

# Optimal Control of a Nonlinear System with White Noise

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## To cite this article:

Georges Kologo, Cédric Kpèbbèwèrè Some, Somdouda Sawadogo. (2025). Optimal Control of a Nonlinear System with White Noise. *American Journal of Applied Mathematics*, 13(2), 153-164. <https://doi.org/10.11648/j.ajam.20251302.15>

**Received:** 19 February 2025; **Accepted:** 10 March 2025; **Published:** 26 March 2025

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**Abstract:** In this paper, we propose a control problem for nonlinear stochastic differential equations with noise. The system proposed for the control problem is a nonlinear system perturbed by standard Brownian motion. We write this problem in a coupled form where the control  $u(t)$  has a value in a convex space. Here, we propose sufficient minimum conditions on the Hamiltonian function to characterize the mean value of the cost function associated with the optimal control problem. The convexity of the Hamiltonian function is a sufficient condition for the existence of the optimal value of the control function. Under certain regularity assumptions, on the control system functional and on the non-differentiability criterion of Brownian motion, the existence and uniqueness results are established by the Cauchy-Lipschitz criteria. We also analyze the mathematical expectation stability of the system to check whether it will converge to an equilibrium point or not. For the study of this stability, the emphasis has been placed on root-mean-square stability. To highlight the results of our work, we apply this control problem to a SIRS-type epidemiological system for the coronavirus epidemic. To study the stability of this epidemiological system, we construct a Lyapunov function associated with the system and then use the results of Lyapunov's theorem to show the convergence of the system to a stable state.

**Keywords:** Stochastic Optimal Control, Optimal Value, Epidemiological System, Stochastic Stability

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## 1. Introduction

Over the last years, several works have been developed to solve stochastic control problems. This work has been applied in various fields such as finance, health, demography, economics, engineering, etc. (see [6, 13, 16, 18]).

Like deterministic control [8], two methods are used to solve stochastic control problems: the stochastic minimum principle and the principle of dynamic programming. In [11] and [6] the authors use the principle of dynamic programming to find the optimal control. For [10], the work revolves around a class of stochastic control problems linked to backward equations with quadratic growth.

In [7], Mezerdi and Farid carried out their work on the Pontryagin maximum principle of a control system. In their work, they consider a control system whose diffusion coefficient is independent of the control. The Principle of

Maximum Stochastic Sufficiency was discussed in [13]. In [3] the author uses the stochastic minimum principle technique to solve problems related to the epidemic.

The stochastic control problem with time change and its theory of the maximum principle was studied by ERKAN (see [4]). In this work, he uses the time-modified Gronwall method to show the existence of a backward stochastic differential equation.

As for applications, in 2023 Flipo De FEO developed his work on the stochastic delay control problem, which he applied to economic and financial models. [6]. In bioscience, Andrés LESNIEWSKI and Nicolas LESNIEWSKI [2] will be working on infectious diseases using stochastic differential equations in 2020. In the same field, Vincus V. L. Albani and Jorge P. Zubelli [16] have worked on stochastic transmission in an epidemiological model. (see [16]).

The main objective of this paper is to find a coupled

optimal solution for a stochastic control problem. Our contribution lies in the characterization of the optimal control for an objective function where the variable states follow a random progression. In order to characterize the control using the convexity of the Hamiltonian function  $H(t, x(t), y(t), u(t), z(t))$  and the terminal cost  $\varphi(x(T))$ , we have proved the non-differentiability of the Brownian motion  $w_t$ . To illustrate the results of our work, we have considered an epidemiological system of the SIRS type, where immunity is not permanently acquired. In this system, a simple  $c(t)$  control

is applied to the infectious states to illustrate the results of our work.

## 2. Problem Formulation and Preliminaries

In this paper, we deal with an optimal control problem defined in a probability space  $(\Omega, \mathcal{F}_t, \mathbb{P})$  and defined as follows:

$$\inf_{u \in \mathcal{U}[0, T]} J(x, u); J(x, u) = E \left[ \varphi(x(T)) + \int_0^T l(t, x(t), u(t)) dt \right] \quad (1)$$

$$s.r : \begin{cases} dx(t) = b(t, x(t), u(t))dt + \sigma(t, x(t), u(t))dw(t), t \in [0, T] \\ x(0) = x_0 \end{cases} \quad (2)$$

$$(x, u) \in \Omega \times \mathcal{U}[0, T] = \{z/z_0 \in \Omega \text{ and } z(t) \in \Omega; v \in L^2(\Omega, \mathcal{F}_t, \mathbb{P})\} \quad (3)$$

Where  $b(t, x(t), u(t)) : [0, T] \times \mathbb{R}^n \times \mathcal{U}[0, T] \rightarrow \mathbb{R}^n$  and  $\sigma : [0, T] \times \mathbb{R}^n \times \mathcal{U}[0, T] \rightarrow \mathbb{R}^{n \times m}$ . In the relation (2)  $b(t, x(t), u(t))$  and  $\sigma(t, x(t), u(t))$  describes respectively the derivation and the diffusion of the system,

$\varphi(t)$  and  $l(t)$  described in the relationship (1) are given functions, verifying certain criteria.

$w(t)$  denotes standard Brownian motion in probability space  $(\Omega, \mathcal{F}_t, \mathbb{P})$ .

$$\mathcal{U}[0, T] = \{v(t) \in L^2(\Omega, \mathcal{F}, \mathbb{P}), v \geq 0 \text{ p.p in } \Omega\}$$

is a bounded closed convex

### 2.1. Existence and Uniqueness of Solutions

Let  $(x_t)_{t \in [0, T]}$  be a random variable solving the control system (2) where  $b(t, x(t), u(t))$  denotes the drift coefficient (deterministic part) and  $\sigma(t, x(t), u(t))$  the diffusion coefficient, which measures the extent of the phenomenon. In the following, we assume the following assumptions.

H1.)  $\mathcal{F}_t$  is a natural filtration generated by Brownian motion  $w_t$ .

H2.)  $(\mathcal{U}, d)$  is a separable metric space and  $T \in [0, \infty)$ .

H3.) The functions  $b(t, x(t), u(t))$ ,  $\sigma(t, x(t), u(t))$ ,  $l(t, (t), u(t))$  and  $\varphi(x(T))$  are measurable functions and there exist  $\delta > 0$  and a continuity modulus  $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that:

$$\begin{cases} |\Lambda(t, x(t), u(t)) - \Lambda(t, z(t), v(t))| < \delta |x(t) - z(t)| + \eta(d(u, v)), \\ \quad \forall t \in [0, T], x, z \in \mathbb{R}^n, u, v \in \mathcal{U}[t, T] \\ \Lambda(t, 0, u(t)) < \delta \forall (t, u) \in [0, T] \times \mathcal{U}[0, T] \end{cases} \quad (4)$$

with  $\Lambda(t, x(t), u(t)) = \{b(t, x(t), u(t)), \sigma(t, x(t), u(t)), l(t, x(t), u(t)), \varphi(x_T)\}$

H4.) The functions  $b(t, x(t), u(t))$ ,  $\sigma(t, x(t), u(t))$ ,  $l(t, x(t), u(t))$ ,  $\varphi(x(T))$  are  $\mathcal{C}^2$  in  $x$  and there exist  $\delta > 0$  and a continuity module  $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that:

$$\begin{cases} |\nabla_x \Lambda(t, x(t), u(t)) - \nabla_x \Lambda(t, z(t), v(t))| < \delta |x(t) - z(t)| + \eta(d(u, v)), \\ \quad \forall t \in [0, T], x, z \in \mathbb{R}^n, u, v \in \mathcal{U}[t, T] \\ |\nabla_{xx} \Lambda(t, x(t), u(t)) - \nabla_{xx} \Lambda(t, z(t), v(t))| < \delta |x(t) - z(t)| + \eta(d(u, v)), \\ \quad \forall t \in [0, T], x, z \in \mathbb{R}^n, u, v \in \mathcal{U}[t, T] \end{cases} \quad (5)$$

with  $\nabla_x \Lambda(\cdot) = \left( \frac{\partial \Lambda}{\partial x_1}(\cdot), \dots, \frac{\partial \Lambda}{\partial x_i}(\cdot) \right)$  and  $\nabla_{xx} \Lambda(\cdot) = \left( \frac{\partial^2 \Lambda}{\partial x_1^2}(\cdot), \dots, \frac{\partial^2 \Lambda}{\partial x_i^2}(\cdot) \right)$

In the following, we assume that assumptions H1.)-H.3) are satisfied for the control problem (1)-(3).

**Proposition 2.1.** In the optimal control problem (1)-(3) the standard Brownian motion  $w_t$  defined in the control system (2) is continuous and non-differentiable.

*Proof.* Let  $w_t$  be an application such that  $\frac{w_{t+s} - w_t}{s}$  is defined: Our aim is to show that  $\frac{w_{t+s} - w_t}{s}$  is not bounded almost surely in the vicinity of zero.

We know that:

$$\begin{aligned}
 P\left(\left|\frac{w_{t+s}-w_t}{s}\right| > \kappa\right) &= P(|w_{t+s}-w_t| > s\kappa) \\
 &= p(|w_s| > s\kappa) \text{ because } w_s \sim \mathcal{N}(0, s) \text{ i.e. } w_{t+s}-w_t \stackrel{\text{law}}{=} w_{t+s-t} \stackrel{\text{law}}{=} w_s \\
 &= P(\{w_s \in ]-\infty; -s\kappa[ \} \cup \{w_s \in [s\kappa; +\infty[ \}) \\
 &= P(w_s < -s\kappa) + P(w_s > s\kappa) - P(w_s < -s\kappa) \cap P(w_s > s\kappa) \\
 &= P(w_s < -s\kappa) + P(w_s > s\kappa) \\
 &= P(w_s < -s\kappa) + 1 - P(w_s < s\kappa) \\
 &= \int_{-\infty}^{-s\kappa} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2s}} + 1 - \int_{-\infty}^{s\kappa} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2s}}
 \end{aligned}$$

since in the vicinity of 0,  $s\kappa = -s\kappa = 0$ ; the relationship becomes:

$$p(|w_s| > s\kappa) = \int_{-\infty}^0 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2s}} + 1 - \int_{-\infty}^0 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2s}} = 1$$

So;  $P\left(\left|\frac{w_{t+s}-w_t}{s}\right| > \kappa\right) = 1$  and therefore  $\frac{w_{t+s}-w_t}{s}$  is unbounded P. almost certainly in the neighbourhood of 0.

**Proposition 2.2.** The Brownian motion  $w_t$  defined in (2) is well integrable

*Proof.* We will prove this by showing that  $\|w_t\|_{L^1} < \infty$

We know that  $\|w_t\|_{L^1} = E[|w_t|]$ .

Since for any  $x \in \mathbb{R}$ ,

$$|x| < 1 + x^2 \Rightarrow \begin{cases} x < 1 + x^2 \text{ si } x > 0 \\ -x < 1 + x^2 \text{ si } x < 0 \end{cases}$$

if  $w_t$  take  $x$ , we have:

$$|w_t| < 1 + w_t^2 \Rightarrow E[|w_t|] < E[1 + w_t^2] < E(1) + E[w_t^2] < 1 + E[w_t^2]$$

We also know that  $Var(w_t) = E[w_t^2] - (E[w_t])^2$  et  $w_t \sim \mathcal{N}(0, t)$

i.e  $Var(w_t) = E[w_t^2]$ .

So

$$E[|w_t|] < 1 + E[w_t^2] < 1 + t < \infty$$

Therefore  $w_t$  is well integrable.

## 2.2. Asymptotic Stability

**Definition 2.1.** Consider the nonlinear control-free system with noise of the control problem (1)-(2):

$$\begin{cases} dx_t = b(t, x_t)dt + \sigma(t, x_t)dw_t, & t \in [0, T] \\ x(0) = x_0, & x_0, x_t \in \Omega \end{cases} \quad (6)$$

$$\Omega := \{z \text{ such that } z_0 \in \Omega \text{ and } z(t) \in \Omega, \forall t \in [0, T]\} \quad (7)$$

We assume that the invariant state variable  $x_t$  is a solution of (6)-(7).

Given a Lyapunov function  $\mathcal{V}(t, x_t) \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R}^n)$ , we work with the diffusion operator  $\mathcal{L}$  defined by:

$$\mathcal{L}\mathcal{V}(t, x_t) = \left[ \frac{\partial \mathcal{V}(t, x_t)}{\partial t} + b(t, x_t) \cdot \frac{\partial \mathcal{V}}{\partial x}(t, x_t) \right] + \left[ \frac{1}{2} \sigma^2(t, x_t) \frac{\partial^2 \mathcal{V}}{\partial x^2}(t, x_t) \right] dt \quad (8)$$

**Proposition 2.3.** Let  $x_t$  be an invariant state vector, solution of (6)-(7), the Lyapunov function associated with (6)-(7) satisfies:

$$d\mathcal{V}(t, x_t) = \mathcal{L}\mathcal{V}(t, x_t) + \sigma(t, x_t) \frac{\partial \mathcal{V}}{\partial x}(t, x_t) dw_t \quad (9)$$

**Theorem 2.1.** The equilibrium point of the stochastic differential system (6)-(7) is asymptotically mean-square stable in the Lyapunov sense if it is mean-square stable and if

$$\lim_{t \rightarrow \infty} E[\mathcal{V}(t, x_t)] = 0 \quad (10)$$

**Theorem 2.2.** ([9] p.3) The equilibrium point of the system (6)-(7) is stable in root mean square if there exists a function  $\mathcal{V}(t, x_t) \in C^{1,2}([0, T] \times \mathbb{R}^n, \mathbb{R})$  defined positive and positive constants  $\kappa_1, \kappa_2, \kappa_3$  such that:

$$H1. E[\mathcal{V}(t, x_t)] \geq \kappa_1 E[x_t^2];$$

$$H2. E[\mathcal{V}(0, x_0)] \leq \kappa_2 x_0^2;$$

$$H3. E[\mathcal{L}\mathcal{V}(t, x_t)] \leq -\kappa_3 x_t^2$$

*Proof.* Let  $\mathcal{V}(t, x_t) = x_t^2$ . From the lemma (2.3) we have:

$$dx_t^2 = \left[ \frac{\partial x_t^2}{\partial t} + b(t, x_t) \frac{\partial x_t^2}{\partial x_t^2} + \frac{1}{2} \sigma^2(t, x_t) \frac{\partial^2 x_t^2}{\partial x_t^2} \right] dt + \sigma(t, x_t) \frac{\partial^2}{\partial x_t^2} dw_t$$

i.e

$$dx_t^2 = \mathcal{L}x_t^2 + \sigma(t, x_t) \frac{\partial^2}{\partial x_t^2} dw_t$$

By integration, we have:

$$x_t^2 - x_0^2 = \int_0^T \mathcal{L}[x^2(\tau)] d\tau + \int_0^T \sigma(\tau, x(\tau)) \frac{\partial^2 x^2(\tau)}{\partial x^2(\tau)} dw_\tau$$

Since the mathematical expectation of a stochastic integral is zero, we have:

$$E[x_t^2 - x_0^2] = \int_0^T E[\mathcal{L}[x^2(\tau)]] d\tau \quad (11)$$

We also have  $E[\mathcal{L}\mathcal{V}(t, x_t)] \leq -\kappa_3 x_t^2 < 0$ .

Thus,

$$E[x_t^2 - x_0^2] \leq -\kappa_3 \int_0^T E[x^2(\tau)] d\tau \quad (12)$$

$$i.e E[\mathcal{V}(t, x_t)] + \kappa_3 \int_0^T E[x^2(\tau)] d\tau \leq E[\mathcal{V}(0, x_0)] \text{ because } x_t^2 = \mathcal{V}(t, x), x_0^2 = \mathcal{V}(0, x_0)$$

Since  $E[\mathcal{V}(t, x_t)] \geq 0$  then,  $\int_0^T E[x^2(\tau)] d\tau \leq \frac{1}{\kappa_3} E[\mathcal{V}(0, x_0)]$  or  $E[\mathcal{V}(0, x_0)] \leq \kappa_2 x_0^2$  thus,  $\int_0^T E[x^2(\tau)] d\tau \leq \frac{\kappa_2}{\kappa_3} E[x_0^2] \leq \frac{\kappa_2}{\kappa_3} x_0^2$

$$i.e \int_0^T E[x^2(\tau)] d\tau \leq \infty$$

$x_t$  is therefore integrable. Applying the lemma (2.3) formally to the process  $x_t^2$  we have:

$$dx_t^2 = 2x_t b(t, x_t) dt + \sigma^2(t, x_t) dt + 2x_t \sigma(t, x_t) dw_t \quad (13)$$

Using the mathematical expectation operator  $E[.]$ , we have:

$$E[dx_t^2] = 2E[x_t b(t, x_t)] dt + E[\sigma^2(t, x_t)] dt + 2E[x_t \sigma(t, x_t)] dw_t$$

$$i.e dE[x_t^2] = 2E[x_t b(t, x_t)] dt + E[\sigma^2(t, x_t)] dt + 2E[x_t \sigma(t, x_t)] dw_t$$

we have:

$$E[x_t b(t, x_t)] \leq (E[|x_t|^2])^{\frac{1}{2}} \times (E[|b(t, x_t)|^2])^{\frac{1}{2}} \leq \|x_t\|_2 \times \|b(t, x_t)\|_2 \leq \xi_1$$

$$E[\sigma^2(t, x_t)] \leq ((E[|\sigma(t, x_t)|^2])^{\frac{1}{2}})^2 \leq \|\sigma(t, x_t)\|_2^2 \leq \xi_2$$

$$E[x_t \sigma(t, x_t)] \leq (E[|x_t|^2])^{\frac{1}{2}} \times (E[|\sigma(t, x_t)|^2])^{\frac{1}{2}} \leq \|x_t\|_2 \times \|\sigma(t, x_t)\|_2 \leq \xi_3$$

We then have  $dE[x_t] \leq 2\xi_1 + \xi_2 + 2\xi_3 \leq \xi$  with  $\xi = 2\xi_1 + \xi_2 + 2\xi_3$ .

In other words

$$E[|x(s_2)|^2 - |x(s_1)|^2] \leq \xi(s_2 - s_1) \quad \forall s_1, s_2 \in [0, T]; s_1 \leq s_2$$

Thus,  $E[x_t^2]$  verifies the Cauchy Lipschitz criterion with

$$\lim_{t \rightarrow \infty} E[|x_t^2|] = 0 \quad (14)$$

Consequently, the differential system (??) is asymptotically stable in root mean square.

In the following section, in addition to the assumptions H1.) – H4.) of section (2.1), we will admit the following assumption.

H5.) The Hamiltonian function  $H(t, x(t), u(t), y(t), z(t))$  is convex in  $u$  and in  $x$ , with  $H(t, x(t), u(t), y(t), z(t))$  defined in the following section (3).

### 3. Principle of the Stochastic Minimum and Illustrative Example

In this section, we characterize optimal control by deriving necessary and sufficient conditions of optimality for the (1)-(3) control problem where the EDS is driven by an invariant state variable  $x(t)$  and normally distributed multidimensional Brownian motion  $w_t$ .

Assuming that  $x(t), u(t)$  is an optimal pair of the stochastic control problem, there exists a pair of adjoint processes  $(y(t), z(t))$  such that:

$$dy(t) = -\nabla_x H(t, x(t), u(t), y(t), z(t))dt + z(t)dw_t \quad (15)$$

$$x(0) = x_0, y(T) = -\nabla_x \varphi(x(T)) \quad (16)$$

where the Hamiltonian function is given by:

$$H(t, x, u, y, z) = y.b(t, x, u) - l(t, x, u) + \frac{1}{2}tr[\sigma(t, x, u)^T . z(t) . \sigma(t, x, u)] \quad (17)$$

**Theorem 3.1.** Under the previous assumptions and the convexity of  $\mathcal{U}[0, T]$ -space, if  $(x^*(t), u^*(t))$  is an optimal pair of the control problem (1)-(3), then there exists a pair of adjoint processes  $(y(t), z(t))$  satisfying the equation (15) -(16) such that:

$$\langle \nabla_u H(t, x^*(t), u^*(t), y(t), z(t)), u(t) - u^*(t) \rangle \leq 0 \quad (18)$$

in other words:

$$H(t, x^*(t), u^*(t)) = \min_{u \in \mathcal{U}} H(t, x^*(t), u(t)) \quad (19)$$

**Remark 3.1.** When the convexity of the control domain  $\mathcal{U}[0, T]$  is not admitted, the local form (18) is substituted by a variational-type inequality.

#### 3.1. Optimality Principle

In order to characterize an optimal solution to the control problem, we combine the control system (2) called Backward and the adjoint system (15)-(16) called forward to arrive at the following stochastic Hamiltonian system (BSDE):

$$dx(t) = \nabla_y H(t, x(t), u(t), y(t), z(t))dt + \nabla_z H(t, x(t), u(t), y(t), z(t))dw(t) \quad (20)$$

$$dy(t) = -\nabla_x H(t, x(t), u(t), y(t), z(t))dt + z(t)dw(t) \quad (21)$$

$$y(T) = -\nabla_x \varphi(x(T)) \quad (22)$$

$$x(0) = x_0 \quad (23)$$

subject to constraints (3).

**Theorem 3.2.** Let  $(\tilde{x}(t), \tilde{u}(t))$  be an admissible pair satisfying the (19) condition and  $(y(t), z(t))$  be a solution of (15)-(16).

Assuming that  $H(t, x, u, y, z)$  and  $\varphi(x(T))$  are convex, then  $(x^*(t), u^*(t))$  is a unique optimal solution of the control problem (1)-(3)

*Proof.* We know that:

$$J(u) = E \left[ \varphi(x(T)) + \int_0^T l(t, x(t), u(t)) dt \right], \text{ and,} \quad (24)$$

$$J(u^*) = E \left[ \varphi(x^*(T)) + \int_0^T l(t, x^*(t), u^*(t)) dt \right] \quad (25)$$

And so,

$$J(u^*) - J(u) = E [\varphi(x^*(T)) - \varphi(x(T))] + E \left[ \int_0^T (l(t, x^*(t), u^*(t)) - l(t, x(t), u(t))) dt \right] \quad (26)$$

From the convexity of  $\varphi(\cdot)$ , we have:

$$\begin{aligned} E[\varphi(x^*(T)) - \varphi(x(T))] &\leq E[x^*(T) - x(T)] \nabla \varphi_x(x^*(T)) \\ &= E[x^*(T) - x(T)] y^*(T) \\ &= E \left[ \int_0^T (x^*(t) - x(t)) dy(t) \right] + E \left[ \int_0^T y^*(t) d(x^*(t) - x(t)) \right] \\ &\quad + E \left[ \int_0^T \text{tr}[\sigma(t, x^*(t), u^*(t)) - \sigma(t, x(t), u(t)) z^*(t)] dt \right] \end{aligned}$$

where,

$$\begin{aligned} E \left[ \int_0^T (x^*(t) - x(t)) dy(t) \right] &= E \left[ \int_0^T (x^*(t) - x(t)) (-\nabla_x H(t, x^*(t), u^*(t), y(t), z(t))) dt \right] \\ &\quad + E \left[ \int_0^T (x^*(t) - x(t)) z^*(t) dw(t) \right] \end{aligned}$$

and

$$\begin{aligned} E \left[ \int_0^T y^*(t) d(x^*(t) - x(t)) \right] &= E \left[ \int_0^T y^*(t) (b(t, x^*(t), u^*(t)) - b(t, x(t), u(t))) dt \right] \\ &\quad + E \left[ \int_0^T y^*(t) (\sigma(t, x^*(t), u^*(t)) - \sigma(t, x(t), u(t)))^T dw(t) \right] \end{aligned}$$

Thus,

$$\begin{aligned} E[\varphi(x^*(T)) - \varphi(x(T))] &\leq E \left[ \int_0^T (x^*(t) - x(t)) (-\nabla_x H(t, x^*(t), u^*(t), y^*(t), z^*(t))) dt \right] \\ &\quad + E \left[ \int_0^T (x^*(t) - x(t)) z^*(t) dw(t) \right] + E \left[ \int_0^T y(t) (b(t, x^*(t), u^*(t)) - b(t, x(t), u(t))) dt \right] \\ &\quad + E \left[ \int_0^T y^*(t) \sigma(t, x^*(t), u^*(t)) - \sigma(t, x(t), u(t))^T dw(t) \right] \end{aligned} \quad (27)$$

From the definition of the Hamiltonian function  $H(\cdot)$  defined by the relation (17), we have:

$$\begin{aligned} E \left[ \int_0^T l(t, x^*(t), u^*(t)) - l(t, x(t), u(t)) dt \right] &= E \left[ \int_0^T H(t, x^*(t), u^*(t), y^*(t), z^*(t)) - H(t, x(t), u(t), y^*(t), z^*(t)) dt \right] \\ &- E \left[ \int_0^T (b(t, x^*(t), u^*(t)) - b(t, x(t), u(t))) y^*(t) \right] - E \left[ \int_0^T \text{tr}[\sigma(t, x^*(t), u^*(t)) - \sigma(t, x(t), u(t))]^T z^*(t) dt \right] \end{aligned} \quad (28)$$

or,

$$J(u^*) - J(u) = E[\varphi(x^*(T)) - \varphi(x(T))] + E \left[ \int_0^T (l(t, x^*(t), u^*(t)) - l(t, x(t), u(t))) dt \right] \quad (29)$$

leads to the following relationship by combining the (3.1) and (3.1) relations:

$$\begin{aligned} J(u^*) - J(u) &\leq E \left[ \int_0^T H(t, x^*(t), u^*(t), y^*(t), z^*(t)) - H(t, x(t), u(t), y^*(t), z^*(t)) dt \right] \\ &- E \left[ \int_0^T (x^*(t) - x(t)) (\nabla_x H(t, x^*(t), u^*(t), y^*(t), z^*(t))) dt \right] \end{aligned} \quad (30)$$

This is because the expectations of the stochastic integrals:

$$E \left[ \int_0^T (x^*(t) - x(t)) z(t) dw(t) \right] \text{ and } E \left[ \int_0^T y^*(t) (\sigma(t, x^*(t), u^*(t)) - \sigma(t, x(t), u(t)))^T dw(t) \right] \text{ are all null.}$$

From the convexity of the Hamiltonian function  $H(t, x(t), u(t), y(t), z(t))$  in  $u$  and the optimality condition (19), we have:

$$\begin{aligned} E \left[ \int_0^T H(t, x^*(t), u^*(t), y^*(t), z^*(t)) - H(t, x(t), u(t), y^*(t), z^*(t)) dt \right] \\ - E \left[ \int_0^T (x^*(t) - x(t)) (\nabla_x H(t, x^*(t), u^*(t), y^*(t), z^*(t))) dt \right] \leq 0 \end{aligned} \quad (31)$$

Therefore,  $J(u^*) \leq J(u)$  i.e.  $u^*(\cdot) \in \arg \min J(u)$

In the next section, assuming that the state variable  $x(t) \in \mathbb{F}r(\Omega)$ , the action of the control function  $u(t)$  is brought inside  $\Omega$ . Our initial control problem (1)-(3) then becomes:

$$\inf_{u \in \mathcal{U}[0, T]} E \left[ \varphi(x(T)) \mathbb{1}_{T \leq \tau} + \int_0^T l(t, x(t), u(t)) dt + \int_0^T \tilde{l}(\tau, x(\tau), u(\tau)) d\xi(t) \right] \quad (32)$$

$$s.r : \begin{cases} dx(t) = b(t, x(t), u(t)) dt + \sigma(t, x(t), u(t)) dw(t) - n_{x(\cdot)}(t, x(t)) d\xi(t), \\ x(0) = x_0, \end{cases} \quad (33)$$

$$(x, u) \in \mathbb{F}r(\Omega) \times \mathcal{U}[0, T] = \{z/z_0 \in \Omega \text{ and } z(t) \in \Omega; v \in L^2_{loc}([0, T]; \mathcal{U}[0, T])\} \quad (34)$$

where  $\tau = \inf \{s \in [0, T]; x(s) \notin \Omega\}$ ;  $\xi(t)$  is a process that grows when  $x(\cdot) \in \mathbb{F}r(\Omega)$ . Here, if  $x(\cdot) \in \Omega$ , then,  $d\xi(t) \rightarrow 0$  and we return to the initial control system (2).  $n_{x(\cdot)}$  denotes the outside normal to the boundary  $\mathbb{F}r(\Omega)$  of  $\Omega$  in  $x$ ,  $\tilde{l}(\cdot)$  denotes the additional cost of reflecting the random variable  $x_t$  back to normal.

In the (32)-(33) problem, the optimal value is given by:

$$u^*(t, x) \equiv (32) \quad (35)$$

Under assumptions H4.)-H.6), if  $b$  and  $\sigma$  are of class  $\mathcal{C}^1$  with respect to  $t$ , then the optimal value of the control problem (32)-(34) is a solution of:

$$\min_{u \in \mathcal{U}[0, T]} \left\{ b(t, x, u) \cdot \nabla u(t, x) + \frac{1}{2} \text{tr} \sigma \sigma^T \cdot \nabla^2 u(t, x) + l(t, x, u) \right\} = 0 \quad (36)$$

$$u(t, x) = \tilde{l}(t, x, u), \quad x \in \mathbb{F}r(\Omega) \quad (37)$$

### 3.2. Example of an Epidemiological Illustration

In this section we consider the SIRS model for the spread of the SARS-CoV-2 virus with additional mortality:

$$\begin{cases} \dot{S}(t) = \Lambda - \beta S(t)I(t) - \mu S(t) + \nu R(t) \\ \dot{I}(t) = \beta S(t)I(t) - \gamma I(t) - (\mu + \mu_c)I(t) \\ \dot{R}(t) = \gamma I(t) - (\nu + \mu)R(t) \\ S(0) > 0, I(0) > 0, R(0) > 0, S, I, R \in \mathcal{X}_+, \\ c \in L^2_{loc}([0, T], \mathcal{U}[0, T]) \end{cases} \quad (38)$$

$$\mathcal{X}_+ = \{S, I, R, \in \mathcal{C}(\mathbb{R}_+, [0, T]); N(t) \leq \frac{\Lambda}{\mu}\} \quad (39)$$

This system is perturbed by environmental noise on the

variable states and receives a  $c$  control on the infectious state. Let  $\mathcal{X}(t)$  be the random variable defined by  $\mathcal{X}(t) = (S(t), I(t), R(t))^T$ . The corresponding stochastic epidemic model can be obtained by applying the infinitesimal transition probabilities.

Consider the normally distributed random vector  $\Delta\mathcal{X}(t)$  defined by:

$$\begin{aligned} \Delta\mathcal{X}(t) &= \mathcal{X}(t + \Delta t) - \mathcal{X}(t) \\ &= (\Delta S(t), \Delta I(t), \Delta R(t))^T \end{aligned}$$

The mathematical expectation  $E[\Delta\mathcal{X}(t)]$  and the covariance matrix  $\Gamma(\Delta\mathcal{X}(t))$  are such that on the basis of the transition probability we have:

$$E[\Delta\mathcal{X}(t)] = \begin{bmatrix} \Lambda - \beta SI - \mu S + \nu R \\ \beta SI - \gamma I - (\mu + \mu_c)I \\ \gamma I - (\nu + \mu)R \end{bmatrix} \Delta t + o(\Delta t) = \mathcal{M}(\mathcal{X}(t))\Delta t + o(\Delta t) \quad (40)$$

$$\Gamma(\Delta\mathcal{X}(t)) = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix} \Delta t + o(\Delta t) = \Gamma(\mathcal{X}(t))\Delta t + o(\Delta t) \quad (41)$$

where the coefficients  $\gamma_{ij}$  are such that:

$$\gamma_{ij} = \text{cov}(\mathcal{X}_i, \mathcal{X}_j) \quad (42)$$

with  $\text{cov}(\mathcal{X}_i, \mathcal{X}_j)$  the covariance of  $(\mathcal{X}_i, \mathcal{X}_j)$ .

$\Delta\mathcal{X}(t) \sim \mathcal{N}(\mathcal{M}(\mathcal{X}(t))\Delta t; \Gamma(\mathcal{X}(t))\Delta t)$ .

Since  $\Gamma(\mathcal{X}(t))$  is positive definite the square root  $\sigma(\mathcal{X}(t)) = \sqrt{\Gamma(\mathcal{X}(t))}$  exists.

Expressing the random variation  $\Delta\mathcal{X}(t)$  using the normal vector  $\eta = (\eta_1, \eta_2, \eta_3)^T \sim \mathcal{N}(0, I)$  with  $I$  a square matrix of size 3, we obtain:

$$\Delta\mathcal{X}(t) := \mathcal{M}(\mathcal{X}(t))\Delta t + \sigma(\mathcal{X}(t))\sqrt{\Delta t}\eta \quad (43)$$

If  $\mathcal{M}$  and  $\sigma$  satisfy the quadratic and Lipschitzian growth conditions (H2.) and H3.) section 1), then  $\mathcal{X}(t)$  converges in the sense of root mean square to the stochastic model:

$$d\mathcal{X}(t) = \mathcal{M}(\mathcal{X}(t))dt + \sigma(\mathcal{X}(t))dw_t \quad (44)$$

i.e;

$$dS(t) = (\Lambda - \beta S(t)I(t) - \mu S(t) + \nu R(t))dt + \sigma(S, I, R)dw_t \quad (45)$$

$$dI(t) = (\beta S(t)I(t) - \gamma I(t) - (\mu + \mu_c)I(t))dt + \sigma(S, I, R)dw_t \quad (46)$$

$$dR(t) = (\gamma I(t) - (\nu + \mu)R(t))dt + \sigma(S, I, R)dw_t \quad (47)$$

Applying the control  $c(t)$  on the infectious  $I(t)$  we obtain the stochastic control problem:

$$\min J(c); J(c) = E \left[ \beta S(T)I(T) + \int_0^T \frac{p}{2} c^2(t)dt \right] \quad (48)$$

$$s.r \begin{cases} dS(t) = (\Lambda - \beta S(t)I(t) - \mu S(t) + \nu R(t))dt + \sigma dw_t \\ dI(t) = (\beta S(t)I(t) - \gamma I(t) - (\mu + c(t) + \mu_c)I(t))dt + \sigma dw_t \\ dR(t) = (\gamma I(t) - (\nu + \mu)R(t))dt + \sigma dw_t \end{cases} \quad (49)$$



where  $\sigma := \sigma(S, I, R)$ . The Hamiltonian function associated with (49) is:

$$H(t, x(t), y(t), z(t), c(t)) = (\Lambda - \beta S(t)I(t) - \mu S(t) + \nu R(t))y_1(t) + (\beta S(t)I(t) - \gamma I(t) - (\mu + c(t) + \mu_c)I(t))y_2(t) + (\gamma I(t) - (\nu + \mu)R(t))y_3(t) - \frac{p}{2}c^2(t) + \frac{1}{2}\sigma^2(z_1(t) + z_2(t) + z_3(t)) \quad (50)$$

where  $(y(t); z(t)) := (y_1(t), y_2(t), y_3(t); z_1(t), z_2(t), z_3(t))^T$  denotes a pair of adjoint states and  $x(t) := (S(t), I(t), R(t))^T$  the random state vector.

Formally applying the previous Hamiltonian to the optimality principle described in section (3.1) we obtain:

$$dS(t) = (\Lambda - \beta S(t)I(t) - \mu S(t) + \nu R(t))dt + \frac{1}{2}\sigma^2(z_2(t) + z_3(t))dw_t \quad (51)$$

$$dI(t) = (\beta S(t)I(t) - \gamma I(t) - (\mu + c(t) + \mu_c)I(t))dt + \frac{1}{2}\sigma^2(z_1 + z_3)dw_t \quad (52)$$

$$dR(t) = (\gamma I(t) - (\nu + \mu)R(t))dt + \frac{1}{2}\sigma^2(z_1(t) + z_2(t))dw_t \quad (53)$$

$$dy_1(t) = (\beta I(t) + \mu)y_1(t) - \beta I(t)y_2(t))dt + z_1(t)dw_t \quad (54)$$

$$dy_2(t) = (\beta S y_1(t) - (\beta S - \gamma - \mu - \mu_c - c(t))y_2(t) - \gamma y_3(t))dt + z_2(t)dw_t \quad (55)$$

$$dy_3(t) = (-\nu y_1(t) + (\nu + \mu)y_3(t))dt + z_3(t)dw_t \quad (56)$$

$$y_1(T) = \beta S(T) \quad (57)$$

To find the endemic equilibrium, we solve the following system:

$$\Lambda - \beta S(t)I(t) - \mu S(t) + \nu R(t) = 0 \quad (58)$$

$$\beta S(t)I(t) - \gamma I(t) - (\mu + \mu_c)I(t) = 0 \quad (59)$$

$$\gamma I(t) - (\nu + \mu)R(t) = 0 \quad (60)$$

From the relationship (59), we have  $I^* = 0$   
where  $S^* = \frac{\gamma + \mu + \mu_c}{\beta}$ .

The relationship  $I^* = 0$  coincides with disease-free equilibrium and is given by:

$$(S^*, 0, 0) = \left( \frac{\Lambda}{\mu}, 0, 0 \right) \quad (61)$$

For the relation  $S^* = \frac{\gamma + \mu + \mu_c}{\beta}$  we find:  $R^* = \frac{\gamma[\Lambda\beta - \mu(\gamma + \mu + \mu_c)]}{\beta[(\gamma + \mu + \mu_c)(\nu + \mu) - \nu\gamma]}$  and,  $I^* = \frac{(\nu + \mu)[\Lambda\beta - \mu(\gamma + \mu + \mu_c)]}{\beta[(\gamma + \mu + \mu_c)(\nu + \mu) - \nu\gamma]}$ .

The endemic equilibrium point is therefore given by:

$$(S^*, I^*, R^*) = (E_1, E_2, E_3) \quad (62)$$

with,

$$E_1 = \frac{\gamma + \mu + \mu_c}{\beta}; E_2 = \frac{(\nu + \mu)[\Lambda\beta - \mu(\gamma + \mu + \mu_c)]}{\beta[(\gamma + \mu + \mu_c)(\nu + \mu) - \nu\gamma]}; E_3 = \frac{\gamma[\Lambda\beta - \mu(\gamma + \mu + \mu_c)]}{\beta[(\gamma + \mu + \mu_c)(\nu + \mu) - \nu\gamma]}$$

$$\mathcal{R}_0 = \frac{\beta\Lambda}{\mu(\gamma + \mu + \mu_c)}$$

*Theorem 3.3.* If  $\mathcal{R}_0 < 1$ ; then the disease-free equilibrium of the noise-free model (38) is stable and there exists an endemic equilibrium  $(S^*, I^*, R^*)$  that is globally asymptotically stable on the invariant set  $\mathcal{X}_+$ .

*Proof.* The proof is obvious: using the characteristic polynomial of the Jacobie matrix in  $E^s$ , we find two negative eigenvalues and third eigenvalue given by

$$X_p = \frac{\beta\Lambda}{\mu} - (\gamma + \mu + \mu_c), \quad (63)$$

The only condition that gives  $X_p < 0$  is:

$$\frac{\beta\Lambda}{\mu} < (\gamma + \mu + \mu_c), \text{ i.e } \frac{\beta\Lambda}{\mu(\gamma + \mu + \mu_c)} < 1 \quad (64)$$

In others words  $\frac{\beta\Lambda}{\mu(\gamma + \mu + \mu_c)} = \mathcal{R}_0 < 1$

Let's consider the previously defined set  $\mathcal{X}_+$ . This set is compact and positively invariant. Let be the Lyapunov function  $\mathcal{V}(t, S, I)$  with value in  $\mathcal{X}_+$  defined by:

$$\mathcal{V}(t, S, I, R) = S - S^* \log(S) + I - I^* \ln(I) + \delta(R - R^* \log(R)) \quad (65)$$

Let's put  $\delta = -\frac{(\gamma + \mu + \mu_c)I^*}{(\nu + \mu)R} - \mu S^*$

We note that  $\mathcal{V}(t, S, I, R) > 0$  and  $\mathcal{V}(t, S^*, I^*, R) = 0$  when  $(S, I, R) = (S^*, I^*, R^*)$

$$\dot{\mathcal{V}}(t, S, I) = \frac{\partial \mathcal{V}}{\partial S} + \frac{\partial \mathcal{V}}{\partial I} + \frac{\partial \mathcal{V}}{\partial R}$$

The derivative of the Lyapunov function  $\dot{\mathcal{V}}(t, S, I)$  is:

$$\dot{\mathcal{V}}(t, S, I) = \left( \dot{S} - S^* \frac{\dot{S}}{S} \right) + \left( \dot{I} - I^* \frac{\dot{I}}{I} \right) + \delta \left( \dot{R} - R^* \frac{\dot{R}}{R} \right)$$

$$\begin{aligned} \dot{\mathcal{V}}(t, S, I) = & \left[ \Lambda - \beta SI - \mu S + \nu R - \Lambda \frac{S^*}{S} + \beta IS^* + \mu S - \nu R \frac{S^*}{S} \right] + \\ & [\beta SI - \gamma I - (\mu + \mu_c)I - \beta SI^* + \gamma I^* + (\mu + \mu_c)I^*] + \delta \left[ \gamma I - (\nu + \mu)R - \gamma I \frac{R^*}{R} + (\nu + \mu)R^* \right] \end{aligned} \quad (66)$$

i.e

$$\begin{aligned} \dot{\mathcal{V}}(t, S, I) = & \left[ \Lambda \left( 1 - \frac{S^*}{S} \right) + \nu R \left( 1 - \frac{S^*}{S} \right) + \mu S \left( \frac{S^*}{S} - 1 \right) + (\gamma + \mu + \mu_c)I \left( \frac{R^*}{R} - 1 \right) \right] + \\ & \delta \left[ \gamma I \left( 1 - \frac{R^*}{R} \right) + (\nu + \mu)R \left( \frac{R^*}{R} - 1 \right) \right] \end{aligned} \quad (67)$$

i.e

$$\dot{\mathcal{V}}(t, S, I) = \left[ (\Lambda + \nu R - \mu S) \left( 1 - \frac{S^*}{S} \right) \right] - \left[ (\gamma + \mu + \mu_c)I \left( 1 - \frac{R^*}{R} \right) \right] + \delta \left[ (\gamma I - (\nu + \mu)R) \left( 1 - \frac{R^*}{R} \right) \right]$$

i.e

$$\begin{aligned} \dot{\mathcal{V}}(t, S, I) \leq & - \left[ \mu S \left( 1 - \frac{S^*}{S} \right) \right] - \left[ (\gamma + \mu + \mu_c)I \left( 1 - \frac{R^*}{R} \right) \right] - \delta \left[ (\nu + \mu)R \left( 1 - \frac{R^*}{R} \right) \right] \\ \leq & -\mu S + (\gamma + \mu + \mu_c)(I^* - I) \left( 1 - \frac{R^*}{R} \right) \leq 0 \end{aligned}$$

For the asymptotic mean-square stability of the (45)-(47) model, it suffices to apply the mean-square convergence of the random variable variation  $\Delta\mathcal{X}(t)$  in connection with the theory described in the (2.2) theorem.

## 4. Conclusion

In this paper, it is shown that a nonlinear stochastic control system is a system perturbed by standard Brownian motion. It is also shown in this paper that the description or characterization of the optimal value of the control function follows from the convexity of the Hamiltonian function. To better apply the control function to the control problem, it was necessary to study the stability conditions of the dynamic system. In this article, root-mean-square stability has been established to guarantee the conditions under which the state variables will converge to a stable or steady state. To better understand this stochastic control theory, a control function  $c(t)$  was applied to the infectious state  $I(t)$  of a SIRS-type epidemiological system for the SARS-CoV-2 epidemic. The cost function associated with this control has also been formulated and the optimality conditions described.

## Abbreviations

S.r	Subject to
min	Minimum
$cov(.,.)$	covariance
$\mathbb{P}r(\Omega)$	The border of $\Omega$
inf	infimum

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## Acknowledgments

The authors would like to thank the anonymous reviewers for their valuable comments and Suggestions.

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## Conflicts of Interest

The author declares no conflicts of interest.

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