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Solutions of One Dimensional Parabolic Partial Differential Equations: An Improved Finite Difference Approach

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Abstract: This paper introduces a finite difference scheme derived from the classical Crank-Nicolson method. The proposed scheme offer an improved spatial accuracy while maintaining the second-order temporal accuracy of the original Crank-Nicolson scheme. The higher order of spatial accuracy leads to improved convergence properties. The consistency and stability of the new scheme are analyzed using Taylor series expansion and von Neumann stability analysis, respectively. To validate the efficiency of the proposed scheme, it is implemented in MATLAB to solve the one-dimensional heat equation. To explore the versatility of the scheme, it is further extended to solve the advection-diffusion equation. Numerical experiments demonstrated on diffusion equation show that the new scheme compares favorably with existing methods in terms of convergence and accuracy. The results of the numerical solutions are presented in tabular form to highlight the accuracy and rates of convergence of the method. In addition, graphical plots of the numerical solutions are provided at different time levels to visualize the behavior of the solution over time and to illustrate the consistency between the numerical and analytical results. These visual and numerical comparisons further emphasize the reliability and precision of the proposed scheme. The combination of improved spatial resolution, solid theoretical foundation, and practical implementation demonstrates the schemeâs potential for solving time-dependent partial differential equations efficiently and accurately. This makes the scheme a valuable contribution to the field of numerical methods for parabolic-type equations.

Keywords: Finite Difference Method, Convergence, Taylor'expansion, von-Newmanns Stability, Heat Equation, Advection-diffusion Equation

1. Introduction

Partial differential equations (PDEs) are fundamental tools in modeling a wide range of physical phenomena. Numerical methods such as finite difference and finite element methods, are essential for obtaining approximate solutions to these equations, especially when analytical solutions are not available. The Crank-Nicolson method is a popular choice for solving parabolic PDEs due to its second order accuracy in both space and time. However, it can be computationally expensive, particularly for large-scale problems. Recent advancement in numerical methods have led to the development of more efficient and accurate schemes.

In this work, we present a high order finite difference scheme derived from the classical Crank-Nicolson method. This scheme leverages a specific spatial discretization technique to achieve a higher order accuracy, surpassing other finite difference methods. By carefully analyzing the dispersion and dissipation properties of the scheme, its potential to accurately capture other problems such as advection-diffusion problems is also demonstrated, which are commonly used to capture transport phenomena.

To evaluate the performance of the proposed scheme, it is compared with the results obtained by [1]. By applying both schemes to a one-dimensional heat equation and advection-diffusion equation respectively, it is shown that the scheme offers an improvement in terms of accuracy, convergence and computational efficiency. Recently, different authors have worked on some interesting aspect of this method. [1], worked on finite difference method and finite element method that uses the Crank-Nicolson method to solve one dimensional heat equation. They compare the results of both methods and discovered that the finite element method that uses Crank-Nicolspn performs better than the other. [2] proves the stability of modified Crank-Nicoldon scheme derived by [1], they shows that the method is stable. [3], worked on the a practical method for the evaluation of the solution of partial differential equations. The results performed better and converges faster. Convergence of finite difference methods for partial differential equation was considered under temporal refinement was considered in [4], the results obtained shows that the method is effective and accurate. [5] proved the convergence and stability of a finite difference scheme, they results obtained shows that the scheme is stable and converges. [6] worked on the convergence of a finite difference method for convection-diffusion equations using singular solutions, results obtained shows the method converges faster. Modified Iterated Crank-Nicolson was used to solve parabolic partial differential equations, there method results into an improved accuracy. [8], worked on Crank-Nicolson and their modified scheme to find the solution of the Convection-Reaction-Diffusion Equation. There method proves to be accurate and efficient for the problem. [9-14] are interesting and relevant texts on finite difference methods.

2. Problem Definition and Methodology

The following heat and advection-diffusion equations given by

$$\frac{\partial\psi}{\partial t} - \alpha\frac{\partial^2\psi}{\partial x^2} = 0 \tag{1}$$

together with

$$\frac{\partial\psi}{\partial t} + v\frac{\partial\psi}{\partial x} = \frac{\partial^2\psi}{\partial x^2} \tag{2}$$

with initial conditions

$$\psi(x,0) = \psi(x)$$

and the boundary conditions

$$\psi(0,t) = 0 = \psi(1,t)$$

are considered. The time derivative is approximated using forward difference, the first order spatial derivative is approximated using central difference as well as the second order spatial derivatives. The generalized variable-weighted implicit approximation given by

$$\frac{\psi_{i,j+1} - \psi_{i,j}}{k} - \frac{1}{h^2}\left[\theta(\delta^2\psi)_{i,j} + (1-\theta)(\delta^2\psi)_{i,j+1}\right] = 0, \quad 0 \le \theta \le 1 \tag{3}$$

is employed, where

$$(\delta^2\psi)_{i,j+1} = \frac{\psi_{i+1,j+1} - 2\psi_{i,j+1} + \psi_{i-1,j+1}}{h^2}$$

and

$$(\delta^2\psi)_{i,j} = \frac{\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j}}{h^2}$$

is considered. In equation (3), the following special cases arises when $\theta = 0, \frac{1}{2}$ and 1, it becomes explicit, Crank-Nicolson and Implicit respectively, this cases has been considered by different authors. For the purpose of this work, the case where $\theta = \left(\frac{1}{2} - \frac{1}{12r}\right)$ will be considered.

2.1. Derivation of the Proposed Finite Difference Scheme I

Substituting $\theta = (\frac{1}{2} - \frac{1}{12r})$ into equation (3) gives

$$\psi_{i,j+1} - \psi_{i,j} = r\left[\left(\frac{1}{2} - \frac{1}{12r}\right)(\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j})\right] + r\left[\left(\frac{1}{2} + \frac{1}{12r}\right)(\psi_{i+1,j+1} - 2\psi_{i,j+1} + \psi_{i-1,j+1})\right] \tag{4}$$

On simplifying we get

$$\psi_{i,j+1} - \psi_{i,j} = \frac{1}{2}r\psi_{i+1,j} - r\psi_{i,j} + \frac{1}{2}r\psi_{i-1,j} - \frac{1}{12}\psi_{i+1,j} + \frac{1}{6}\psi_{i,j} - \frac{1}{12}\psi_{i-1,j}$$

$$+\frac{1}{2}r\psi_{i+1,j+1} - r\psi_{i,j+1} + \frac{1}{2}r\psi_{i-1,j+1} + \frac{1}{12}\psi_{i+1,j+1} - \frac{1}{6}\psi_{i,j+1} + \frac{1}{12}\psi_{i-1}j + 1 \tag{5}$$

separating terms with future steps and present time steps gives

$$\left(\frac{7}{6} + r\right)\psi_{i,j+1} - \frac{1}{2}r\psi_{i+1,j+1} - \frac{1}{12}\psi_{i+1,j+1} - \frac{1}{2}r\psi_{i-1,j+1}$$

$$= \left(\frac{7}{6} - r\right)\psi_{i,j} + \frac{1}{2}r\psi_{i+1,j} - \frac{1}{12}\psi_{i+1,j} + \frac{1}{2}r\psi_{i-1,j} - \frac{1}{12}\psi_{i-1,j} \tag{6}$$

re-arranging equation (7) gives

$$\left(\frac{7}{6} + r\right)\psi_{i,j+1} - \left(\frac{1}{2}r + \frac{1}{12}\right)(\psi_{i+1,j+1} + \psi_{i-1,j+1})$$
$$= \left(\frac{7}{6} - r\right)\psi_{i,j} + \left(\frac{1}{2}r - \frac{1}{12}\right)(\psi_{i+1,j} + \psi_{i-1,j}) \tag{7}$$

Equation (7) is our proposed finite difference scheme and can be written in the following matrix form

$$\begin{bmatrix} \frac{7}{6}+r & -(\frac{1}{2}r+\frac{1}{12}) & 0 & \cdots & 0 \\ -(\frac{1}{2}r+\frac{1}{12}) & \frac{7}{6}+r & -(\frac{1}{2}r+\frac{1}{12}) & \cdots & 0 \\ 0 & -(\frac{1}{2}r+\frac{1}{12}) & \frac{7}{6}+r & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & -(\frac{1}{2}r+\frac{1}{12}) \\ 0 & 0 & 0 & 0 & \frac{7}{6}+r \end{bmatrix} \begin{bmatrix} \psi_{1,j+1} \\ \psi_{2,j+1} \\ \psi_{3,j+1} \\ \vdots \\ \psi_{i,j+!} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{7}{6}-r & (\frac{1}{2}r+\frac{1}{12}) & 0 & \cdots & 0 \\ (\frac{1}{2}r+\frac{1}{12}) & \frac{7}{6}-r & (\frac{1}{2}r+\frac{1}{12}) & \cdots & 0 \\ 0 & -(\frac{1}{2}r+\frac{1}{12}) & \frac{7}{6}-r & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & (\frac{1}{2}r+\frac{1}{12}) \\ 0 & 0 & 0 & (\frac{1}{2}r+\frac{1}{12}) & \frac{7}{6}-r \end{bmatrix} \begin{bmatrix} \psi_{1,j} \\ \psi_{2,j} \\ \psi_{3,j} \\ \vdots \\ \psi_{i,j} \end{bmatrix} \tag{8}$$

2.2. Derivation of the Proposed Finite Difference Scheme II

For the derivation of the advection-diffusion equation, we consider the following equation

$$\frac{\partial \psi}{\partial t} + v\frac{\partial \psi}{\partial x} = \alpha \frac{\partial^2 \psi}{\partial x^2} \tag{9}$$

using the generalized equation (3), and substituting the value of θ into it gives

$$\frac{\psi_{i,j+1} - \psi_{i,j}}{k} =$$
$$\left(\frac{1}{2} - \frac{1}{12r}\right)(\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j})$$
$$+\left(\frac{1}{2} + \frac{1}{12r}\right)(\psi_{i+1,j+1} - 2\psi_{i,j+1} + \psi_{i-1,j+1})$$
$$+v\frac{\psi_{i+1,j} - \psi_{i-1,j}}{2h} \tag{10}$$

On simplifying gives

$$\frac{2h(\psi_{i,j+1} - \psi_{i,j}) + v(\psi_{i+1,j} - \psi_{i-1,j})}{2hk}$$
$$= \frac{1}{2}\psi_{i+1,j} - \psi_{i,j} +$$
$$\frac{1}{2}\psi_{i-1,j} - \frac{1}{12}\psi_{i+1,j} + \frac{1}{6r}\psi_{i,j}$$
$$-\frac{1}{12r}\psi_{i-1,j} + \frac{1}{2}\psi_{i+1,j+1} - \psi_{i,j+1}$$
$$+\frac{1}{2}\psi_{i-1,j+1} + \frac{1}{12r}\psi_{i+1,j+1} - \frac{1}{6r}\psi_{i,j+1} + \frac{1}{12r}\psi_{i-1,j+1} \tag{11}$$

simplifying equation (11) further using $r = \frac{k}{h^2}$ and $\alpha = v\frac{k}{h}$ gives

$$\psi_{i,j+1} - \psi_{i,j} + \frac{\alpha}{2}\psi_{i+1,j} - \frac{\alpha}{2}\psi_{i-1,j} = \frac{1}{2}\psi_{i+1,j} - \psi_{i,j} + \frac{1}{2}\psi_{i-1,j}$$
$$-\frac{1}{12}\psi_{i+1,j} + \frac{1}{6r}\psi_{i,j}$$
$$-\frac{1}{12r}\psi_{i-1,j} + \frac{1}{2}\psi_{i+1,j+1} - \psi_{i,j+1} +$$
$$+\frac{1}{2}\psi_{i-1,j+1} + \frac{1}{12r}\psi_{i+1,j+1} - \frac{1}{6r}\psi_{i,j+1} + \frac{1}{12r}\psi_{i-1,j+1} \tag{12}$$

separating terms with future steps and present time steps gives

$$\left(\frac{7}{6} + r\right)\psi_{i,j+1} - \left(\frac{r}{2} + \frac{1}{12}\right)[\psi_{i+1,j+1} + \psi_{i-1,j+1}]$$
$$= \left(\frac{7}{6} - r\right)\psi_{i,j} + \left(\frac{r}{2} + \frac{1}{12}\right)[\psi_{i+1,j} + \psi_{i-1,j}]$$
$$+\left(-\frac{\alpha}{2}\psi_{i+1,j} + \frac{\alpha}{2}\psi_{i-1,j}\right) \tag{13}$$

Equation (13) is the proposed finite difference scheme for the advection-diffusion equation.

2.3. Order and Consistency of Finite Difference Scheme I

The derivation of the order of finite difference scheme I employs Taylor's series expansion for the local truncation error. The procedure follows from the approximation of the following:

1. $\psi_{i+1,j} = \psi_{i,j} + h\frac{\partial\psi}{\partial x} + \frac{1}{2}h^2\frac{\partial^2\psi}{\partial x^2} + \frac{1}{6}h^3\frac{\partial^3\psi}{\partial x^3} + ...$
2. $\psi_{i-1,j} = \psi_{i,j} - h\frac{\partial\psi}{\partial x} + \frac{1}{2}h^2\frac{\partial^2\psi}{\partial x^2} - \frac{1}{6}h^3\frac{\partial^3\psi}{\partial x^3} + ...$
3. $\psi_{i,j+1} = \psi_{i,j} + k\frac{\partial\psi}{\partial t} + \frac{1}{2}k^2\frac{\partial^2\psi}{\partial t^2} + \frac{1}{6}k^3\frac{\partial^3\psi}{\partial t^3} + ...$
4. $\psi_{i+1,j+1} = \psi_{i,j} + h\frac{\partial\psi}{\partial x} + k\frac{\partial\psi}{\partial t} + \frac{1}{2}h^2\frac{\partial^2\psi}{\partial x^2} + \frac{1}{2}k^2\frac{\partial^2\psi}{\partial t^2} + \frac{1}{6}h^3\frac{\partial^3\psi}{\partial x^3} + \frac{1}{6}k^3\frac{\partial^3\psi}{\partial t^3} + ...$
5. $\psi_{i-1,j+1} = \psi_{i,j} - h\frac{\partial\psi}{\partial x} + k\frac{\partial\psi}{\partial t} - \frac{1}{2}h^2\frac{\partial^2\psi}{\partial x^2} + \frac{1}{2}k^2\frac{\partial^2\psi}{\partial t^2} - \frac{1}{6}h^3\frac{\partial^3\psi}{\partial x^3} + \frac{1}{6}k^3\frac{\partial^3\psi}{\partial t^3} + ...$

substituting the Taylor's expansion into equation (4), on simplifying and canceling common terms, the leading order truncation error becomes

$$T_{ij} = \left(\frac{\partial\psi}{\partial t} - \frac{\partial^2\psi}{\partial x^2}\right) + \frac{1}{2}k\frac{\partial^2\psi}{\partial t^2} + \frac{1}{6}k^2\frac{\partial^2\psi}{\partial t^3} - \frac{1}{12}h^2\frac{\partial^4\psi}{\partial x^4} - \frac{1}{180}h^4\frac{\partial^6\psi}{\partial x^6} \tag{14}$$

comparing with the original equation (1) and taking limit as $h, k \to 0$ equation (14) gives

$$\frac{\partial\psi}{\partial t} - \frac{\partial^2\psi}{\partial x^2} = 0$$

showing that the scheme is consistent.

Considering the generalized scheme (3) with the condition $0 \leq r \leq 1$, then equation (14) can be written as

$$T_{ij} = \frac{1}{12}h^2\left(6\frac{k}{h^2}\frac{\partial^2\psi}{\partial t^2} - \frac{\partial^4\psi}{\partial x^4}\right) + O(k^2) + O(h^4)$$

showing that the order of the scheme is $O(k^2) + O(h^4)$

2.3.1. Stability of the Proposed Finite Difference Scheme I Using von-Newmann Method

The stability of the method is investigated using von-Newmann's method. von-Newmann's stability method is the most widely used procedure for determining stability of a finite difference approximation. The method introduces an initial line of errors as represented by finite Fourier series and considers the growth of these errors as x increases.

Given equation (7), let

$$E_{i,j} = e^{\gamma ih}e^{z\beta jk} = \xi e^{z\beta jk} \tag{15}$$

substituting (15) into (7) gives

$$\begin{aligned}\left(\frac{7}{6}+r\right)\xi^{i+1}e^{z\beta jk} - \left(\frac{r}{2}+\frac{1}{12}\right)(\xi^{i+1}e^{z\beta(j+1)k} + \xi^{i+1}e^{z\beta(j-1)k})\\ = \left(\frac{7}{6}-r\right)\xi^{i}e^{z\beta jk} + \left(\frac{r}{2}-\frac{1}{12}\right)(\xi^{i}e^{z\beta(j+1)k} + \xi^{i}e^{z\beta(j-1)k})\end{aligned} \tag{16}$$

using some basic mathematics principles and factoring common terms, equation (16) results to

$$\left[\left(\frac{7}{6}+r\right) - \left(\frac{r}{2}-\frac{1}{12}\right)(e^{z\beta k} + e^{-z\beta k})\right] \tag{17}$$

Applying the following trigonometry identities

$$e^{z\beta k} + e^{-z\beta k} = 2\cos\beta k$$

and

$$1 - \cos\beta k = 2\sin^2\left(\frac{\beta k}{2}\right)$$

in equation (17) gives

$$\left(\frac{7}{6}+r\right) + \left[-r - \frac{1}{6} + \left(\frac{6r+1}{3}\right)sin^2\left(\frac{\beta k}{2}\right)\right]\xi = \left[\left(\frac{7}{6}-r\right) + r - \frac{1}{6} - \left(\frac{6r-1}{3}\right)sin^2\left(\frac{\beta k}{2}\right)\right] \tag{18}$$

from algebra we have that

$$\xi = \frac{1 - \left(\frac{6r-1}{3}\right)sin^2\left(\frac{\beta k}{2}\right)}{1 + \left(\frac{6r+1}{3}\right)sin^2\left(\frac{\beta k}{2}\right)} \tag{19}$$

Equation (19) shows that the proposed scheme is unconditionally stable.

2.4. Order of the Proposed Finite Difference Scheme II

The derivation of the order of finite difference scheme II employs also Taylor's series expansion for the local truncation error. The procedure follows is the same as that described for the proposed finite difference scheme I. Expanding expand ψ in Taylor series about the point (x_i, t_j), time expansion gives

$$\psi_{i,j+1} = \psi_{i,j} + ku_t + \frac{k^2}{2}\psi_{tt} + \mathcal{O}(k^3),$$
$$\psi_{i\pm1,j+1} = \psi_{i,j} \pm h\psi_x + \frac{h^2}{2}\psi_{xx} \pm \frac{h^3}{6}\psi_{xxx} + \frac{h^4}{24}\psi_{xxxx} + k\psi_t + \mathcal{O}(k^2, h^5).$$

Similarly, spatial expansion gives

$$\psi_{i\pm1,j} = \psi_{i,j} \pm h\psi_x + \frac{h^2}{2}\psi_{xx} \pm \frac{h^3}{6}\psi_{xxx} + \frac{h^4}{24}\psi_{xxxx} + \mathcal{O}(h^5),$$
$$\psi_{i+1,j} + \psi_{i-1,j} = 2\psi_{i,j} + h^2\psi_{xx} + \frac{h^4}{12}\psi_{xxxx} + \mathcal{O}(h^6),$$
$$\psi_{i+1,j} - psi_{i-1,j} = 2hpsi_x + \frac{h^3}{3}\psi_{xxx} + \mathcal{O}(h^5).$$

On substituting into the scheme we get

$$\text{LHS} = \left(\frac{7}{6} + r\right)\left(\psi_{i,j} + k\psi_t + \frac{k^2}{2}\psi_{tt}\right) - \left(\frac{r}{2} + \frac{1}{12}\right)\left(2\psi_{i,j} + h^2\psi_{xx} + \frac{h^4}{12}\psi_{xxxx} + 2k\psi_t\right) + \mathcal{O}(k^2, h^4).$$

Also,

$$\text{RHS} = \left(\frac{7}{6} - r\right)\psi_{i,j} + \left(\frac{r}{2} + \frac{1}{12}\right)$$
$$\left(2\psi_{i,j} + h^2\psi_{xx} + \frac{h^4}{12}\psi_{xxxx}\right) - \alpha h\psi_x - \frac{\alpha h^3}{6}\psi_{xxx} + \mathcal{O}(h^4).$$

Simplifying and canceling of common terms, the leading order truncation error becomes:

$$\text{LHS} - \text{RHS} = k\psi_t - h^2\psi_{xx} + \alpha h\psi_x + \mathcal{O}(k^2, h^2),$$

Hence, the scheme approximates the advection-diffusion equation with truncation error of the form:

$$\mathcal{O}(k^2) + \mathcal{O}(h^2),$$

Showing that it is second order accurate both in time and space.

2.4.1. Stability of the Proposed Finite Difference Scheme II Using von-Newmann Method

Following the procedure in section (2.3.1), the amplification factor is given by

$$\xi = \frac{1 - \left(\frac{6r-1}{3}\right)sin^2\left(\frac{\beta k}{2}\right) - \alpha sinbk}{1 + \left(\frac{6r+1}{3}\right)sin^2\left(\frac{\beta k}{2}\right)} \tag{20}$$

Showing that the amplification factor is unconditionally stable.

3. Numerical Experiment

In order to demonstrate the efficiency and accuracy of scheme I, equation (1) together with the initial condition $\psi(x,0) = x^2 + x$ and boundary conditions $\psi(0,t) = 0 = \psi(1,t)$ are considered. For the problem, mesh size $h = 0.1$, and time step $k = 0.01$ are considered. To test for convergence, the time step is refined to be $k = 0.001$. The numerical solutions of the problem are presented in table 1 for $k = 0.01$, table 2 is the numerical solutions at $k = 0.001$ while table 3 and 4 present the exact solutions for both $k = 0.01$ and $k = 0.001$ respectively.

3.1. Solutions Using the Proposed Finite Difference Methods

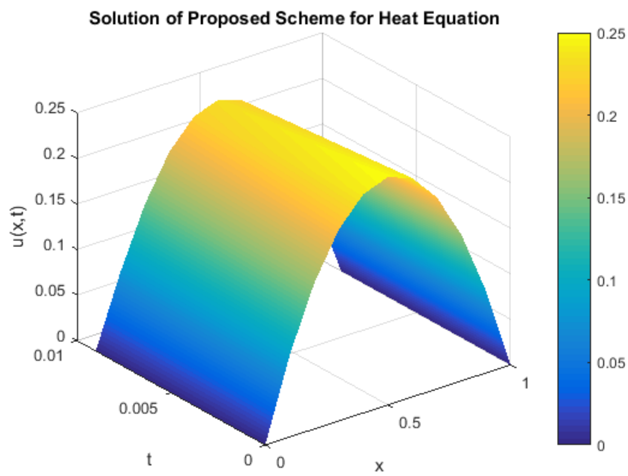

Figure 1. Solution of temperature distribution at k = 0.001.

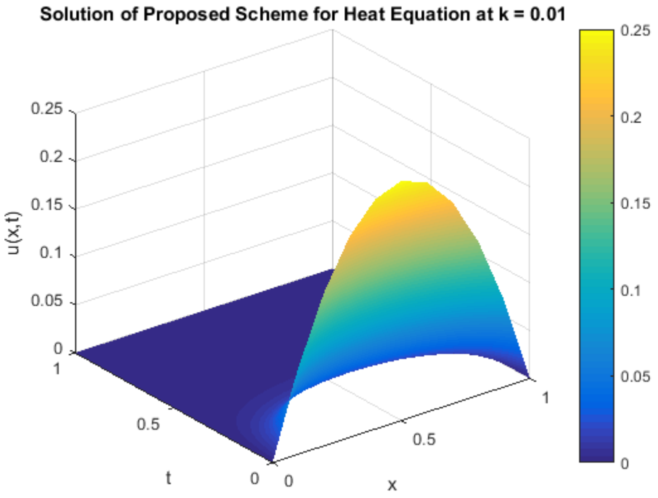

Figure 2. *Solution of proposed scheme I at t = 1 for t = 0.001.*

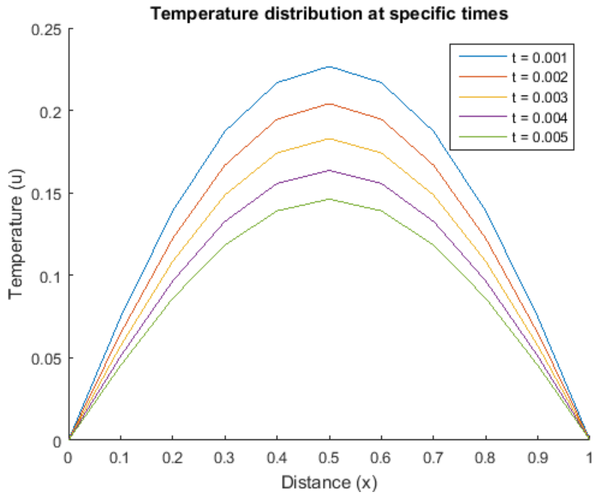

Figure 3. *Solution graph at different time.*

Table 1. *Numerical solutions at* $k = 0.01,\ h = 0.1.$

t	x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.00	0	0.0900	0.1600	0.2100	0.2400	0.2500	0.2400	0.2100	0.1600	0.0900	0.0000
0.01	0	0.0758	0.1417	0.1905	0.2202	0.2301	0.2202	0.1905	0.1417	0.0758	0.0000
0.02	0	0.0708	0.1329	0.1800	0.2089	0.2187	0.2089	0.1800	0.1329	0.0708	0.0000
0.03	0	0.0663	0.1250	0.1700	0.1981	0.2076	0.1981	0.1700	0.1250	0.0663	0.0000
0.04	0	0.0623	0.1177	0.1607	0.1877	0.1969	0.1877	0.1607	0.1177	0.0623	0.0000
0.05	0	0.0587	0.1110	0.1519	0.1778	0.1866	0.1778	0.1519	0.1110	0.0587	0.0000
0.06	0	0.0553	0.1048	0.1437	0.1683	0.1768	0.1683	0.1437	0.1048	0.0522	0.0000
0.07	0	0.0522	0.0990	0.1359	0.1593	0.1674	0.1593	0.1359	0.0990	0.0522	0.0000
0.08	0	0.0493	0.0935	0.1285	0.1508	0.1584	0.1508	0.1285	0.0935	0.0493	0.0000
0.09	0	0.0465	0.0884	0.1215	0.1427	0.1500	0.1427	0.1215	0.0884	0.0465	0.0000

Table 2. *Numerical solutions for refined time step size at* $k = 0.001,\ h = 0.1.$

t	x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.00	0	0.0900	0.1600	0.2100	0.2400	0.2500	0.2400	0.2100	0.1600	0.0900	0.0000
0.01	0	0.0882	0.1580	0.2080	0.2380	0.2480	0.2380	0.2080	0.1580	0.0882	0.0000
0.02	0	0.0860	0.1554	0.2053	0.2353	0.2453	0.2353	0.2053	0.1554	0.0860	0.0000
0.03	0	0.0840	0.1529	0.2027	0.2327	0.2427	0.2327	0.2027	0.1529	0.0840	0.0000
0.04	0	0.0821	0.1504	0.2001	0.2300	0.2400	0.2300	0.2001	0.1504	0.0821	0.0000
0.05	0	0.0804	0.1480	0.1975	0.2274	0.2373	0.2974	0.1975	0.1480	0.0804	0.0000
0.06	0	0.0788	0.1456	0.1949	0.2247	0.2347	0.2247	0.1949	0.1456	0.0788	0.0000
0.07	0	0.0773	0.1434	0.1923	0.2221	0.2320	0.2221	0.1923	0.1434	0.0773	0.0000
0.08	0	0.0759	0.1411	0.1898	0.2194	0.2294	0.2194	0.1898	0.1411	0.0759	0.0000
0.09	0	0.0746	0.1390	0.1873	0.2168	0.2267	0.2168	0.1873	0.1390	0.0746	0.0000

3.2. Exact Solutions

Applying the variable separable method on equation (1) together with the initial and boundary conditions given, we get

$$\psi(x,t) = \sum_{n=1}^{\infty} \left(\frac{-4}{n^3\pi^3}\right)(-1^n - 1)sin(n\pi x)e^{(-n^2\pi^2 t)}$$

The values of $u(x_i, t_j), i = 1, 2, 3, ..., 10$ for $k = 0.01$ and $k = 0.001$ is given in tables 3 and 4 respectively.

Table 3. *Exact Solutions at* $k = 0.01$.

t	x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.00	0	0.0900	0.1600	0.2100	0.2400	0.2500	0.2400	0.2100	0.1600	0.0900	0.0000
0.01	0	0.0756	0.1411	0.1902	0.2200	0.2300	0.2200	0.1902	0.1411	0.0756	0.0000
0.02	0	0.0668	0.1260	0.1718	0.2005	0.2102	0.2005	0.1718	0.1260	0.0668	0.0000
0.03	0	0.0598	0.1134	0.1554	0.1821	0.1912	0.1821	0.1554	0.1134	0.0598	0.0000
0.04	0	0.0539	0.1024	0.1407	0.1652	0.1736	0.1652	0.1407	0.1024	0.0539	0.0000
0.05	0	0.0488	0.0927	0.1275	0.1497	0.1574	0.1497	0.1275	0.0927	0.0488	0.0000
0.06	0	0.0441	0.0839	0.1155	0.1357	0.1427	0.1357	0.1155	0.0839	0.0441	0.0000
0.07	0	0.0400	0.0760	0.1046	0.1230	0.1293	0.1230	0.1046	0.0760	0.0400	0.0000
0.08	0	0.0362	0.0689	0.0948	0.1114	0.1171	0.1114	0.0948	0.0689	0.0362	0.0000
0.09	0	0.0328	0.0624	0.0859	0.1009	0.1061	0.1009	0.0859	0.0624	0.0328	0.0000

Table 4. *Exact Solutions at* $k = 0.001$.

t	x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.00	0	0.0900	0.1600	0.2100	0.2400	0.2500	0.2400	0.2100	0.1600	0.0900	0.0000
0.01	0	0.0880	0.1580	0.2080	0.2380	0.2480	0.2380	0.208-	0.1580	0.0880	0.0000
0.02	0	0.0861	0.1560	0.2060	0.2360	0.2460	0.2360	0.2060	0.1560	0.0861	0.0000
0.03	0	0.0845	0.1540	0.2040	0.2340	0.2440	0.2340	0.2040	0.1540	0.0845	0.0000
0.04	0	0.0829	0.1520	0.2020	0.2320	0.2420	0.2320	0.2020	0.1520	0.0829	0.0000
0.05	0	0.0815	0.1501	0.2000	0.2300	0.2400	0.2300	0.2000	0.1501	0.0815	0.0000
0.06	0	0.0802	0.1482	0.1980	0.2280	0.2380	0.2280	0.1980	0.1482	0.0802	0.0000
0.07	0	0.0789	0.1464	0.1960	0.2260	0.2360	0.2260	0.1960	0.1464	0.0789	0.0000
0.08	0	0.0778	0.1446	0.1941	0.2240	0.2340	0.2240	0.1941	0.1446	0.0778	0.0000
0.09	0	0.0767	0.1428	0.1921	0.2220	0.2230	0.2220	0.1921	0.1428	0.0767	0.0000

Table 5. *Absolute Error.*

Numerical Solutions	Exact Solutions	Absolute Error
0.2500	0.2500	0.0000
0.2480	0.2480	0.0000
0.2453	0.2460	0.0007
0.2427	0.2440	0.0013
0.2400	0.2420	0.0020
0.2373	0.2400	0.0027
0.2347	0.2380	0.0033
0.2320	0.2360	0.0040
0.2294	0.2340	0.0046

4. Discussion

Here, the proposed finite difference scheme is compared with [1]. The comparison was conducted for $k = 0.001$. Clearly from table 5, it is observed that the numerical solutions demonstrate convergent behavior towards the exact solution. The error are minimal and increases gradually, suggesting a consistent and stable numerical scheme. We also display the temperature distributions graph at specific time in figure (3). The results of the numerical solutions are presented in tables (1) and (2) at $k = 0.01$ and $k = 0.001$ respectively. Similarly, the exact solutions are presented in tables (3) and (4) also for $k = 0.01$ and $k = 0.001$. Figures (1) and (2) presents the 3D graphical solutions of the scheme at $k = 0.01$ and $k = 0.001$ respectively. The smoothness of the solution confirms that the numerical solutions approximate the analytical solutions.

5. Conclusion

From the results, it is observed that the scheme gives a good approximates solution and converges faster when compared with other implicit schemes in literature. Also, comparing with [1], it is observed that the proposed scheme performs well and also converges faster to the exact solutions. Hence, we conclude that the scheme can be used for solving problems on heat equations, advection-diffusion equations and other engineering problems.

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Conflicts of Interest

The authors declare no conflicts of interest.

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