(x_0,t_0) \leq 0$$

and

$$I(v)(-\triangle)^s \tilde{v}(x_0,t_0) = I(v) C_{n,s} P.V \int_{R^n} \frac{\tilde{v}(x_0,t_0) - \tilde{v}(y,t_0)}{|x_0 - y|^{n+2s} dy} < 0$$

contradicts the previous statement. Therefore, the theorem is proven.

Theorem 2.3. (Asymptotic strong maximum principle for antisymmetric functions)

For sufficiently large $\bar{t}$, assume $w_\kappa(x,t) \in \left(C^{1,1}_{loc}(\Sigma_\kappa) \cap L_{2s}\right) \times C^1\left([\bar{t},+\infty)\right)$ is bounded and satisfies

$$\begin{cases} \frac{\partial w_\kappa}{\partial t}(x,t) + I(u)(-\Delta)^s w_\kappa(x,t) = c_\kappa(x,t) w_\kappa(x,t), & (x,t) \in \Sigma_\kappa \times [\bar{t},+\infty), \\ w_\kappa(x,t) = -w_\kappa(x^\kappa,t), & (x,t) \in \Sigma_\kappa \times [\bar{t},+\infty), \\ \lim\limits_{t \to +\infty} w_\kappa(x,t) \geq 0, & x \in \Sigma_\kappa, \end{cases} \tag{16}$$

where $c_\kappa(x,t)$ is bounded. Assume $\Upsilon_\kappa > 0$ in some areas in Σ_κ, then $\Upsilon_\kappa(x) > 0$ in Σ_κ.

Proof: Due to the definition of ϖ, for any $\varpi \in w(u)$, there exists t_k such that as $t_k \to +\infty$, $w_\kappa(x,t_k) \to \Upsilon_\kappa(x)$. Let

$$w_k(x,t) = w_\kappa(x, t + t_k - 1),$$

we obtain that

$$\frac{\partial w_k}{\partial t}(x,t) + I(u)(-\Delta)^s w_k(x,t) = c_k(x,t) w_k(x,t), \quad (x,t) \in \Sigma_\kappa \times [\bar{t},+\infty)$$

where $c_k(x,t) = c_\kappa(x, t + t_k - 1)$. By using the standard parabolic regularity estimate in [17], we can deduce that there exists a subsequence $w_k(x,t)$ of function $w_\infty(x,t)$ that uniformly converges to region $\Sigma_\kappa \times [0,2]$. As $k \to +\infty$, we have

$$\frac{\partial w_k}{\partial t}(x,t) + I(u)(-\Delta)^s w_k(x,t) \to \frac{\partial w_\infty}{\partial t}(x,t) + I(u)(-\Delta)^s w_\infty(x,t),$$

$$c_k(x,t) \to c_\infty(x,t).$$

Since c_κ is bounded, it can be obtained that c_∞ is bounded. It can be inferred from Fernández-Real, Ros-Oton [17] that $w_\infty(x,t)$ is Hölder continuous in x and t. Especially for $k \to +\infty$,

$$w_\kappa(x,t_k) = w_k(x,1) \to w_\infty(x,1) = \Upsilon_\kappa(x).$$

Choose $m > 0$ such that

$$c_\infty(x,t) + m > 0.$$

We establish a new function

$$\tilde{w}(x,t) = e^{mt} w_\infty(x,t).$$

By the third condition of Theorem 2.3,

$$\tilde{w}(x,t) \geq 0, \quad (x,t) \in \Sigma_\kappa \times [0,2].$$

Thus,

$$\frac{\partial \tilde{w}}{\partial t}(x,t) + I(w)(-\Delta)^s \tilde{w}(x,t) = (m + c_\infty(x,t))\tilde{w}(x,t) \geq 0, \quad (x,t) \in \Sigma_\kappa \times [0,2]. \tag{17}$$

Because there is $\Upsilon_\kappa > 0$ on Σ_κ, through continuity, there exists a subset $D \subset\subset \Sigma_\kappa$ such that

$$\Upsilon_\kappa(x) > c > 0, \quad x \in D \tag{18}$$

where C is a constant. Through the continuity of $w_\infty(x,t)$, there exists $0 < \epsilon_0 < 1$,

$$w_\infty(x,t) > \frac{c}{2}, \quad (x,t) \in D \times [1-\epsilon_0, 1+\epsilon_0].$$

For simplicity, assuming

$$w_\infty(x,t) > \frac{c}{2}, \quad D \times [0,2]. \tag{19}$$

For any point $\bar{x}$ on $\Sigma_\kappa \backslash D$, choose $\delta = \min\{\text{dist}(\bar{x}, D), \text{dist}(\bar{x}, T_\kappa)\} > 0$, then $B_\delta(\bar{x}) \subset \Sigma_\kappa \backslash D$. Next, we construct the lower solution on region $B_\delta(\bar{x}) \times [0,2]$. Let

$$\underline{w}(x,t) = \chi_{D \bigcup D_\kappa}(x)\tilde{w}(x,t) + \epsilon\eta(t)g(x)$$

where D_κ is the reflection plane of region D with respect to plane T_κ. $\eta(t) \in C^\infty(0,2)$ is defined as

$$\eta(t) = \begin{cases} 1 & t \in \left[1 - \frac{\epsilon_0}{2}, 1 + \frac{\epsilon_0}{2}\right], \\ 0 & t \notin \left[1 - \frac{\epsilon_0}{2}, 1 + \frac{\epsilon_0}{2}\right]. \end{cases}$$

and

$$g(x) = \left(\delta^2 - |x - \bar{x}|^2\right)_+^s - \left(\delta^2 - |x - \bar{x}|^{\kappa 2}\right)_+^s.$$

It is obvious that $g(x^\kappa) = -g(x)$,

$$(-\triangle)^s g(x) \leq c_0. \tag{20}$$

This prove can be found in Chen [15]. By using the fractional Laplacian operator and (19), we can obtain that for any fixed $t \in [0,2]$ and any fixed $x \in B_\delta(\bar{x})$,

$$(-\Delta)^s \left(\chi_{D \bigcup D_\kappa}(x)\tilde{w}(x,t)\right) \leq -c_1. \tag{21}$$

This prove can also be found in Chen [15]. From $D \subset\subset \Sigma_\kappa$, continuity, (20) and (21), it can be concluded that for region $(x,t) \in B_\delta(\bar{x}) \times [0,2]$,

$$\begin{aligned} \tfrac{\partial \underline{w}(x,t)}{\partial t} + I(u)(-\Delta)^s \underline{w}(x,t) &= \epsilon\eta'(t)g(x) + I(u)(-\Delta)^s(\chi_{D\bigcup D_\kappa \tilde{w}(x,t)}) + \varepsilon\eta(t)(-\Delta)^s g(x) \\ &\leq \epsilon\eta'(t)g(x) - c_1 + \epsilon\eta(t)c_0. \end{aligned}$$

Choose $\epsilon > 0$ to be sufficiently small such that for region $(x,t) \in B_\delta(\bar{x}) \times [0,2]$,

$$\frac{\partial \underline{w}(x,t)}{\partial t} + I(u)(-\Delta)^s \underline{w}(x,t) \leq 0. \tag{22}$$

Let $v(x,t) = \tilde{w}(x,t) - \underline{w}(x,t)$, we can derive that $v(x,t) = -v(x^\kappa, t)$. From (17) and (22), it can be concluded that

$$\frac{\partial v(x,t)}{\partial t} + I(u)(-\Delta)^s v(x,t) \geq 0$$

for region $(x,t) \in B_\delta(\bar{x}) \times [0,2]$. Because $\frac{\partial v}{\partial t} = \frac{\partial \tilde{w}}{\partial t} - \frac{\partial \underline{w}}{\partial t}$, $I(u)(-\Delta)^s v = I(\tilde{w} - \underline{w})(-\Delta)^s(\tilde{w} - \underline{w})$. Then,

$$\frac{\partial \tilde{w}}{\partial t} - \frac{\partial \underline{w}}{\partial t} + I(u)(-\Delta)^s \tilde{w} - I(u)(-\Delta)^s \underline{w} = \frac{\partial \tilde{w}}{\partial t} + I(u)(-\Delta)^s \tilde{w} - (\frac{\partial \underline{w}}{\partial t} + I(u)(-\Delta)^s \underline{w}).$$

Similarly, according to the definition of $\underline{w}(x,t)$, there are

$$v(x,t) \geq 0 \quad \text{and} \quad v(x,0) \geq 0, \quad x \in \Sigma_\kappa$$

in region $(\Sigma_\kappa \backslash B_\delta(\bar{x})) \times [0,2]$.

By using asymptotic strong maximum principle for antisymmetric functions,

$$v(x,t) \geq 0, \quad (x,t) \in B_\delta(\bar{x}) \times [0,2].$$

It can be concluded that

$$v(x,t) = e^{mt} w_\infty(x,t) - \epsilon g(x)\eta(t) \geq 0$$

in region $(x,t) \in B_\delta(\bar{x}) \times [0,2]$. Especially for

$$w_\infty(x,1) \geq e^{-m}\epsilon g(x), \quad x \in B_\delta(\bar{x}).$$

Since $g(\bar{x}) = \delta^{2s}$,

$$\Upsilon_\kappa(\bar{x}) = w_\infty(\bar{x},1) \geq e^{-m}\epsilon\delta^{2s} > 0. \tag{23}$$

By considering the arbitrariness of $\bar{x}$ in region $\Sigma_\kappa \backslash D$, we can combine (18) and (23) to obtain

$$\Upsilon_\kappa(x) > 0, \quad x \in \Sigma_\kappa.$$

3. Asymptotic Symmetry of Solutions on the Unit Sphere

Let

$$\Omega_\kappa = \Sigma_\kappa \bigcap B_1(0) = \{x \in B_1(0) | x_1 < \kappa\}.$$

Using

$$\frac{\partial u}{\partial t}(x,t) + \left(a + b\int_{\mathbb{R}^n} \left|(-\Delta)^{\frac{s}{2}} u(x,t)\right|^2 dx\right)(-\Delta)^s u(x,t) = f(t, u(x,t)),$$

and $I(u) = I(u_\kappa)$, we have

$$\frac{\partial u_\kappa(x,t)}{\partial t}(x,t) + \left(a + b\int_{\mathbb{R}^n} \left|(-\Delta)^{\frac{s}{2}} u_\kappa(x,t)\right|^2 dx\right)(-\Delta)^s u_\kappa(x,t) = f(t, u_\kappa(x,t))$$

and

$$-\frac{\partial u(x,t)}{\partial t}(x,t) - \left(a + b\int_{\mathbb{R}^n} \left|(-\Delta)^{\frac{s}{2}} u(x,t)\right|^2 dx\right)(-\Delta)^s u(x,t) = f(t, u(x,t)).$$

From the above, it can be inferred that

$$\frac{\partial w_\kappa(x,t)}{\partial t}(x,t) + I(u(x,t))(-\Delta)^s w_\kappa(x,t) = f(t, u_\kappa(x,t)) - f(t, u(x,t)).$$

Denote $c_\kappa(x,t) = \frac{f(t,u_\kappa(x,t)) - f(t,u(x,t))}{u_\kappa(x,t) - u(x,t)}$, then

$$\frac{\partial w_\kappa(x,t)}{\partial t}(x,t) + I(u(x,t))(-\Delta)^s w_\kappa(x,t) = c_\kappa(x,t) w_\kappa(x,t).$$

Therefore,

$$\begin{cases} \frac{\partial w_\kappa(x,t)}{\partial t} + I(u(x,t))(-\Delta)^s w_\kappa(x,t) = c_\kappa(x,t) w_\kappa(x,t), & (x,t) \in \Omega_\kappa \times (0, +\infty), \\ w_\kappa(x,t) = -w_\kappa(x^\kappa, t), & (x,t) \in \Omega_\kappa \times (0, +\infty). \end{cases} \tag{24}$$

Proof of theorem 1.1: For any $\varpi \in \omega(u)$, if $\varpi(x) \equiv 0$, then this theorem is obvious. Without loss of generality, we can assume that for any $\varpi \in \omega(u)$ in $B_1(0)$, $\varpi \not\equiv 0$.

Step 1: For $\kappa > -1$ and sufficiently close to -1, as well as for all $\varpi \in \omega(u)$, $x \in \Omega_\kappa$. There is

$$\Upsilon_\kappa(x) \geq 0. \tag{25}$$

The Lipschitz continuity of f makes $c_\kappa(x,t)$ bounded. For $x \in \Sigma_\kappa \backslash \Omega_\kappa$ and $t \in (0, +\infty)$,

$$w_\kappa(x,t) \geq 0.$$

Because $u(x,t) = 0$, we have $(x,t) \in B_1^c(0) \times (0, +\infty)$. For $(x,t) \in \Omega_\kappa \times (0, +\infty)$. This combine (24) and narrow region principles to derive (25).

Step 2: We demonstrate that $\kappa_0 = \sup\{\kappa \leq 0 | \Upsilon_u(x) \geq 0,\ \forall \varpi \in w(u),\ x \in \Omega_\mu,\ \mu \leq \kappa\}$ satisfies

$$\kappa_0 = 0.$$

The following proves that T_{κ_0} can continue to move slightly to the right when $\kappa_0 < 0$. But it contradicts with the definition κ_0. Thus, $\kappa_0 \geq 0$. Since the definition κ_0, for any $\varpi \in \omega(u)$ and $x \in \Omega_{\kappa_0}$,

$$\Upsilon_{\kappa_0}(x) \geq 0.$$

Firstly, it is proven that there exists $x_\varpi \in \Sigma_{\kappa_0}$ such that

$$\Upsilon_{\kappa_0}(x_\varpi) > 0$$

for $\forall \varpi \in \omega(u)$. If it false, there is $\bar{\varpi} \in \omega(u)$ such that

$$\bar{\Upsilon}_{\kappa_0}(x) = \bar{\varpi_{\kappa_0}}(x) - \bar{\varpi}(x) \equiv 0$$

in Σ_{κ_0}. Similarly, through the external conditions of u, we have $\bar{\varpi}(x) \equiv 0$ in $B_1^c(0) \cap \Sigma_{\kappa_0}$. Thus, there exists $x_0 \in B_1(0)$ such that $\bar{\varpi}(x_0) = 0$. For this $\bar{\varpi}$, there is t_k such that $u(x, t_k) \to \varpi(\bar{x})$ as $t_k \to +\infty$. Through regularity process,

$$\frac{\partial u_\infty(x,t)}{\partial t} + I(u)(-\Delta)^s u_\infty(x,t) = f(t, u_\infty(x,t))$$

where $u_\infty(x,1) = \bar{\varpi}(x)$. Since $u_\infty(x,t) \geq 0$, we conclude that $\frac{\partial u_\infty}{\partial t}(x_0, 1) \leq 0$ and

$$(-\Delta)^s u_\infty(x_0, 1) = C_{n,s} P.V \int_{B_1(0)} \frac{-u_\infty(y,1)}{|x_0 - y|^{n+2s}} dy < 0.$$

The last inequality holds because $u_\infty(y,1) \not\equiv 0$ in $B_1(0)$. Thus, $\tilde{f}(x, 1, u_\infty(x_0, 1)) = \tilde{f}(x, 1, 0) < 0$, This contradicts the conditions of the Theorem. Thus, $\Upsilon_{\kappa_0}(x_\varpi) > 0$.

By applying the asymptotic strong maximum principle of antisymmetric functions, for any $\varpi \in \omega(u)$, $x \in \Omega_{\kappa_0}$,

$$\Upsilon_{\kappa_0}(x) > 0.$$

Therefore, for any small $\delta > 0$ and any Υ_{κ_0}, there exists a constant $C_\varpi > 0$ such that

$$\Upsilon_{\kappa_0}(x) \geq C_\varpi > 0, \quad x \in \overline{\Omega_{\kappa_0 - \delta}}. \tag{26}$$

The following statement states that for all $\varpi \in \omega(u)$, there exists a constant C_0 such that

$$\Upsilon_{\kappa_0}(x) \geq C_0 > 0, \quad x \in \overline{\Omega_{\kappa_0 - \delta}}. \tag{27}$$

Otherwise, there exists a sequence of function $\Upsilon_{\kappa_0^k}$ and a sequence of points $x^k \subset \overline{\Omega_{\kappa_0-\delta}}$ such that

$$\Upsilon^k_{\kappa_0}(x^k) < \frac{1}{k}. \tag{28}$$

Due to the compactness of $C_0(\overline{B_1(0)})$, there exists $\Upsilon^0_{\kappa_0}$ and $x_0 \in \overline{\Omega_{\kappa_0-\delta}}$ such that

$$\Upsilon^k_{\kappa_0}\left(x^k\right) \to \Upsilon^0_{\kappa_0}\left(x^0\right)$$

as $k \to +\infty$. By (28), we have

$$\Upsilon^0_{\kappa_0}(x^0) = 0.$$

It contradicts with (26). Since $\varpi^0 \in \omega(u)$, (27) is true the continuity of Υ_κ and κ. For any Υ_κ, there is $\epsilon_\varpi > 0$ such that $\forall\kappa \in (\kappa_0, \kappa_0 + \epsilon_\varpi)$ in region $x \in \overline{\Omega_{\kappa_0-\delta}}$,

$$\Upsilon_\kappa \geq \frac{C_0}{2} > 0.$$

Through a compactness process similar to (27), we can obtain that there exist consistent $\epsilon > 0$ such that

$$\Upsilon_\kappa(x) \geq \frac{c_0}{2} > 0 \tag{29}$$

for any Υ_κ in region $x \in \overline{\Omega_{\kappa_0-\delta}}$, $\forall\kappa \in (\kappa_0, \kappa_0 + \epsilon)$. As a result, for sufficiently large t,

$$w_\kappa(x,t) \geq 0$$

in region $x \in \overline{\Omega_{\kappa_0-\delta}}$, $\forall\kappa \in (\kappa_0, \kappa_0 + \epsilon)$. Because $\delta > 0$ is small enough, by using (27), one can choose a sufficiently small $\epsilon > 0$ such that $\Omega_\kappa \backslash \Omega_{\kappa_0-\delta}$.

Because $\kappa \in (\kappa_0, \kappa_0 + \epsilon)$ is a narrow region, we can obtain

$$\Upsilon_\kappa(x) \geq 0 \tag{30}$$

for any $x \in \Omega_\kappa \backslash \Omega_{\kappa_0-\delta}$ by using asymptotic narrow region principle. Combining (29) and (30), one can conclude that

$$\Upsilon_\kappa(x) \geq 0$$

for any $x \in \Omega_\kappa$, any $\kappa \in (\kappa_0, \kappa_0 + \epsilon)$ and any $\varpi \in \omega(u)$. It contradicts with the definition κ_0. Thus, we must have $\kappa_0 = 0$. For any $\varpi \in \omega(u)$,

$$\Upsilon_0(x) \geq 0, \quad x \in \Omega_0$$

or for any $\varpi \in \omega(u)$

$$\varpi(-x_1, \ldots, x_N) \leq \varpi(x_1, \ldots, x_N) \tag{31}$$

in region $0 < x_1 < 1$. Because the direction of x_1 can be chosen arbitrarily, (31) can deduce that $\varpi(x)$ is radially symmetrical about the origin.

4. Asymptotic Symmetry of Solutions in the Entire Space

In this section, we using moving plane method to prove asymptotic symmetry of (1.5). Let $T_\kappa, \Sigma_\kappa, x^\kappa, u_\kappa, w_\kappa, \varpi$ and Υ_κ is the symbol defined in the first section. By using the mean value theorem,

$$c_\kappa(x,t) = f_u(t, \iota_\kappa(x,t))$$

where $\iota_\kappa(x,t)$ is a value between $u_\kappa(x,t)$ and $u(x,t)$. Then,

$$\begin{cases} \frac{\partial w_\kappa(x,t)}{\partial t} + I(u(x,t))(-\Delta)^s w_\kappa(x,t) = c_\kappa(x,t) w_\kappa(x,t), & (x,t) \in \Sigma_\kappa \times (0,+\infty), \\ w_\kappa(x,t) = -w_\kappa(x^\kappa, t), & (x,t) \in \Sigma_\kappa \times (0,+\infty). \end{cases} \tag{32}$$

Proof of theorem 1.2: Step 1: we want to prove that

$$\Upsilon_\kappa(x) > 0 \tag{33}$$

for sufficiently small κ and $\forall\varpi \in \omega(u)$ in Σ_κ. The following proof shows that

$$\Upsilon_\kappa(x) \geq 0 \tag{34}$$

in Σ_κ for $\forall\varpi \in \omega(u)$.

Because (12), we have $\lim\limits_{|x|\to+\infty} u(x,t) = 0$ for sufficiently large t. Thus, by using the definition in (12), there exists $R > 0$ such that for sufficiently large t,

$$0 < u(x,t) < \beta.$$

Since the condition (G) : $f_u(t,0) < -\delta$ and f_u is continuous near $u = 0$, using the definition of continuous function, there exists $\beta^{'} > 0$ such that for any $0 \leq \eta < \beta$,

$$f_u(x,t,\eta) < -\delta. \tag{35}$$

Choose $\beta = \beta^{'}$, then for sufficiently large t,

$$0 < u(x,t) < \beta^{'}. \tag{36}$$

For points of $w_\kappa(x,t) < 0$ and the definition of w_κ,

$$u_\kappa(x,t) \leq \xi(x,t) \leq u(x,t). \tag{37}$$

Thus, for sufficiently large t and the points which make $w_\kappa(x,t) < 0$ in $|x| > R$, combining (35), (36) and (37), we can conclude that

$$c_\kappa(x,t) = f_u(x,t,\xi_\kappa(x,t)) < -\delta \quad in|x| > R. \tag{38}$$

Due to (12), for sufficiently large t,

$$\lim_{|x|\to+\infty} w_\kappa(x,t)=0 \tag{39}$$

for $\kappa\in\mathbb{R}$. Therefore, let $\Omega=\Sigma_\kappa$ and we applying asymptotic maximum principle near infinity for$\kappa\le -R$, one can derive (34). Without loss of generality, assume there exists $\varpi\in\omega(u)$ which is positive in some parts of $\mathbb{R}^n$. For all $\varpi\in\omega(u)$,

$$\varpi(x)>0$$

by using strong maximum principle.

To obtain (33), the strong maximum principle of antisymmetric functions is required. We just need to prove that Υ_κ is negative in some places. By the strong maximum principle of antisymmetric functions, there exists $C_{r,\varpi}>0$ such that for any $r>0(r<R)$, $\varpi\in\omega(u)$ and$x\in B_r(0)$,

$$\varpi(x)\ge C_{r,\varpi}>0.$$

Since(12), for $C_{r,\varpi}>0$, $x\in B_r(D^{\kappa_\varpi})$, there is $\kappa_\varpi\le -R$ such that

$$\varpi(x)\le\frac{C_{r,\varpi}}{2}.$$

Since the compactness of $\omega(u)$ in $C_0(\mathbb{R}^n)$, one can choose $C_{r,\varpi}$ and κ_ϖ are consistent regarding ϖ. We denote these as C_r and κ. Then for $\forall\varpi\in\omega(u)$,

$$\Upsilon_\kappa(x)=\varpi(x)-\varpi_\kappa(x)\ge\frac{C_r}{2}>0,\quad x\in B_r(0^\kappa). \tag{40}$$

Because (33), (40) and $c_\kappa(x,t)$ is bounded, one can derive that

$$\Upsilon_\kappa(x)=\varpi(x)-\varpi_\kappa(x)\ge\frac{C_r}{2}>0$$

by using strong maximum principle. The first step has been completed.

We will give the proof of asymptotic strong maximum principle for antisymmetric function. For our aim, a lemma and the proof of this lemma need to be provided in the following:

Lemma 4.1. Assume u is the positive solution of (11). It also satisfies the condition (G) and $\lim\limits_{|x|\to+\infty}u(x,t)=0$ for sufficiently large t. Assume for some $\varpi\in\omega(u)$ and $\varpi\not\equiv 0$ or we can assume

$$\overline{\lim_{t\to+\infty}}\parallel u(x,t)\parallel_{L^\infty(\mathbb{R}^n)}>0, \tag{41}$$

then

$$\underline{\lim_{t\to+\infty}}\parallel u(x,t)\parallel_{L^\infty(\mathbb{R}^n)}>0. \tag{42}$$

From the proof, it can be seen that if $\mathbb{R}^n$ in (42) and (41) are replaced with any bounded field Ω or unbounded field, We can also draw this conclusion.

By contradiction, if (42) is false, then there exists $t_k\to+\infty$ such that

$$\|u(x,t_k)\|_{L^\infty(R^n)}\to 0.$$

Since (G), there is sufficently small $\epsilon_0>0$ such that $f_u(t,\eta)<-\delta$ for any $0<\eta<\epsilon_0$. In $\mathbb{R}^n$, there is $k_0\in N$ for ϵ_0 such that for any $k\ge k_0$,

$$u(x,t_k)<\epsilon_0.$$

Now, we fix a $t_k(k\ge k_0)$ and let $u_k(x,t)=u(x,t+t_k)$. Then,

$$\frac{\partial u_k}{\partial t}+I(u_k)(-\Delta)^s u_k=f(x,t+t_k,u_k),\ (x,t)\in\mathbb{R}^n\times(0,+\infty).$$

If we can construct a supersolution $\xi(t)$ such that$\xi(t)\ge u(x,t)$ and $\lim\limits_{|x|\to+\infty}\xi(t)=0$. Then it contradicts with (41).

We construct $\xi(t)$. Let $\xi(t)$ be the solution of the following ordinary differential equation:

$$\begin{cases}\frac{d\xi(t)}{dt}=f(t+t_k,\xi(t)) & t\in(0,+\infty)\\ \xi(0)=\epsilon_0.\end{cases}$$

Using mean value theorem,

$$f(t+t_k,\xi(t))=f_u(t+t_k,r(t))\xi(t),\ r(t)\in(0,\xi(t))$$

and $f(t,0)=0$. Since the solutions of the first order linear homogeneous differential equation $y'+f(x)y=0$, $\xi(t)=\epsilon_0 e^{\int_0^t f_u(\tau+t_k,r(\tau))d\tau}$. Using (G) and $t=0,\ r(0)<\xi(0)=\epsilon_0$, one can conclude that

$$\frac{d\xi}{dt}(0)=f_u(t_k,r(0))\xi(0)<-\sigma\epsilon(0)<-\sigma\epsilon_0.$$

Therefore, $\xi(t)$ monotonically decreases near $t=0$, and for sufficiently small $t>0$, $0<r(t)<\xi(t)<\epsilon_0 e^{-\sigma t}<\epsilon_0$. Since F, Repeat the above steps for any $t>0$, we have

$$0<\xi(t)<\epsilon_0 e^{-\sigma t},\quad\forall t>0.$$

Thus,

$$当\ t\to+\infty,\ \xi(t)\to 0. \tag{43}$$

Next, we compare $\xi(t)$ and $u_k(x,t)$ in $\mathbb{R}^n\times(0,+\infty)$. Let $v(x,t)=\xi(t)-u_k(x,t)$, then

$$\begin{cases} \frac{\partial v}{\partial t} + I(u)(-\Delta)^s v = c_{\kappa,k}(x,t)v, & (x,t) \in \mathbb{R}^n \times (0,+\infty), \\ v(x,0) \geq 0, & x \in \mathbb{R}^n \end{cases} \tag{44}$$

where $c_{\kappa,k}(x,t) = \frac{f(t+t_k,x,\xi(t)) - f(t+t_k,x,u_k(x,t))}{\xi(t)-u_k(x,t)}$. Since $f(t,x,u)$ is Lipschitz continuous in the function u with respect to t and x, one can derive that $c_\kappa, k(x,t)$ is bounded. Using maximum principle,

$$v(x,t) \geq 0. \quad (x,t) \in \mathbb{R}^n \times [0,T], \quad \forall T > 0.$$

Let $T \to +\infty$ and (43), we have $u(x,T) \to 0$. Thus, it contradicts with (41).

The proof of the strong maximum principle is similar to the proof of the strong maximum principle for antisymmetric functions. Thus, it will not be proven here.

Step 2: Move the plane to its limit position. Let

$$\kappa_0^- = \sup\{\kappa | \Upsilon_\mu(x) > 0, \ \forall \varpi \in \omega(u), \ x \in \Sigma_\mu, \ \mu \leq \kappa\}.$$

Firstly, we prove

(i) At least some $\varpi(x) \in \omega(u)$ are symmetric with the limit plane T_{κ^-} which means for some $\varpi(x) \in \omega(u)$,

$$\Upsilon_{\kappa_0^-}(x) \equiv 0, \ x \in \Sigma_{\kappa_0^-}. \tag{45}$$

(ii) For any $\varpi \in \omega(u)$,

$$\partial_{x_1}\varpi(x) > 0, \ x \in \Sigma_{\kappa_0^-}.$$

By contradiction, if (45) is false, then one can prove that there exists ϵ_0 such that for all $\varpi \in \omega(u)$, $x \in \Sigma_\kappa$ and $\forall \kappa \in (\kappa_0^-, \kappa_0^- + \epsilon_0)$,

$$\Upsilon_\kappa(x) > 0. \tag{46}$$

For any $\varpi \in \omega(u)$, there is $x_\varpi \in \Sigma_{\kappa_0^-}$ such that $\Upsilon_{\kappa_0^-}(x_\varpi) > 0$. Since $\Upsilon_{\kappa_0}^-(x) \geq 0$ in $\Sigma_{\kappa_0^-}$, we have

$$\Upsilon_{\kappa_0^-} > 0 \tag{47}$$

for $\forall x \in \Sigma_{\kappa_0^-}$ and $\forall \varpi \in \omega(u)$ by using strong maximum principle for antisymmetric function. Let R be R in (38). Using (47) and the process of the second part for the moving plane method in theorem (1.1), one can conclude that there exists $C_0 > 0$ and $\epsilon_0 > 0$ such that for $x \in \overline{\Sigma_{\kappa_0^- - \delta} \cap B_R(0)}$, $\kappa \in ({\kappa_0}^-, {\kappa_0}^- + \epsilon_0)$ and $\forall \varpi \in w(u)$,

$$\Upsilon_\kappa(x) \geq \frac{C_0}{2} > 0 \tag{48}$$

for sufficiently large t. Thus for $x \in \overline{\Sigma_{\kappa_0^- - \delta} \cap B_R(0)}$, $\kappa \in ({\kappa_0}^-, {\kappa_0}^- + \epsilon_0)$ and $\forall \varpi \in \omega(u)$,

$$w_\kappa(x) \geq 0.$$

By using (38), (39), $\kappa \leq -R$, the boundness of $c_\kappa(x,t)$, narrow theorem and comparison principle at infinity. one have

$$\Upsilon_\kappa(x) \geq 0$$

for $x \in \overline{\Sigma_{\kappa_0^- - \delta} \cap B_R(0)}$, $\kappa \in ({\kappa_0}^-, {\kappa_0}^- + \epsilon_0)$ and $\forall \varpi \in w(u)$. Combining (48) and (49),

$$\Upsilon_\kappa(x) \geq 0 \tag{49}$$

for all $\varpi \in \omega(u)$, $x \in \Sigma_\kappa$, $\kappa \in ({\kappa_0}^-, {\kappa_0}^- + \epsilon_0)$ and $\forall \varpi \in w(u)$. Then, Using strong maximum principle for antisymmetric function, we have (46).

Next, we want to prove (ii). For any $\varpi \in \omega(u)$,

$$\partial_{x_1}\varpi(x) > 0, \ x \in \Sigma_{\kappa_0^-}.$$

We first give Hopf Lemma and proof for antisymmetric function.

Lemma 4.2. Let $w_\kappa(x,t) \in \left(C^{1,1}_{loc}(\Sigma_\kappa) \cap L_{2s} \cap H^s\right) \times C^1(0,+\infty)$ is bounded and satisfies

$$\begin{cases} \frac{\partial w_\kappa}{\partial t} + I(u)(-\Delta)^s w_\kappa = c_\kappa(x,t)w_\kappa, & (x,t) \in \tilde{\Sigma}_\kappa \times (0,+\infty), \\ w_\kappa(x^\kappa,t) = -w_\kappa(x,t), & (x,t) \in \tilde{\Sigma}_\kappa \times (0,+\infty), \\ \lim\limits_{t\to+\infty} w_\kappa(x,t) \geq 0, & x \in \tilde{\Sigma}_\kappa \end{cases} \tag{50}$$

where for any sufficiently large t, $\underline{lim}_{X\to\partial\tilde{\Sigma}_\kappa} c_\kappa(x,t) = o\left(\frac{1}{[dist(x,\partial\tilde{\Sigma}_\kappa)]^2}\right)$. If $\Upsilon_\kappa > 0$ in some parts in Σ_κ, then

$$\frac{\partial \Upsilon_\kappa}{\partial v}(x) < 0$$

for any $x \in \partial\tilde{\Sigma}_\kappa$ where v is unit outer normal vector.

Proof: Analogous to the proof of the strong maximum principle for antisymmetric function, without loss of generality $\kappa = 0$, $\tilde{\Sigma}_\kappa = \{x \in R^n | x_1 > \kappa\}$, we only need to prove

$$\frac{\partial \Upsilon_\kappa}{\partial x_1}(0) > 0. \tag{51}$$

Let $g(x) = x_1\xi(x)$,

$$\xi(x) = \xi(|x|) = \begin{cases} 1, & |x| \le \epsilon, \\ 0, & |x| \ge 2\epsilon \end{cases}$$

and $0 \le \iota(x) \le 1$, $\iota(x) \in C_0^\infty(B_{2\epsilon}(0))$. Obviously, $g(x)$ is an antisymmetric function about plane T_0 which means

$$g(-x_1, x_2, \dots x_n) = -g(x_1, x_2, \dots, x_n).$$

Let $\underline{w}(x,t) = \chi_{D\bigcup D_\kappa}(x)\tilde{w}(x,t) + \delta\eta(t)g(x)$,

$$\chi_{D\bigcup D_\kappa}(x) = \begin{cases} 1, & x \in D\bigcup D_\kappa, \\ 0, & x \notin D\bigcup D_\kappa. \end{cases}$$

$\eta(t) \in C_0^\infty([1-\varepsilon_0, 1+\epsilon_0])$ satisfies

$$\eta(t) = \begin{cases} 1, & t \in [1-\frac{\varepsilon_0}{2}, 1+\varepsilon_0 2], \\ 0, & t \notin [1-\varepsilon_0, 1+\varepsilon_0]. \end{cases}$$

Since $g(x)$ is $C_0^\infty(B_{2\epsilon}(0))$,

$$|(-\Delta)^s g(x)| \le C_0. \tag{52}$$

For any $t \in [0,2]$ and any $x \in B_{2\epsilon(0)}\bigcap\tilde{\Sigma}_\kappa$, using the method in Chen [15],

$$(-\Delta)^s(\chi_{D\bigcup D_\kappa}\tilde{w}(x,t)) \le -C_1. \tag{53}$$

By (52) and (53),

$$\begin{aligned}
&\frac{\partial \underline{w}(x,t)}{\partial t} + I(u(x,t))(-\Delta)^s\underline{w}(x,t) \\
&= \frac{\partial(\chi_{D\bigcup D_\kappa(x)}\tilde{w}(x,t) + \delta\eta(t)g(x))}{\partial t} + I(u(x,t))(-\Delta)^s(\chi_{D\bigcup D_\kappa(x)}\tilde{w}(x,t) + \delta\eta(t)g(x)) \\
&= \delta\eta^{'}(t)g(x) + I(u(x,t))(-\Delta)^s(\chi_{D\bigcup D_\kappa}(x)\tilde{w}(x,t) + \delta\eta(t)g(x)) \\
&= \delta\eta^{'}(t)g(x) + I(u(x,t))(-\Delta)^s\chi_{D\bigcup D_\kappa}(x)\tilde{w}(x,t) + I(u(x,t))(-\Delta)^s(\delta\eta(t)g(x)) \\
&\le \delta\eta^{'}(t)g(x) + I(u(x,t))(-C_1) + I(u(x,t))C_0\eta(t)\delta.
\end{aligned}$$

Because $I(u) > 0$, choose sufficiently small δ, then for any $(x,t) \in (B_{2\epsilon}(0)\bigcap\tilde{\Sigma}_\kappa) \times [0,2]$,

$$\frac{\partial \underline{w}(x,t)}{\partial t} + I(u(x,t))(-\Delta)^s\underline{w}(x,t) \le 0. \tag{54}$$

Let $v(x,t) = \tilde{w}(x,t) - \underline{w}(x,t)$, then $v(x,t) = -v(x^\kappa,t)$. Analogous to the proof of (3.34) in [18],

$$\frac{\partial\tilde{w}}{\partial t} + I(u)(-\Delta)^s\tilde{w} = (m + c_\infty)\tilde{w} \ge 0 \tag{55}$$

for $(x,t) \in \tilde{\Sigma}_\kappa \times [0,2]$. Thus, by using(54) and (55),

$$\frac{\partial v}{\partial t} + I(u)(-\Delta)^s\tilde{v} \ge 0$$

for $(x,t) \in \tilde{B}_{2\epsilon}(0) \cap \tilde{\Sigma}_\kappa \times [0,2]$. Since the definition of $\underline{w}(x,t)$, $v(x,t) \ge 0$ as $(x,t) \in \left(\tilde{\Sigma}_\kappa \backslash B_{2\epsilon}(0) \cap \tilde{\Sigma}_\kappa\right) \times [0,2]$ and

$$v(x,0) \ge 0$$

for $x \in \tilde{\Sigma}_\kappa$. Thus, applying the maximum principle for antisymmetric function which means

$$v(x,0) \ge 0$$

for $(x,t) \in (B_{2\epsilon}(0) \cap \tilde{\Sigma}_\kappa) \times [0,2]$. We have

$$e^{mt}w_\infty(x,t) - \delta g(x)\eta(t) \ge 0,$$

$$w_\infty(x,1) \ge e^{-m}\delta g(x)$$

for $(x,t) \in \left(B_{2\epsilon}(0)\bigcap\tilde{\Sigma}_\kappa\right) \times [0,2]$ and

$$w_\infty(x,1) \ge e^{-m}\delta x_1$$

for $x \in B_\epsilon(0)\bigcap\tilde{\Sigma}_\kappa$. Since $w_\infty(x,1) \equiv 0$, $x \in T_0$ and $w_\infty(0,1) = 0$.

$$\frac{w_\infty(x,1) - 0}{x_1 - 0} \ge e^{-m}\delta > 0$$

for $x \in B_\epsilon(0) \cap \tilde{\Sigma}_\kappa$. We can derive that

$$\frac{\partial\Upsilon_\kappa}{\partial x_1}(0) > 0.$$

Therefore,

$$\frac{\partial\Upsilon_\kappa}{\partial x_1}(0) < 0$$

for $x \in \tilde{\Sigma}_\kappa$.

Using Hopf lemma,

$$\partial_{x_1}\Upsilon_\kappa(x)|_{x\in T_\kappa} < 0$$

for $\forall\varpi \in w(u)$. By

$$\begin{aligned}
\partial_{x_1}\Upsilon_\kappa(x)|_{x\in T_\kappa} &= \partial_{x_1}\varpi_\kappa(x)|_{x\in T_\kappa} - \partial_{x_1}\varpi(x)|_{x\in T_\kappa} \\
&= -2\partial_{x_1}\varpi(x)|_{x\in T_\kappa},
\end{aligned}$$

one can derive that

$$\partial_{x_1}\Upsilon(x)|_{x\in T_\kappa} > 0, \quad \forall\varpi\in w(u).$$

Thus, we have (51).

Step 3: All w limit function are radially symmetric which means for all $\varpi(x)$,

$$\Upsilon_{\kappa_0^-}(x)\equiv 0.$$

If we can prove

$$\kappa_0^- = \kappa_0^+, \tag{56}$$

then for all $\varpi(x)$, $\Upsilon_{\kappa_0^-}(x)\equiv 0$.

We prove (56) by contradiction in the following. Assume (56) is false, then $\kappa_0^- < \kappa_0^+$. For any $\kappa\in(\kappa_0^-,\kappa_0^+)$, we want to prove

$$\hat{\Upsilon}_\kappa(x) > 0, \quad x\in\tilde{\Sigma}_\kappa \tag{57}$$

for all $\forall\kappa\in(\kappa_0^-,\kappa_0^+)$ and

$$\bar{\Upsilon}_\kappa(x) < 0, \quad x\in\tilde{\Sigma}_\kappa. \tag{58}$$

Since $\bar{\Upsilon}_{\kappa_0^+}\equiv 0$, then $\bar{\Upsilon}_\kappa(x) = \bar{\varpi}(x^\kappa) - \bar{\varpi}(x) = \bar{\varpi}(x_{\kappa_0^+}) - \bar{\varpi}(x) < 0$. The last inequality is known by Hopf Lemma that on the set of $x_1 > \kappa_0^+$, $\bar{\varpi}$ is decreasing with respect to x_1. Thus, one can conclude (58). Similarly, it can be concluded that(57). By (58) and (57), for any compact subset $D\subset\subset\tilde{\Sigma}_\kappa$, there exists $q>0$ such that

$$\tilde{\Upsilon}_\kappa(x) > q, \quad x\in\bar{D}, \tag{59}$$

and

$$\bar{\Upsilon}_\kappa(x) < -q \quad x\in\bar{D}. \tag{60}$$

Using $\hat{\varpi}$ and $\bar{\varpi}\in\omega(u)$, there exists sequences $\{t_n\}$ and $\{\bar{t}_n\}$ of $t_n<\bar{t}_n$ such that

$$u(\cdot,t_n)\to\hat{\varpi}(\cdot),\ u(\cdot,\bar{t}_n)\to\bar{\varpi}(\cdot).$$

By (59) and (60) , For sufficiently large n, $x\in\bar{D}$, we have $w_\kappa(x,t_n) > q$ and $w_\kappa(x,\bar{t}_n) < -q$ in $x\in\bar{D}$. Thus, there is $T_n\in(t_n,\bar{t}_n)$ such that

$$w_\kappa(x,t) > 0, \quad x\in\bar{D},\ t\in[t_n,T_n), \tag{61}$$

and

$$w_\kappa(\cdot,T_n) \text{ has some points which are equal to } 0 \ in\ \partial D. \tag{62}$$

If it can be concluded that this contradicts with (62), then we finish the third step of proof. Let

$$\tilde{w}(x,t) = e^{-m(t-t_n)}w_\kappa(x,t),$$

then

$$\tilde{L}\tilde{w}\equiv\frac{\partial\tilde{w}}{\partial t} + I(u)(-\Delta)^s\tilde{w} - \tilde{C}(x,t)\tilde{w}$$

where $\tilde{C}(x,t) = c_\kappa(x,t)-m$. We choose suitable $m>0$ such that

$$\tilde{C}(x,t) < 0.$$

By

$$\frac{\partial u}{\partial t} + I(u)(-\Delta)^s u = f(x,t,u),$$

$$\frac{w_\kappa}{\partial t} + I(u)(-\Delta)^s w_\kappa = c_\kappa(x,t)w_\kappa.$$

Then,

$$\begin{aligned}
\tilde{L}\tilde{w} &\equiv e^{-m(t-t_n)}\frac{\partial w_\kappa}{\partial t} + e^{-m(t-t_n)}(-m)w_\kappa + I(u)(-\Delta)^s(e^{-m(t-t_n)}w_\kappa) - (c_\kappa - m)e^{-m(t-t_n)}w_\kappa\\
&= e^{-m(t-t_n)}\frac{\partial w_\kappa}{\partial t} + I(u)(-\Delta)^s(e^{-m(t-t_n)}w_\kappa) - c_\kappa e^{-m(t-t_n)}w_\kappa\\
&= e^{-m(t-t_n)}\frac{\partial w_\kappa}{\partial t} + I(u)e^{-m(t-t_n)}{-\Delta}^s w_\kappa - c_\kappa e^{-m(t-t_n)}w_\kappa\\
&= e^{-m(t-t_n)}\frac{\partial w_\kappa}{\partial t} + e^{-m(t-t_n)}(I(u)(-\Delta)^s w_\kappa - c_\kappa w_\kappa)\\
&= 0.
\end{aligned}$$

Thus $\tilde{I}\tilde{w}=0$. Let

$$\chi_{D_0}(x) = \begin{cases}1, & x\in D_0,\\ 0, & x\notin D_0,\end{cases}$$

$\underline{w} = \chi_{D_0}(x)\tilde{w}(x,t) + 2(\phi_\delta-1)\epsilon_n e^{-(m+\theta)(t-t_n)}$. Then,

$$
\begin{aligned}
\tilde{L}\underline{w} &= \frac{\underline{w}}{t} + I(u)(-\Delta)^s\underline{w} - \tilde{c}(x,t)\underline{w} \\
&= \chi_{D_0}(x)\frac{\partial\tilde{w}}{\partial t} + (2\phi_\delta - 1)\epsilon_n e^{-(m+\theta)(t-t_n)} - (m+\theta) \\
&+ I(u)\left((-\Delta)^s\left(\chi_\infty(x)\tilde{w} + (2\phi_\delta - 1)\epsilon_n e^{-(m+\theta)(t-t_n)}\right)\right) \\
&- \tilde{c}(x,t)\left(\chi_D(x)\tilde{w} + (2\phi_\delta - 1)\epsilon_n e^{-(m+\theta)(t-t_n)}\right) \\
&= (2\phi_\delta - 1)\epsilon_n e^{-(m+\theta)(t-t_n)} - (m+\theta) + I(u)\Big((-\Delta)^s(\chi_\infty(x)\tilde{w} \\
&+ I(u)(-\Delta)^s\Big((2\phi_\delta - 1)\epsilon_n e^{-(m+\theta)(t-t_n)}\Big)\Big) - c(2\phi_\delta - 1)\epsilon_n e^{-(m+\theta)(t-t_n)} \\
&\le (2\phi_\delta - 1)\epsilon_n e^{-(m+\theta)(t-t_n)} - (m+\theta) + I(u)(-\Delta)^s(\chi_\infty(x)\tilde{w}) \\
&+ I(u)2a_\delta\epsilon_n e^{-(m+\theta)(t-t_n)} - C_1\epsilon_n e^{-(m+\theta)(t-t_n)} \\
&\le I(u)(-\Delta)^s(\chi_{\infty(x)}\tilde{w}) + c_1\epsilon_n e^{-(m+\theta)(t-t_n)}.
\end{aligned}
$$

By Chen [15], one can conclude

$$(-\Delta)^s(\chi_{\infty(x)}\tilde{w}) \le -C_2 e^{-(m+\theta)(t-t_n)}$$

and

$$I(u)(-\Delta)^s(\chi_{\infty(x)}\tilde{w}) \le -C_2 I(u) e^{-(m+\theta)(t-t_n)}.$$

Then,

$$\tilde{L}\underline{w}(x,t) \le -C_2 I(u) e^{-(m+\theta)(t-t_n)} + c_1\epsilon_n e^{-(m+\theta)(t-t_n)} = (c_1\epsilon_n - c_2 I(u))e^{-(m+\theta)(t-t_n)}.$$

For sufficiently large n,

$$\tilde{L}\underline{w}(x,t) \le 0.$$

Thus for $(x,t) \in B_\delta(\bar{x}) \times [t_n, T_n]$, $\tilde{L}(\underline{w}(x,t) - \tilde{w}(x,t)) \le 0$. By maximum principle, for $(x,t) \in B_\delta(\bar{x}) \times [t_n, T_n]$,

$$\underline{w}(x,t) \le \tilde{w}(x,t).$$

In the case of boundary,

$$\tilde{w}(\bar{x}, T_n) \ge \underline{w}(\bar{x}, T_n) = \epsilon_n e^{-(m+\theta)(T_n - t_n)}.$$

Thus,

$$w_\kappa(\bar{x}, T_n) \ge \epsilon_n e^{-\theta}(T_n - t_n) > 0$$

contradict with (62).

In order to prove(ii)through the principle of maximum principle of antisymmetric function which means there exists $D_0 \subset\subset D$ and constant $C_0 > 0$ such that

$$w_\kappa(x,t) \ge e^{-\theta(t-t_n)}C_0, \quad t_n \le t \le T_n, \quad x \in D_0.$$

It is required that

$$w_\kappa(x,t) \ge \Upsilon(x,t) \quad x \in D, \quad t_n \le t \le T_n$$

which means

$$(w_\kappa - \Upsilon) \ge 0.$$

The key lies in verifying the conditions $\frac{\partial v}{\partial t} + I(u) - \Delta^s v(x,t) \ge c_\kappa(x,t)v(x,t)$ for the maximum principle for antisymmetric functions. We can refer to Chen [15] for the verification of other conditions. Therefore, it is necessary to verify $\frac{\partial(w_\kappa-\Upsilon)}{\partial t}+I(u)-\Delta^s(w_\kappa-\Upsilon)-c_\kappa(x,t)(w_\kappa-\Upsilon)\geq 0$ which means

$$L_\kappa w_\kappa(x,t)-L_\kappa\Upsilon(x,t)\geq 0.$$

Since $L_\kappa w_\kappa(x,t)=\frac{\partial w_\kappa}{\partial t}+I(u)(-\Delta)^s w_\kappa(x,t)-C_\kappa(x,t)w_\kappa(x,t)=0$, it is necessary to verify $L_\kappa\Upsilon(x,t)\leq 0$ where $\Upsilon(x,t)$ denote $q\frac{\iota(x,t)}{\|\iota(x,t_n)\|_{L^\infty(D)}}$. Thus, we only need differential inequality

$$L_\kappa\iota(x,t)\leq 0,\quad \forall x\in D,\quad t_n\leq t\leq T_n.$$

It will be proved in the following:

$$\begin{aligned}
L_\kappa\iota(x,t) &= \frac{\partial\iota}{\partial t}(x,t)+I(u)-\Delta^s\iota(x,t)-c_\kappa(x,t)\iota(x,t)\\
&= -\theta e^{-\theta(t-t_n)}\left(\tilde{w}_\mu(x,t)-\tau h\right)+e^{-\theta(t-t_n)}\frac{\partial\tilde{w}_\mu}{\partial t}\\
&+I(u)(-\Delta)^s\left(e^{-\theta(t-t_n)}(\tilde{w}_\mu-\tau h)\right)-c_\kappa(x,t)e^{-\theta(t-t_n)}(\tilde{w}_\mu(x,t)-\tau h)\\
&= -\theta e^{-\theta(t-t_n)}\left(\tilde{w}_\mu(x,t)-\tau h+e^{-\theta(t-t_n)}\frac{\partial\tilde{w}_\mu}{\partial t}\right)\\
&+I(u)e^{-\theta(t-t_n)}(-\Delta)^s\left((\tilde{w}_\mu-\tau(-\Delta)^s h)\right)-c_\kappa(x,t)e^{-\theta(t-t_n)}(\tilde{w}_\mu(x,t)-\tau h)\\
&= e^{-\theta}(t-t_n)-\theta\tilde{w}_\mu+\theta\tau+\frac{\partial w_\mu}{\partial t}+I(u)-\Delta^s\tilde{w}_\mu-I(u)(-\Delta)^s w_\mu+I(u)(-\Delta)^s w_\mu\\
&-\tau(-\Delta)^s h(x)-c_\kappa w_\mu+\tau c_\kappa\\
&= e^{-\theta}(t-t_n)-\theta\tilde{w}_\mu+\theta\tau+I(u)[-\Delta^s\tilde{w}_\mu-(-\Delta)^s w_\mu]-\tau(-\Delta)^s h(x)+(c_\mu-c_\kappa)w_\mu\\
&+\tau c_\kappa.
\end{aligned}$$

By Chen [15],

$$(-\Delta)^s\tilde{w}_\mu-(-\Delta)^s w_\mu<0,\ x_1>\mu,\ t>0.$$

Thus $I(u)\left((-\Delta)^s\tilde{w}_\mu-(-\Delta)^s w_\mu\right)<0$ and

$$L_\kappa\iota(x,t)\quad\leq e^{-\theta(t-t_n)}[-\theta+c_\mu-c_\kappa]w_\mu+\tau(\theta+c_\kappa)-\tau(-\Delta)^s h.$$

Finally, by Chen [15], one can derive that

$$L_\kappa\iota(x,t)\leq 0,\quad \mu<x_1<d,\quad t_n\leq t\leq T_n$$

and

$$L_\kappa\iota\leq 0,\quad \forall x\in D,\quad t_n\leq t\leq T_n.$$

5. Conclusion

In this article, we establish different narrow region principles and various maximum principles to apply moving planes to study the symmetry of positive solutions on the unit sphere and in the entire space. The ideas and methods presented in this article have significant value in studying the properties of solutions to partial differential equations.

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Author Contributions

Yubo Ni is the sole author. The author read and approved the final manuscript.

Conflicts of Interest

The authors declare no conflicts of interest.

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