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Research Article

A Mathematical Analysis of Mamadu-Njoseh Polynomials and Their Differential Properties

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Abstract

This study develops a mathematical framework for the Mamadu-Njoseh polynomial (MNP) family and investigates its fundamental properties and differential characteristics on a specified interval with an associated weight function. The polynomials are constructed using a weighted Gram-Schmidt orthogonalization process and normalized according to a prescribed endpoint condition. Their principal properties, including orthogonality, parity structure, norms, three-term recurrence relations, polynomial coefficients, and shifted forms, are derived and examined. Furthermore, an exact Euler-type differential equation is obtained from the parity structure of the MNP family, providing a degree-dependent differential characterization of its polynomial members. The established properties provide a theoretical basis for investigating the approximation capabilities of MNPs and their potential use in numerical analysis. In particular, the polynomials can be explored as basis functions in weighted spectral and collocation methods for approximating solutions of ordinary and partial differential equations. Their possible extension to fractional differential equations also presents an avenue for further research. However, the practical effectiveness of these applications depends on additional analysis of approximation errors, convergence, numerical stability, and computational performance. The results therefore establish foundational analytical properties of the Mamadu-Njoseh polynomial family and identify directions for future research in approximation theory, differential equations, and numerical computation.

Keywords

Mamadu-Njoseh Polynomials, Orthogonal Polynomials, Weighted Gram-Schmidt Orthogonalization, Three-Term Recurrence, Parity, Spectral Approximation

1. Introduction

Orthogonal polynomials form an important class of functions with extensive applications in approximation theory, numerical analysis, spectral methods, interpolation, quadrature, differential equations, mathematical physics, and related areas. The importance arises from the combination of algebraic structure, orthogonality, recurrence relations, differential properties, zero distributions, and approximation capabilities [1-10]. The general theory of orthogonal polynomials is

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closely related to positive measures, moment functionals, Gram-Schmidt orthogonalization, three-term recurrence relations, and Gaussian quadrature [11, 12]. These properties provide the theoretical foundation for constructing efficient polynomial bases for analytical and computational applications.

For a positive weight function, the corresponding orthogonal polynomial sequence can be constructed from the monomial basis by an orthogonalization procedure. The polynomials obtained satisfy a three-term recurrence relation, one of the basic structural properties of orthogonal polynomial systems [13, 14]. Such recurrence relations are important for the generation of successive polynomial members, for the analysis of their zeros, norms, Christoffel-Darboux kernels, Gaussian quadrature rules and their approximation properties [15-17]. Therefore, orthogonal polynomial bases have become the basis of high-order approximation and spectral discretization methods [12, 18-21].

The classical theory of orthogonal polynomials is primarily concerned with the Jacobi, Laguerre and Hermite families. These systems have rich structures related to recurrence relations, Rodrigues-type representations, generating functions and differential equations [1, 2, 4]. Their defining weights satisfy Pearson-type differential equations and the polynomial members satisfy second-order differential equations with order independent of the polynomial degree [4, 22]. These properties provide an important reference framework for investigating newly constructed orthogonal polynomial systems and determining whether analogous structural properties can be established.

The characterization of orthogonal polynomial families via differential equations is particularly significant. Classical families are closely related to second-order differential operators and Sturm-Liouville-type problems, a connection which underlies many of their analytical and computational applications. This theory has motivated the investigation of broader classes of orthogonal polynomials for which the corresponding differential equations may involve more complicated coefficients or higher-order structures [23, 24]. Thus, in recent years, much attention has been paid to non-classical and semi-classical orthogonal polynomial systems. For these systems, recurrence coefficients, structure relations, differential equations, and asymptotic properties may show considerably richer behavior than for classical families [25]. For example, Wang *et al*. [25] studied orthogonal polynomials associated with a semi-classical weight and considered the connection between recurrence coefficients and differential equations. Yakubovich [26] studied composition-orthogonality methods for the construction of new systems of orthogonal polynomials and proved the corresponding recurrence, differential, generating-function and Rodrigues-type properties. These studies demonstrate the continuing development of systematic approaches for constructing and analysing non-classical orthogonal polynomial families.

The study of differential equations associated with orthogonal polynomials remains particularly important. For semi-classical systems, the relationship between the differential properties of the weight and the equations satisfied by the associated polynomials provides an important means of understanding their structure. Dzhamay *et al*. [27] demonstrated that differential systems associated with recurrence coefficients of semi-classical orthogonal polynomials can be connected with nonlinear differential equations of Painlevé type. Similarly, Zhu *et al*. [28] considered orthogonal polynomials associated with modified weights and established second-order linear differential equations, including equations related to Heun-type structures. Bracciali *et al*. [29] further studied semi-classical orthogonal polynomials through Pearson-type equations and obtained structure relations, difference equations, and differential equations for the associated polynomial families. These developments highlight the importance of identifying differential equations that characterize newly constructed or generalized orthogonal polynomial systems.

Orthogonal polynomials also play a fundamental role in approximation theory and numerical computation. Their use as basis functions in spectral methods provides highly accurate approximations for differential and integral equations, particularly when the underlying solutions have sufficient regularity [12]. Alternative orthogonal systems are therefore of interest when classical bases are not ideally suited to a particular weight function, boundary condition, differential operator, or numerical formulation. Recent studies have demonstrated the application of orthogonal polynomial bases to fractional integral and differential equations, quadrature, and other problems involving non-standard operators [30-32].

Within this broader development, the Mamadu-Njoseh polynomials (MNPs) constitute a non-classical orthogonal polynomial family introduced for numerical approximation and computational applications. The polynomials were constructed as orthogonal basis functions and applied as trial functions for the numerical solution of fifth-order boundary-value problems. Their numerical experiments considered both linear and nonlinear problems and demonstrated the applicability of the MNP basis to boundary-value problems [33-35]. The MNP family has subsequently been employed in several numerical formulations. Mamadu and Njoseh [36] investigated the use of orthogonal polynomials in orthogonal collocation methods for Fredholm integro-differential equations, thereby demonstrating the relevance of the polynomial basis to collocation procedures. Further applications have included iterative and collocation approaches for differential equations, as well as numerical approximation of fractional models. In particular, Mamadu *et al*. [37] developed a finite-element orthogonal collocation approach for the time-fractional telegraph equation using MNPs as basis functions, while Mamadu *et al*. [38] also investigated an implicit finite-difference method with MNP basis functions and provided an associated error analysis.

The application of MNPs has also been extended to nonlinear and stochastic problems. Akponana and Njoseh [39] in-

vestigated MNP-based iterative approaches for nonlinear differential equations, while Onyeoghane and Njoseh [32] employed MNPs as modifier and basis functions in enhanced iterative methods for the Heston stochastic partial differential equation. These studies demonstrate the continuing use of MNPs in numerical schemes involving ordinary, partial, fractional, and stochastic differential equations. More recent work has also explored MNP-based spectral and approximation formulations, further indicating the potential of the family as a computational basis.

Despite these developments, the existing MNP literature has mainly focused on the use of the polynomials as basis, trial, modifier or approximation functions in numerical schemes. It is essential to perform a systematic investigation of their intrinsic mathematical structure. In particular, we need a uniform treatment of their orthogonality, normalization, parity, norm relations, recurrence structure, coefficient representation, generating function, shifted formulation and differential properties, so as to lay a firmer theoretical foundation for later applications. The question of differential structure is especially important because the existence of a three-term recurrence or a degree-dependent differential identity does not automatically imply the existence of a fixed-order differential equation for the entire polynomial family.

Motivated by this gap, the present study develops a systematic mathematical framework for the Mamadu-Njoseh polynomial family. The MNPs are constructed through weighted Gram-Schmidt orthogonalization and normalized at the right endpoint. Their fundamental properties, including orthogonality, parity, norm relations, recurrence relations, coefficient representations, and shifted forms, are established. The three-term recurrence provides an efficient mechanism for generating successive polynomial members, while the associated coefficient recurrence provides an algebraic procedure for computing their coefficients.

The analysis relies heavily on the parity of the defining weight. Since the weight is even, the resulting MNPs inherit an alternating parity: even-indexed polynomials are even functions, odd-indexed polynomials are odd functions. This property provides a simplified representation of the polynomial family and facilitates the derivation of coefficient and differential relations. The first ten MNP members are explicitly constructed to demonstrate the consistency of the proposed formulation and to provide concrete basis functions for subsequent theoretical and numerical investigations. A further objective is to investigate the differential structure of the MNP family. An exact Euler-type differential relation is derived directly from the parity-based polynomial representation. This relation provides a rigorous differential characterization of each individual MNP. However, because its order depends on the polynomial degree, it should not be identified with a fixed second-order differential equation for the entire family. Establishing whether a genuine fixed-order differential equation exists for the MNPs under the present weight and normalization constitutes a separate mathematical problem and requires independent analysis.

The orthogonality and approximation properties of the MNPs also establish their potential as basis functions for weighted spectral and projection methods. Their existing applications to boundary-value problems, integro-differential equations, fractional telegraph equations, and stochastic partial differential equations further demonstrate their computational relevance. The present work therefore provides a mathematical foundation for extending MNP-based techniques to approximation schemes, collocation methods, spectral methods, fractional differential equations, and other computational problems.

The main contributions of this study are the systematic construction of the Mamadu-Njoseh polynomial family, the establishment of its fundamental orthogonality and parity properties, the derivation of its recurrence and coefficient relations, the explicit construction of its initial members, and the investigation of its differential structure. Particular emphasis is placed on distinguishing exact degree-dependent polynomial annihilating relations from genuine fixed-order differential equations. This distinction provides a mathematically precise basis for further theoretical investigations of the MNP family and its applications in numerical analysis.

2. Mamadu-Njoseh Polynomials

Let φ_n denote the vector space of all algebraic polynomials of degree at most n. Consider the interval [33].

$$I = [-1,1]$$

and the weight function

$$w(x) = 1 + x^2$$

Since

$$1 + x^2 > 0, \qquad x \in [-1,1],$$

we define the weighted inner product

$$\langle u, v\rangle_w = \int_{-1}^{1} (1 + x^2)u(x)v(x)\,dx. \tag{1}$$

The Mamadu-Njoseh polynomial of degree n, denoted by $\varphi_n(x)$, is the degree-n polynomial satisfying [30].

$$\varphi_n(x) = \sum_{r=0}^{n} C_r^{(n)}\, x^r, \tag{2}$$

together with

$$\langle \varphi_m, \varphi_n\rangle_w = 0, \qquad m \neq n, \tag{3}$$

and the normalization

$$\varphi_n(1) = 1. \tag{4}$$

Thus, the MNP sequence $\{\varphi_n(x)\}_{n=0}^{\infty}$ is an orthogonal polynomial system on $[-1,1]$ with respect to $w(x) = 1 + x^2$.

2.1. Construction by Gram-Schmidt Orthogonalization

The MNPs can be constructed from the monomial sequence $1, x, x^2, x^3, \ldots$ using the Gram-Schmidt process under [1-10].

$$\langle u, v\rangle_w = \int_{-1}^{1}(1 + x^2)u(x)v(x)\,dx. \tag{5}$$

Let $\varphi_0(x) = 1$. For $n \geq 1$, suppose $\varphi_0, \varphi_1, \ldots, \varphi_{n-1}$, have already been constructed. Define

$$q_n(x) = x^n - \sum_{j=0}^{n-1} \frac{\langle x^n, \varphi_j\rangle_w}{\langle \varphi_j, \varphi_j\rangle_w}\varphi_j(x). \tag{6}$$

Then

$$\langle q_n, \varphi_j\rangle_w = 0, \qquad j = 0,1, \ldots, n-1.$$

Finally, normalize by

$$\varphi_n(x) = \frac{q_n(x)}{q_n(1)}.$$

Consequently,

$$\varphi_n(1) = 1.$$

This provides an algorithmic definition of the MNPs.

2.2. General Properties

2.2.1. Orthogonality

The Mamadu-Njoseh polynomials satisfy [10].

$$\int_{-1}^{1}(1 + x^2)\varphi_m(x)\varphi_n(x)\,dx = h_n\delta_{mn}, \tag{7}$$

where

$$\delta_{mn} = \begin{cases} 1, & m = n, \\ 0, & m \neq n, \end{cases}$$

and

$$h_n = \| \varphi_n \|_w^2.$$

By construction, the polynomial φ_n is orthogonal to every polynomial of lower degree. Since $\deg(\varphi_m) = m < n$, we obtain $\langle \varphi_m, \varphi_n\rangle_w = 0$. For $m = n$, $\langle \varphi_n, \varphi_n\rangle_w = h_n$. Hence $\langle \varphi_m, \varphi_n\rangle_w = h_n\delta_{mn}$.

2.2.2. Parity Properties

The weight function satisfies

$$w(-x) = 1 + (-x)^2 = 1 + x^2 = w(x)$$

Therefore, the weight is symmetric about the origin. Consequently, the MNPs can be chosen with the parity property [1, 5, 10].

$$\varphi_n(-x) = (-1)^n\varphi_n(x). \tag{8}$$

Hence

$$\varphi_{2k}(-x) = \varphi_{2k}(x),$$

while

$$\varphi_{2k+1}(-x) = -\varphi_{2k+1}(x).$$

Thus, $\varphi_0, \varphi_2, \varphi_4, \ldots$ are even, and $\varphi_1, \varphi_3, \varphi_5, \ldots$ are odd.

2.2.3. Norm and Inner-Product Relations

Define

$$h_n = \int_{-1}^{1}(1 + x^2)\varphi_n^2(x)\,dx. \tag{9}$$

Then

$$\| \varphi_n \|_w = \sqrt{h_n}.$$

The corresponding normalized polynomial is

$$\hat{\varphi}_n(x) = \frac{\varphi_n(x)}{\sqrt{h_n}}. \tag{10}$$

It follows that

$$\langle \hat{\varphi}_m, \hat{\varphi}_n\rangle_w = \delta_{mn}.$$

Hence $\{\hat{\varphi}_n\}$ is an orthonormal system, whereas the conventional MNPs satisfy the endpoint normalization

$$\varphi_n(1) = 1.$$

For example,

$$h_0 = \int_{-1}^{1}(1 + x^2)\,dx = \frac{8}{3}.$$

For $\varphi_1(x) = x$,

$$h_1 = \int_{-1}^{1}(1 + x^2)x^2\,dx = \frac{16}{15}.$$

2.2.4. Recurrence Relation

Let $\varphi_n(x)$ denote the Mamadu-Njoseh polynomial, so that

the coefficient of x^n is 1. For the weight $w(x) = 1 + x^2, -1 \le x \le 1,$ the monic polynomials satisfy [1-10].

$$\varphi_{n+1}(x) = x\varphi_n(x) - \beta_n\varphi_{n-1}(x), \qquad n \ge 1, \tag{11}$$

where

$$\beta_n = \frac{\int_{-1}^{1}(1+x^2)\varphi_n^2(x)\,dx}{\int_{-1}^{1}(1+x^2)\varphi_{n-1}^2(x)\,dx}. \tag{12}$$

The normalized MNPs are then

$$\varphi_n(x) = \frac{\varphi_n(x)}{\varphi_n(1)}. \tag{13}$$

Consequently, (11) can be concisely written as

$$\varphi_{n+1}(x) = A_n x\varphi_n(x) - B_n\varphi_{n-1}(x), \tag{14}$$

where

$$A_n = \frac{\varphi_n(1)}{\varphi_{n+1}(1)},$$

and

$$B_n = \beta_n\frac{\varphi_{n-1}(1)}{\varphi_{n+1}(1)}.$$

This is the recurrence (11) or (14) we will actually use to construct the normalized MNPs. Thus, the first ten MNPs for $n = 0,1,\ldots,9,$

$$\phi_0 = 1,$$

$$\phi_1 = x,$$

$$\phi_2 = \frac{5x^2-2}{3},$$

$$\phi_3 = \frac{14x^3-9x}{5},$$

$$\phi_4 = \frac{357x^4-322x^2+37}{72},$$

$$\phi_5 = \frac{1221x^5-1410x^3+325x}{136},$$

$$\phi_6 = \frac{17589x^6-24750x^4+8685x^2-460}{1064},$$

$$\phi_7 = \frac{65780x^7-109109x^5+51590x^3-6125x}{2136},$$

$$\phi_8 = \frac{4052477x^8-7739732x^6+4610606x^4-879956x^2+26621}{70016},$$

and

$$\phi_9 = \frac{15968797x^9-34505172x^7+24814062x^5-6609876x^3+477981x}{145792}.$$

For practical computation, the recurrence becomes

$\varphi_{n+1}(x) = A_n x\varphi_n(x) - B_n\varphi_{n-1}(x).$ (15)

For $n = 1,\ldots,8,$ the coefficients are:

n	A_n	B_n
1	$\frac{5}{3}$	$\frac{2}{3}$
2	$\frac{42}{25}$	$\frac{17}{25}$
3	$\frac{85}{48}$	$\frac{37}{48}$
4	$\frac{3663}{2023}$	$\frac{1640}{2023}$
5	$\frac{9061}{4921}$	$\frac{4140}{4921}$
6	$\frac{61180}{32841}$	$\frac{28339}{32841}$
7	$\frac{7566513}{4025920}$	$\frac{3540593}{4025920}$
8	$\frac{39524579}{20885843}$	$\frac{18638736}{20885843}$

For example, the first recurrence is

$$\varphi_2 = \frac{5}{3}x\varphi_1 - \frac{2}{3}\varphi_0.$$

Since

$$\varphi_1 = x, \qquad \varphi_0 = 1,$$

we obtain

$$\varphi_2 = \frac{5}{3}x^2 - \frac{2}{3} = \frac{5x^2-2}{3}.$$

The next is

$$\varphi_3 = \frac{42}{25}x\varphi_2 - \frac{17}{25}\varphi_1.$$

Substituting φ_2 and φ_1,

$$\begin{aligned}\varphi_3 &= \frac{42}{25}x\left(\frac{5x^2-2}{3}\right) - \frac{17}{25}x\\ &= \frac{14}{5}x^3 - \frac{9}{5}x\\ &= \frac{14x^3-9x}{5}.\end{aligned}$$

This demonstrates exactly how the recurrence generates the subsequent members.

Now, due to the parity relation, write

$$\varphi_n(x) = \sum_{j=0}^{\left\lfloor\frac{n}{2}\right\rfloor} c_{n,j}\,x^{n-2j}, \tag{16}$$

with leading coefficient

$$c_{n,0} = 1.$$

Substituting (16) into (11), we have,

$$\sum_{j=0}^{\left\lfloor\frac{n+1}{2}\right\rfloor} c_{n+1,j}\, x^{n+1-2j} = \sum_{j=0}^{\lfloor n/2\rfloor} c_{n,j}\, x^{n+1-2j} - \beta_n \sum_{j=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} c_{n-1,j}\, x^{n-1-2j}. \quad (17)$$

Re-index the second sum to obtain,

$$n-1-2j = n+1-2(j+1). \quad (18)$$

Therefore,

$$c_{n+1,j} = c_{n,j} - \beta_n c_{n-1,j-1}, \qquad j \geq 1. \quad (19)$$

with

$$c_{n,0} = 1,$$

which is a coefficient recurrence for the MNPs. Thus the MNP recurrence has a direct algebraic-to-coefficient formulation.

Now, let differentiate (11) for the first time yields

$$\varphi'_{n+1} = \varphi_n + x\varphi'_n - \beta_n \varphi'_{n-1}. \quad (20)$$

Differentiating again for the second time,

$$\varphi''_{n+1} = 2\varphi'_n + x\varphi''_n - \beta_n \varphi''_{n-1}. \quad (21)$$

These are exact differential-difference relations. Notice that (11) is a recurrence in the degree n, whereas φ'_n, φ''_n are derivatives in the variable x. Therefore, the three-term recurrence by itself does not immediately imply a second-order ODE.

2.2.5. Generating-Function Formulation

Define the formal generating function [1, 10] for the MNPs by

$$G(x,t) = \sum_{n=0}^{\infty} \varphi_n(x) t^n. \quad (22)$$

Multiply (11) by t^{n+1} and sum for $n \geq 1$, we obtain,

$$\sum_{n=1}^{\infty} \varphi_{n+1}\, t^{n+1} = x \sum_{n=1}^{\infty} \varphi_n\, t^{n+1} - \sum_{n=1}^{\infty} \beta_n\, \varphi_{n-1} t^{n+1}. \quad (23)$$

The left side is $G - 1 - xt$, and the first term on the right is $xt(G-1)$. For the last term, put $k = n-1$, such taht

$$\sum_{n=1}^{\infty} \beta_n\, \varphi_{n-1} t^{n+1} = t^2 \sum_{k=0}^{\infty} \beta_{k+1}\, \varphi_k t^k.$$

Therefore

$$G - 1 - xt = xt(G-1) - t^2 \sum_{k=0}^{\infty} \beta_{k+1}\, \varphi_k t^k.$$

Hence

$$(1-xt)G(x,t) = 1 - t^2 \sum_{k=0}^{\infty} \beta_{k+1}\, \varphi_k(x) t^k. \quad (24)$$

This is the exact generating-function identity implied by the MNP recurrence.

2.2.6. Shifted Mamadu-Njoseh Polynomials

The MNPs are naturally defined on $[-1,1]$. For a problem defined on $[a,b]$, introduce the linear transformation [33].

$$t = \frac{2x-a-b}{b-a}.$$

Then

$$x = a \quad \Rightarrow \quad t = -1,$$

and

$$x = b \quad \Rightarrow \quad t = 1.$$

The shifted MNP is therefore defined by

$$\varphi_n^{[a,b]}(x) = \varphi_n\left(\frac{2x-a-b}{b-a}\right). \quad (25)$$

This transformation is used in applications of MNPs to boundary-value problems on other intervals.

3. Exact MNP Differential Equation

Here we can construct an exact differential equation for every MNP using the parity structure.

Let us introduce the Euler differential operator

$\Theta = x\frac{d}{dx}$. (26)

Its action on a monomial is $\Theta x^k = kx^k$. Since

$$P_n(x) = c_{n,0}x^n + c_{n,1}x^{n-2} + \cdots + c_{n,m}x^{n-2m},$$

where $m = \left\lfloor\frac{n}{2}\right\rfloor$, we have

$$\Theta x^{n-2j} = (n-2j)x^{n-2j}.$$

Consequently,

$$[\Theta - n]x^n = 0,$$

$$[\Theta - (n-2)]x^{n-2} = 0,$$

and so on. Therefore every term in φ_n is annihilated by $\prod_{j=0}^{m}\ [\Theta - (n-2j)]$. Hence we obtain the exact MNP differential equation

$$\left[\prod_{j=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\left(x\frac{d}{dx} - (n-2j)\right)\right]\varphi_n(x) = 0, \quad (27)$$

or,

$$\left[\prod_{j=0}^{\lfloor n/2 \rfloor}\left(x\frac{d}{dx}-(n-2j)\right)\right]\frac{\varphi_n(x)}{\varphi_n(1)}=0, \tag{28}$$

in the normalized MNP. This is a genuine differential equation, not a conjecture. Let us verify it for the first few MNPs.

When $n=0$, $\phi_0(x)=1$. The equation is $\Theta\phi_0=0$. Indeed, $x\frac{d}{dx}(1)=0$. Therefore, $\Theta\phi_0=0$.

When $n=1$, $\phi_1(x)=x$. The equation is $(\Theta-1)\phi_1=0$. Since $\Theta x=x$, we obtain $(\Theta-1)x=0$. Hence, $\left(x\frac{d}{dx}-1\right)\phi_1=0$.

When $n=2$, $\phi_2(x)=\frac{5x^2-2}{3}$. The differential equation is $\Theta(\Theta-2)\phi_2=0$. Indeed, $\Theta\phi_2=\frac{10x^2}{3}$, and $(\Theta-2)\phi_2=\frac{4}{3}$. Therefore, $\Theta(\Theta-2)\phi_2=\Theta\left(\frac{4}{3}\right)=0$.

When $n=3$, $\phi_3(x)=\frac{14x^3-9x}{5}$. The equation becomes $\Theta(\Theta-1)(\Theta-3)\phi_3=0$. Indeed, the two powers occurring are x^3, x, and $(\Theta-3)x^3=0$, while $(\Theta-1)x=0$. Thus both terms are annihilated.

Equation (27) is exact; however, it does not yet constitute a fixed second-order differential equation of the standard type commonly associated with classical polynomial families. In a classical setting, a polynomial sequence satisfies a differential equation whose order remains fixed, independently of the polynomial degree. For example, Legendre polynomials satisfy.

$$(1-x^2)y''-2xy'+n(n+1)y=0,$$

where the differential order remains 2 for every n.

In contrast, although (27) provides an exact differential relation for the Mamadu-Njoseh polynomials, further analysis is required to determine whether it can be reduced to a fixed second-order differential equation with coefficients independent of (n) in the standard classical sense. Establishing such an equation, if it exists, constitutes an important direction for future research.

4. Approximation Theorem

Let

$$L_w^2(-1,1)=\left\{f:\ \int_{-1}^{1}(1+x^2)|f(x)|^2\,dx<\infty\right\}. \tag{29}$$

Since the MNPs constitute an orthogonal polynomial system, a function f can be approximated by

$$f_N(x)=\sum_{n=0}^{N}a_n\,\varphi_n(x). \tag{30}$$

The best weighted L^2-projection is obtained by choosing

$$a_n=\frac{\langle f,\varphi_n\rangle_w}{h_n}=\frac{\int_{-1}^{1}(1+x^2)f(x)\varphi_n(x)\,dx}{\int_{-1}^{1}(1+x^2)\varphi_n^2(x)\,dx}. \tag{31}$$

Thus, the approximation is

$$f_N(x)=\sum_{n=0}^{N}\frac{\langle f,\varphi_n\rangle_w}{h_n}\varphi_n(x). \tag{32}$$

The weighted projection error satisfies

$$\|f-f_N\|_w^2=\|f\|_w^2-\sum_{n=0}^{N}h_n\,a_n^2,$$

provided the expansion is interpreted as the orthogonal projection in L_w^2. This identity is particularly useful in convergence studies.

5. Fractional Mamadu-Njoseh Polynomials

A fractional extension of the MNP family has been introduced. For $\alpha\geq 0$, define

$$\varphi_n^{(\alpha)}(x)=\sum_{r=0}^{n}C_r^{(n)}\frac{x^{r\alpha}}{\Gamma(r\alpha+1)}, \tag{33}$$

where,

$$\Gamma(z)=\int_0^{\infty}t^{z-1}e^{-t}\,dt,$$

is the Gamma function.

The fractional basis is designed so that fractional differentiation produces a particularly simple structure. A published formulation gives the corresponding fractional orthogonality and derivative properties. For example, under the appropriate fractional derivative D^{α},

$$D^{\alpha}\varphi_n^{(\alpha)}(x)=-\varphi_{n-1}^{(\alpha)}(x), \tag{34}$$

with higher derivatives satisfying

$$D^{a\alpha}\varphi_n^{(\alpha)}(x)=(-1)^a\varphi_{n-a}^{(\alpha)}(x), \tag{35}$$

under the conventions and normalization of that fractional formulation. This makes the fractional MNP basis attractive for fractional differential equations.

6. Discussion

The construction and analysis of the Mamadu-Njoseh polynomials (MNPs) presented in this study establish a systematic orthogonal polynomial family on the interval $[-1,1]$ with respect to the positive weight function $w(x)=1+x^2$. Since $1+x^2>0$ throughout the interval, the associated weighted inner product defines a valid positive-definite inner-product space. The Gram-Schmidt procedure applied to the

monomial basis consequently produces a unique sequence of orthogonal polynomials up to a multiplicative constant, which is fixed here by the normalization $\varphi_n(1) = 1$. The derived MNPs exhibit a clear parity structure resulting from the symmetry of the weight function. Specifically, the even-indexed polynomials are even functions, whereas the odd-indexed polynomials are odd functions. This property significantly simplifies their algebraic representation and provides an efficient means of constructing the polynomial coefficients. The first several members obtained from the recurrence confirm this structure explicitly.

A three-term recurrence relation was established for the MNP sequence. This recurrence provides an efficient computational mechanism for generating higher-degree polynomials without repeatedly performing the complete Gram-Schmidt orthogonalization process. The coefficient recurrence obtained from the parity representation further demonstrates that the algebraic structure of the MNPs can be generated recursively.

An exact differential relation for every MNP was obtained using the Euler operator $\Theta = x\frac{d}{dx}$. Since each φ_n contains only powers $x^n, x^{n-2}, x^{n-4}, \ldots$, every member is annihilated by the operator $\prod_{j=0}^{\lfloor n/2 \rfloor}[\Theta - (n-2j)]$. Thus,

$$\prod_{j=0}^{\lfloor n/2 \rfloor}\left(x\frac{d}{dx} - (n-2j)\right)\varphi_n(x) = 0$$

This provides an exact Euler-type differential equation for each individual MNP. However, its order depends on n, namely $\lfloor n/2 \rfloor + 1$. Consequently, it should be distinguished from a fixed second-order differential equation of the type satisfied by classical polynomial families.

The weighted orthogonality of the MNPs also provides a natural approximation framework. Any function in the weighted space $L_w^2(-1,1)$ can be approximated through finite MNP expansions. The associated coefficients are obtained by weighted projection, thereby providing a direct spectral approximation procedure. This property makes the MNP family potentially useful for numerical methods for ordinary, partial, and fractional differential equations.

The fractional extension introduces a basis involving powers of x^α scaled by Gamma functions. Such a formulation can be useful in fractional differential-equation problems, although the precise fractional derivative, normalization, orthogonality, and associated operational properties must be specified and independently established before these properties are treated as general consequences of the classical MNP construction.

7. Conclusion

This study develops a systematic mathematical framework for the Mamadu-Njoseh polynomial (MNP) family on the interval $[-1,1]$ with weight function $w(x) = 1 + x^2$, constructed through weighted Gram-Schmidt orthogonalization and normalized by $\varphi_n(1) = 1$. The fundamental properties of the MNPs, including orthogonality, parity, norms, three-term recurrence, coefficient representation, shifted forms, and the first ten polynomi)al members, are established. An exact Euler-type differential relation is derived from their parity structure, providing a rigorous degree-dependent differential characterization, although it does not constitute a fixed second-order differential equation for the entire family. The results establish the MNPs as a useful polynomial basis for weighted approximation and provide a foundation for their further application in spectral, collocation, numerical, and fractional differential equation methods.

Author Contributions

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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