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Exponentiated Inverse Unit Teissier Distribution and Its Application to Survival Data

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Abstract: Probability distribution theory is fundamental to statistical modeling, especially in survival analysis, where correct representation of time-to-event data is critical. Classical distributions such as the Weibull and exponential have been used with great success but fall behind in modeling complex datasets with heavy-tailed behavior. The Inverse Unit Teissier Distribution (IUTD) presents a good solution to the issue; however, it is one-parameter-tailed. The authors introduced a new distribution called the Exponentiated Inverse Unit Teissier Distribution (EIUTD) as a modification of the IUTD to tackle the single-parameter constraint by incorporation of a shape parameter via exponentiation of the baseline IUTD. The present work developed the cumulative distribution function (CDF) and probability density function (PDF) of the EIUTD in a systematic way, investigated its statistical properties such as moments, quantile function, order statistics, Shannon and Renyi entropy, and skewness and kurtosis, estimated parameters using Maximum Likelihood Estimation (MLE), and performed simulation studies that showed the consistency and efficiency of the estimators for different sample sizes. The modeling capability exhibited by the EIUTD model across two different public health applications confirmed its strong performance. For the Kenya DHS 2022 child mortality data set ($n = 77$), the EIUTD produced a substantially better statistical fit than the IUTD base model with respect to both AIC (529.19 vs. 653) and BIC (533.88 vs. 655.35), while demonstrating acceptable goodness-of-fit based on the Kolmogorov-Smirnov test (KS $p = 0.1289$). For the COVID-19 recovery times of vaccinated individuals in Kenya ($n = 107$), the EIUTD model provided competitive performance (AIC = 471.96, $p = 0.1825$) when compared with both the Lognormal and Gamma models and additionally provided a clearer hazard interpretation via the α-β parameterization than either of the other models. The overall flexible parameterization capability offered by the EIUTD model suggests that it is an appropriate survival analysis method for demographic health research and also for infectious disease epidemiology.

Keywords: Exponentiated Distribution, Inverse Unit Teissier, Survival Analysis, Heavy-tailed Data, Maximum Likelihood Estimation

1. Introduction

Over the past century, statistical modeling has developed fast, driven by the need to analyze and interpret increasingly complex datasets. The core of this process is given by probability distributions, mathematically describing the behavior of random variables [25]. New distributions and the refinement of existing ones have been and will continue to be driven by real data problems, particularly in survival analysis, reliability engineering, and actuarial science.

Early statistical models, mostly produced before the mid-20th century, were chiefly parametric. These models consider that the data follow a specific probability distribution with an appropriate functional form but with unknown parameters to be estimated from the observed data. Distributions like the normal [45], exponential [34], and Poisson [22] were central to these early developments and found applicability in various scientific endeavors.

The latter half of the 20th century witnessed the rise of non-parametric methods; unlike their parametric counterparts, these methods do not assume a given probability distribution.

That is, they use ranks or order of the data to proceed with statistical analysis employing non-parametric techniques. This makes them robust to deviations from distributional assumptions [19]. For analyzing data when the distribution of underlying data is unknown or complicated, non-parametric techniques such as the Wilcoxon rank-sum test [47] and the Kaplan-Meier estimator [27] became invaluable.

This section summarizes the progression of statistical modeling, by focusing on parametric and non-parametric methods, and presenting their limitations.

Parametric methods assume specific probability distributions of the data with a prescribed functional form. They are widely applied as they have a reputation for being simple, interpretable, and efficient in estimating parameters. Some common parametric distributions include the normal, exponential, Weibull, and gamma distributions. Each has a set of characteristics that allows it to be applied to specific problems.

The normal distribution is often used to characterize continuous symmetric data, while the exponential distribution has commonly been used for survival analysis where the hazard is assumed to be constant [30]. The Weibull distribution, presented by Weibull [46], has been especially revered for its versatility in characterizing different hazard rates and has thus become the preferred choice in reliability engineering and survival analysis. Likewise, the gamma distribution is suited for modeling waiting times and events that occur after a certain delay.

Regardless of how much these methods have gained respect and popularity, they have serious limitations. The major weakness to these methods is their reliance on strong assumptions about the underlying distribution of the data. Unless those assumptions are satisfied, said models will usually get it all wrong, if not mislead. For example, while the exponential distribution's assumption of a constant hazard rate may work with a wide range of data, it sometimes fails in the real scenario where rates keep varying over time [30].

Non-parametric methods as opposed to parametric, need no particular distributional shape for the description of the data. Essentially, they rely on the data itself to determine the structure of the model. That flexibility is useful in assessing complex datasets where the unknown or really difficult to determine form of distribution exists. Some common non-parametric methods include Kaplan-Meier for survival analysis and Cox's proportional hazards model, which allows analysis of covariates without stating the baseline hazard function [14].

The Kaplan-Meier estimator [27] is widely employed in estimating survival functions for lifetime data, especially in medical research. Cox proportional hazards model, on the contrary, is a semi-parametric approach combining the flexibility of non-parametric with the interpretability of parametric models.

Non-parametric methods face criticism or have certain disadvantages. They often require larger sample sizes for obtaining the equivalent degree of precision of the parametric methods, which may have reduced efficiency in parameter estimation. Also, non-parametric models may lack the interpretability of parametric models, making it difficult to draw meaningful conclusions from the data [32]. For example, while the Kaplan-Meier estimator is a good estimator of a survival function, it does not tell us much about the variables affecting survival time [30].

Distribution Theory also referenced as the theory of generalized functions, is most employed in mathematical analysis, probability theory, and statistical modeling. This enables a rigorous framework to handle functions that are not ordinarily defined, like Dirac's delta function [15]. The roots of distribution theory date back to the 18th and 19th centuries when probability distributions were developed by mathematicians, such as Pierre-Simon Laplace and Carl Friedrich Gauss, to describe random variables. The normal distribution, introduced by Gauss, soon became a fundamental tool in statistical analysis and error theory [18].

In the beginning of the 20th century, mathematicians sought to extend the functions so as to accommodate singularities and discontinuities. Paul Dirac, however, was the first to introduce the delta function in quantum mechanics as an idealized function whose integral equals one while being zero elsewhere, leading to challenges in traditional analysis [15]. In the 1940s, the French mathematician Laurent Schwartz captured what we now call the distributions, or generalized functions. This framework would permit the differentiation of non-smooth functions and established a rigorous foundation from which to solve differential equations with singularities. This inversion opened up a whole new avenue for mathematical physics and functional analysis.

Schwartz's theory became an essential tool for solving PDEs, especially in approachable areas, such as physics and engineering, in which wave equations, heat equations, and fluid dynamics progressed during this time considerably [23]. In probability theory, distribution functions describe the likelihood of outcomes in stochastic processes. Development of characteristic functions, moment-generating functions, and Levy processes further enriched the understanding of statistical distributions [10].

Nowadays, distribution theory has applications in machine learning, signal processing, and financial mathematics. It also finds its place in Bayesian inference, econometrics, and survival analysis through generalized distributions [11]. Distribution theory has moved, from probability distributions to Schwartz's generalized functions, to be a powerful mathematical construction. Its influence spans across pure and applied mathematics thus making impact in modern analysis and computational techniques.

Flexible Models are a new class of methods that, in part, advance beyond traditional parametric and non-parametric methods and, as a primary goal, these methods demonstrate an increased ability to represent complex data sets, particularly for data sets with heavy-tailed structures. Heavy-tailed distributions, characterized by a higher probability of extreme values, are often encountered in survival analysis, reliability engineering, and actuarial science. Conventional distributions, such as Weibull and exponential, are unable to sufficiently account for these extreme values; thus, their predictions and conclusions are deficient in accuracy.

The inverse unit Teissier distribution (IUTD) represents a fairly recent contribution to probability distributions, introduced by [5]. The IUTD is derived from the Unit Teissier Distribution (UTD) via inverse transformation; thus, it is well-defined to model data on $(1, \infty)$. However, while the IUTD satisfies certain application areas, it is encumbered in flexibility in modeling complex datasets due to its single-parameter structure. This limitation has led to the development of more flexible distributions, such as the exponentiated IUTD, which introduces an additional parameter (shape parameter) through exponentiation.

The development of new probability distributions has long occupied an eminent place within statistical research simply because of the model necessity for more elaborated datasets. While parametric and non-parametric are powerful tools to analyze data, they often fall short when faced with specific data characteristics, such as heavy tails. The Exponentiated Inverse Unit Teissier Distribution (EIUTD), proposed to mitigate the aforementioned limitations, aims at being a powerful model while modeling heavy-tailed survival data.

1.1. Problem Statement

Alsadat et al. [5] introduced the inverse unit Teissier distribution (IUTD) through the inverse transformation of the unit Teissier distribution. The IUTD was supported by the interval $(1, \infty)$. Despite some advantages, the IUTD introduced limitations on flexibility for modeling more complicated data primarily due to its single-parameter structure. While it performed well in certain applications, its ability to capture a wide range of real-life phenomena was constrained by this simplicity. To address this, we proposed the exponentiated inverse unit Teissier distribution (EIUTD).

It was theorized that exponentiation enhanced the model flexibility and applicability of the original IUTD by introducing a shape parameter, allowing a wider range of real-life phenomena. The new model captured the different behaviors that characterized heavy-tailed distributions. In this research work, statistical properties including moments, quantile function, hazard rate, and entropy for the EIUTD were derived, and parameter estimation was based upon Maximum Likelihood Estimation (MLE).

The EIUTD was also rigorously tested against real-world survival datasets while also comparing with other models including the Weibull, Lognormal, Gamma, and Exponential. This comparative study showed that the EIUTD had better accuracy and applicability for modeling heavy-tailed survival data.

1.2. Objectives

1.2.1. General Objective

To develop the Exponentiated Inverse Unit Teissier Distribution, derive its statistical properties, estimate the parameters and investigate goodness of fit of the distribution.

1.2.2. Specific Objectives

1) To develop the EIUTD model by extending the IUTD through exponentiation.
2) To derive the statistical properties of the EIUTD.
3) To estimate the parameters of the EIUTD.
4) To assess the performance of the estimators through simulation and goodness of fit of the distribution.

1.3. Significance of the Study

The development and analysis of new probability distributions is one of the most fundamental areas of statistical research, justified by the ever-increasing complexity and diversity of data encountered in different areas of application. While existing distributions represent a very rich toolbox for modeling data, they often failed in scenarios involving complex data characteristics, such as heavy tails in survival analysis [30].

Introduced by [5], the Inverse Unit Teissier Distribution (IUTD) was a notable development. However, given that it was based on a single parameter, it was relatively limited in the degree of freedom offered to describe the peculiarity of intricate real-life phenomena [21]. This was particularly important in survival analysis because proper representation of tail behavior could facilitate explanation and prediction of the occurrence of events of interest. In general, survival data tended to have some amount of tail weightiness [16], meaning there were extreme values that were more likely than lighter tailed models would suggest. The IUTD, in its original form, did not suffice when faced with implant performance and extremes of survival data.

This study was therefore justified by the need for more flexible models capable of accurately representing heavy-tailed survival data. The proposed Exponentiated Inverse Unit Teissier Distribution (EIUTD) addressed this need by introducing a shape parameter through exponentiation [34, 35]. In this way, the EIUTD was established, statistical properties were indicated, and applicability can be shown using survival datasets.

The EIUTD was a valuable resource for researchers working in various areas, including use in medical research, reliability engineering, and others working with time to event statistics. The EIUTD made great advances to both the prediction given and data interpretation from survival experiments. Accordingly, this content provided better decision making to the above fields from the research work completed on the EIUTD.

1.4. Scope of the Study

This study aimed to develop the Exponentiated Inverse Unit Teissier Distribution (EIUTD), which was obtained by exponentiating the IUTD. The mathematical development was achieved with an expression for the EIUTD CDF (cumulative distribution function) and PDF (probability density function), plus the main statistical properties of the EIUTD such as moments, quantile functions, hazard functions and entropy to elaborate practical use of the EIUTD was completed. Estimation of parameters for the EIUTD was performed using maximum likelihood estimation (MLE). The performance of these estimates was assessed through simulation studies. The fit of the EIUTD was tested with actual survival data. The performance of the EIUTD was compared with existing distributions through statistical criteria such as AIC, BIC, and the Kolmogorov-Smirnov test. The EIUTD was applied to real survival data to substantiate its practical utility in modeling heavy-tailed situations. In the analysis, the results were compared against those obtained from other distributions commonly used in survival analysis. The scope of this study was limited to the analysis of complete (uncensored) observations. The extension of the EIUTD to accommodate censored survival data is identified as a direction for future research.

1.5. Introduction

This chapter explores the development of methods for generating new distributions by studying both classical and more modern approaches. The leading emphasis is on methods of exponentiation, which turns out to be central to the development of the EIUTD. The chapter also reviews some of the properties and applications of some related distributions: for instance, the IUTD, and other exponentiated families. Also, the chapter indicates the gaps this study intends to fill by establishing the area where more flexible models are required for an accurate modelling of survival data, particularly the heavy tails. This review serves as a foundation for understanding the motivation behind the EIUTD and its potential contributions to the field of statistical modeling.

1.6. Methods of Generating Distributions

Constructing new probability distributions has always fascinated statisticians owing to the increasing complexity of data in all fields. Various methods have been developed to construct new distributions on the basis of previously established distributions, each with its own advantages and disadvantages. Some authors have classified these under several general approaches that date back to before and after 1980.

1.6.1. Pre-1980 Methods

For much of the period before 1980, the development of new probability distributions frequently rested on certain basic techniques based on transformations of existing distributions. Classical methods have had great significance in the early development of distribution theory and still have their place in a few contexts. They generally served as stepping stones toward more sophisticated techniques and still offer valuable insights into the relationships among various families of distributions.

The quantile method, formalized in statistical literature by the mid-20th century and notably discussed by [40], provides a framework for generating probability distributions through their quantile functions, which are the inverses of cumulative distribution functions (CDFs). The quantile method, known also as inverse transform sampling, is useful in situations where the quantile function has a simple analytical form, even if the corresponding CDF or probability density function (PDF) is complex. Parzen [40] draws attention to the fact that the quantile function, $Q(u) = F^{-1}(u)$ for $u \in (0,1)$, leads to generation of random variables $X = Q(U)$ where $U \sim \text{Uniform}(0,1)$. The quantile method was developed in response to the need for flexible simulation methods in the early days of computing in statistics. The method is useful to model specific tail behavior, or skewness, that ordinary distributions cannot model. For example, Tukey's lambda distribution, introduced by Joiner and Rosenblatt (1971), is a good example where the quantile method was used for generating a specific distribution, with different shapes, demonstrating its utility in robust statistics and data modeling.

The translation method, formalized by [24], is used to create new distributions with a transformation of a standard random variable. This is typically expressed as $Y = g(X)$, where X as a random variable is known to have a distribution such as the normal distribution. Johnson's work was directed at the normal distribution as a reference distribution and he proposed various transformations including $Y = \mu + \sigma \sinh^{-1}((X - \gamma)/\lambda)$ in order to create the Johnson system of distributions, which includes the lognormal, bounded, and unbounded families. This work was prompted by a need for alternative distributions that would describe non-normal data, especially when data in Engineering and Biology exhibited varying levels of skewness and kurtosis. The translation method describes the location-scale family, which allows the statistician to shift the mean and variance of the distribution and retain the shape of the reference distribution. due to its simplicity and interoperability, the translation method became a cornerstone in pre-1980 distribution theory, as noted by [25], who later extended its applications to multivariate settings.

The differential equation method, pioneered by [41], constructs probability distributions by defining their PDFs as solutions to specific differential equations. Pearson's system of distributions, which includes types such as the normal, beta, and gamma distributions, was developed to model a wide range of empirical data shapes encountered in biometric and economic studies. The method consists of defining differential equation of the form $\frac{1}{f(x)}\frac{df(x)}{dx} = \frac{x-a}{b_0+b_1x+b_2x^2}$, where $f(x)$ is the PDF, and the parameters a, b_0, b_1, b_2 determine the distribution type based on the roots of the denominator. The development of the approach was inspired by a need to systematize the classification of distributions by their moment properties, namely skewness and kurtosis. Pearson's development gave a unifying framework for constructing distributions that could capture different kinds of empirical data. Kendall and Stuart [28] developed this technique in greater detail, particularly in a statistical inference context, showing that it could be used to produce distributions with analytically solvable moment expressions.

1.6.2. Post-1980 Methods

In the years following 1980, the development of the new distribution generation methods took off dramatically. This was against a myriad of backgrounds, such as massive increases in computational power, the availability of complex datasets, and a greater need for flexible and specialized models that are capable of capturing intricate data patterns. In these ways, modern techniques have tremendously widened the arsenal available to statisticians and resulted in the formulation of a plethora of new distribution families.

The Exponentiated Family of Distributions was proposed by [20]. They proposed a model that incorporated all types of failure rates and was more flexible. The motivation to model survival data that is characterized by non-monotonic failure rates was that for a disease, the aspect of mortality in a study population attains a peak after some period of time then starts to decline. They developed the exponentiated Weibull distribution and studied its mathematical properties. Nadarajah & Kotz [37] extended the idea of R. Gupta et al. [20] by introducing four more exponentiated type of distributions that generalized the Weibull, gamma, Gumbel and the Frechet distributions. They studied mathematical properties of the distributions.

The Marshall-Olkin family of distributions was proposed by [34]. Their model was to extend distributions by simply adding a parameter and to get a new CDF, $G(x) = \frac{\alpha F(x)}{1-(1-\alpha)F(x)}$ using some baseline CDF, $F(x)$ and $\alpha > 0$. It was motivated by the need to be able to model dependent random variables in reliability engineering, for example, systems that are exposed to random external shocks. The new model they proposed could model data with all sorts of hazard rate shapes including increasing, decreasing, or bathtub-shaped hazard rates. Al-Saiari et al. [4] used this model with a Burr XII distribution and Mansour et al. (2018) did the same using a Fréchet distribution. The Marshall-Olkin family of distributions represents an important contribution to the application and development of survival analysis.

The method of compounding distributions was proposed by [1]. They created the exponential-geometric distribution, a mixing of an exponential distribution and a geometric parameter distribution. Their motivation was to create a model for lifetime data that had declining failure rates to reflect biological systems that have early failures. This compounding approach, where one distribution's parameter follows another, allowed a way to model overdispersion in count data. Adamidis et al. [2] extended this idea to other geometric mixtures, while Chahkandi and Ganjali [12] developed exponential power series distributions and these expanded the applications of this approach into other areas such as medical and ecological modeling.

Eugene, Lee and Famoye [17] proposed the beta-generated family of distributions. They introduced a model which used the beta distribution as a generator and applied the incomplete beta function to a baseline CDF $F(x)$ producing a new CDF $G(x) = I_{F(x)}(a, b)$, motivated by the need to model skewed, symmetric, or bimodal data in statistical applications, such as those found in hydrology and finance. The beta-normal distribution was a specific instance of this generator that could demonstrate this flexibility. Jones [26] explored some theoretical properties of the family, and Zografos and Balakrishnan [48] generalized it to gamma-generated distributions further extending its statistical application.

The concept of transmutation was proposed by [43]. They presented the quadratic transmuted family by composing a distribution's CDF with the quantile function of another, thereby forming distributions similar to the transmuted extreme value distribution. This was motivated by the need to model extreme market events in financial mathematics with tailored asymmetry and tail behavior. The transmutation method gave the possibility to adjust kurtosis and skewness of symmetric distributions. Aryal and Tsokos [9] created the transmuted Weibull distribution, while Merovci [35] developed for the exponential distribution, subsequently increasing the application of transmutation methods to risk analysis.

The Kumaraswamy generated family of distributions was proposed by [13]. They introduced a model that used the Kumaraswamy distribution to generate new distributions, substituting its CDF for the beta CDF in a process analogous to beta-generated families, motivated by the need for computationally efficient models for bounded-support data in hydrology and engineering, where beta-generated models were complex. If X follows a Kumaraswamy distribution with parameters a and b, and $F(x)$ is the baseline CDF, the resulting CDF offered simplicity and flexibility. Nadarajah et al. [39] studied its properties, and Hussain (2013) explored applications in reliability modeling.

The T-X family of distributions, introduced by [6], generates new distributions by transforming a baseline cumulative distribution function (CDF) $F(x)$ through a function $W(F(x))$, yielding a new CDF $G(x) = \int_a^{W(F(x))} r(t)\, dt$, where $r(t)$ is the density of a random variable T, motivated by the need for a flexible framework to model complex data shapes in statistical applications like reliability and finance. This general approach includes distributions like the Weibull-normal, where T has a Weibull distribution and $F(x)$ is normal. Alzaatreh et al. [7] broadened this approach to T-normal distributions, which build on the normal distribution and permit more flexibility in skewness or heavy-tailed data. Lee, Famoye, and Alzaatreh [33] reviewed methods for generating distributions, including the T-X family, highlighting its broad applicability in statistical modeling.

1.7. Research Gap

One of the central gaps in existing statistical modeling was the lack of flexible distributions tailored to heavy-tailed survival data. Most existing models lacked the flexibility in tail behavior necessary to effectively represent the intricacies of such data, which made it difficult to draw accurate inferences and predictions. Although the IUTD was a promising distribution, its single-parameter form limited its capacity to model heavy tails comprehensively. In this study, we addressed this gap by proposing the Exponentiated Inverse Unit Teissier Distribution (EIUTD), which generalized the IUTD via exponentiation to incorporate a shape parameter. This provided improved flexibility in tail behavior as well as improved analysis of the heavy-tailed survival data. This project explored the properties of EIUTD, developed parameter estimation methods, and compared the results with the existing models. Importantly, it demonstrated increased accuracy and utility in this area of statistical modeling.

2. Methodology

2.1. Introduction

In this chapter, the research methodology used in the development and analysis of the Exponentiated Inverse Unit Teissier Distribution (EIUTD) is presented. The chapter is organized to understand the distribution as a whole, statistical properties of the distribution, estimation of parameters, and goodness-of-fit tests. The EIUTD distribution is obtained by exponentiating the Inverse Unit Teissier Distribution (IUTD), and its properties are analyzed in depth for assessing its suitability for modeling actual data.

2.2. Exponentiated Inverse Unit Teissier Distribution

Here the EIUTD is introduced. This starts with giving the CDF and PDF of the IUTD, followed by a summary of the exponentiated family of distributions. Then the process of generating the EIUTD is given, along with the CDF and PDF of the derived distribution.

2.2.1. CDF and PDF of the Inverse Unit Teissier Distribution (IUTD)

The Inverse Unit Teissier Distribution (IUTD) is a new lifetime distribution proposed by [5]. It is defined for $x > 1$ and contains a single shape parameter $\beta > 0$. The IUTD is quite useful for modeling lifetime data, but being single-parameter, it lacks the capability to model intricate patterns in actual data. The cumulative distribution function (CDF) and probability density function (PDF) of IUTD are as follows:

CDF of IUTD:

$$R(x;\beta) = 1 - e^{1-x^{\beta}} x^{\beta}, \quad x > 1, \beta > 0. \tag{1}$$

PDF of IUTD:

$$r(x;\beta) = \beta e^{1-x^{\beta}} x^{\beta-1}(x^{\beta} - 1), \quad x > 1, \beta > 0. \tag{2}$$

The IUTD is obtained by applying the inverse transformation method to the unit Teissier distribution. Although it is quite suitable for some kinds of data, its single-parameter nature limits its capacity to model more intricate phenomena. In order to remove this drawback, we introduce the Exponentiated Inverse Unit Teissier Distribution (EIUTD) by adding a shape parameter $\alpha > 0$ to gain flexibility.

2.2.2. CDF and PDF of the Exponentiated Family of Distributions

The exponentiated family of distributions is a method of generalization that adds flexibility to a base distribution through the addition of a new shape parameter $\alpha > 0$. If $R(x)$ is the CDF of the base distribution, then the CDF and PDF of the exponentiated distribution are as follows:

CDF of Exponentiated Distribution:

$$G(x;\alpha) = [R(x)]^{\alpha}, \quad \alpha > 0. \tag{3}$$

PDF of Exponentiated Distribution:

$$g(x;\alpha) = \alpha [R(x)]^{\alpha-1} r(x), \quad \alpha > 0, \tag{4}$$

where $r(x)$ is the PDF of the baseline distribution.

2.2.3. Generation of the Exponentiated Inverse Unit Teissier Distribution (EIUTD)

The Exponentiated Inverse Unit Teissier Distribution (EIUTD) is generated by applying the exponentiated family approach to the IUTD. Let $R(x;\beta)$ be the CDF of the IUTD as defined in Equation (1). The CDF and PDF of the EIUTD are obtained by exponentiating the CDF of the IUTD and applying the exponentiated family formulas given in Equations (3) and (4). The resulting CDF and PDF of the EIUTD are:

CDF of EIUTD:

$$G(x;\alpha,\beta) = \left[1 - e^{1-x^{\beta}} x^{\beta}\right]^{\alpha}, \quad x > 1, \beta > 0, \alpha > 0. \tag{5}$$

PDF of EIUTD:

$$g(x;\alpha,\beta) = \alpha\beta e^{1-x^{\beta}} x^{\beta-1}(x^{\beta} - 1)\left[1 - e^{1-x^{\beta}} x^{\beta}\right]^{\alpha-1}, \quad x > 1, \beta > 0, \alpha > 0. \tag{6}$$

Introduction of this new parameter α considerably expands the utility of the distribution and allows it to describe more types of real-world phenomena. The EIUTD can now model more complex patterns in the data and is thus appropriate for reliability analysis, survival analysis, and others where flexibility for lifetime distributions are required.

2.3. Statistical Properties

We consider statistical properties of the Exponentiated Inverse Unit Teissier Distribution (EIUTD) in this section. These are needed in order to explain the nature of the distribution and could prove valuable when performing data fitting in practical applications. Moments, moment generating function (MGF), quantile function, entropy, order statistics, skewness and kurtosis, are among what we study.

2.3.1. Moments

The moments of a distribution provide insights into its central tendency, variability, and shape. The r-th moment of the EIUTD is given by:

$$E(X^r) = \int_1^\infty x^r g(x;\alpha,\beta)\,dx, \tag{7}$$

where $g(x;\alpha,\beta)$ is the PDF of the EIUTD as defined in Equation (6). The moments can be used to derive other important properties, such as the mean ($E(X)$) and variance ($\text{Var}(X) = E(X^2) - [E(X)]^2$).

2.3.2. Moment Generating Function (MGF)

The moment generating function (MGF) is a useful tool for deriving moments and other properties of a distribution. The MGF of the EIUTD is defined as:

$$M_X(t) = E(e^{tX}) = \int_1^\infty e^{tx} g(x;\alpha,\beta)\,dx, \tag{8}$$

where t is a real number. The MGF can be used to compute moments by taking derivatives with respect to t:

$$E(X^r) = \left.\frac{d^r M_X(t)}{dt^r}\right|_{t=0}.$$

2.3.3. Quantile Function

The quantile function is the inverse of the cumulative distribution function (CDF). In the case of the EIUTD, the quantile function $Q(p)$ is given through solving $G(Q(p);\alpha,\beta) = p$ for $Q(p)$, where $G(x;\alpha,\beta)$ is the CDF of the EIUTD as defined in Equation (5). The quantile function is given by:

$$Q(p) = G^{-1}(p;\alpha,\beta), \quad 0 < p < 1. \tag{9}$$

The quantile function is useful for generating random samples and computing percentiles.

2.3.4. Entropy

Entropy measures the uncertainty or randomness associated with a distribution. For the EIUTD, the entropy is given by:

$$H(X) = -\int_1^\infty g(x;\alpha,\beta)\log g(x;\alpha,\beta)\,dx, \tag{10}$$

where $g(x;\alpha,\beta)$ is the PDF of the EIUTD. Higher entropy values indicate greater uncertainty in the distribution.

2.3.5. Order Statistics

Order statistics are useful for analyzing the behavior of extreme values in a dataset. For a random sample $X_1, X_2, \ldots, X_n$ from the EIUTD, the PDF of the k-th order statistic $X_{(k)}$ is given by:

$$\begin{aligned} f_{X_{(k)}}(x) = {} & \frac{n!}{(k-1)!(n-k)!}[G(x;\alpha,\beta)]^{k-1} \\ & \times[1-G(x;\alpha,\beta)]^{n-k} g(x;\alpha,\beta), \end{aligned} \tag{11}$$

where $G(x;\alpha,\beta)$ and $g(x;\alpha,\beta)$ are the CDF and PDF of the EIUTD, respectively.

2.3.6. Skewness and Kurtosis

Skewness and kurtosis are measures of the asymmetry and tail heaviness of a distribution, respectively. For the EIUTD, skewness and kurtosis are derived from the moments of the distribution:

$$\text{Skewness} = \frac{E[(X-\mu)^3]}{\sigma^3}, \tag{12}$$

$$\text{Kurtosis} = \frac{E[(X-\mu)^4]}{\sigma^4}, \tag{13}$$

where $\mu = E(X)$ is the mean and $\sigma = \sqrt{\text{Var}(X)}$ is the standard deviation.

2.4. Parameter Estimation

This section considers MLE parameter estimation for the EIUTD. It is the most widely used statistical method for estimating the parameters of a distribution by maximizing its likelihood function. MLE is a well-known statistical method for parameter estimation of a probability distribution by maximizing a likelihood function. MLE parameter values are those that maximize the likelihood function, or that make the observed data most likely to occur.

2.4.1. Maximum Likelihood Estimation (MLE)

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from the EIUTD with parameters α and β. The likelihood function, denoted by $L(\alpha,\beta|x_1,x_2,\ldots,x_n)$, is defined as the joint probability density function (PDF) of the sample, treated as a function of the parameters:

$$L(\alpha,\beta|x_1,x_2,\ldots,x_n) = \prod_{i=1}^n r(x_i;\alpha,\beta) \tag{14}$$

where $r(x_i;\alpha,\beta)$ is the probability density function (PDF) of the EIUTD. For ease of maximization, one usually deals with the natural logarithm of the likelihood function, called the log-likelihood function, and is denoted as $\ell(\alpha,\beta|x_1,x_2,\ldots,x_n)$:

$$\ell(\alpha,\beta|x_1,x_2,\ldots,x_n) = \sum_{i=1}^n \log r(x_i;\alpha,\beta). \tag{15}$$

The MLEs of α and β, $\hat\alpha$ and $\hat\beta$ respectively, are found by maximizing the log-likelihood function. This is generally accomplished by taking partial derivatives of $\ell(\alpha,\beta)$ with respect to α and β, equating them equal to zero, and solving the consequent system of equations:

$$\frac{\partial \ell(\alpha,\beta)}{\partial \alpha} = 0, \quad \frac{\partial \ell(\alpha,\beta)}{\partial \beta} = 0. \tag{16}$$

As the PDF of EIUTD is of complicated form, it is not likely to have closed-form solutions for these equations. Thus, numerical optimization methods, i.e., BFGS (Broyden–Fletcher–Goldfarb–Shanno), Newton-Raphson or other gradient-based algorithms, are utilized to maximize the log-likelihood function. These algorithms initialize with some starting values for the parameters and recursively update them until convergence to a local maximum is obtained. The information matrix observed, the negative Hessian matrix of the log-likelihood function at the MLEs, may be used to estimate the variance-covariance matrix of the estimators, giving estimates of their standard errors.

2.5. Goodness of Fit Criterion

One of the key steps in model validation is checking the good fit of the statistical model with observed data-evaluating how well the chosen model represents the underlying data generating process. There are several ways of assessing goodness of fit, each having its own advantages and drawbacks.

2.5.1. Methods for Assessing Goodness of Fit

This section describes various methods that will be used in this research for evaluating the fit of the EIUTD against survival data.

Akaike Information Criterion (AIC): Akaike's Information Criterion is a model selection criterion broadly used to balance the goodness of fit with the model complexity [3]. AIC is

$$AIC = -2\ell(\hat{\theta}) + 2k \tag{17}$$

where $\ell(\hat{\theta})$ is the maximized log-likelihood of the model, and k is the number of parameters in the model. A smaller value of the AIC implies a better model fit. AIC penalizes the parametric models to avoid overfitting. Among a set of models, the model with the lowest AIC is selected.

Bayesian Information Criterion (BIC): BIC is another criterion for model selection with similar considerations for goodness of fit and model complexity [42]. It is defined as:

$$BIC = -2\ell(\hat{\theta}) + k\log(n) \tag{18}$$

where n is the sample size. BIC penalizes model complexity more than the AIC, especially for larger sample sizes. Like AIC, lower BIC values indicate a better fit. BIC is particularly useful in comparing models with different numbers of parameters and more so when the sample size is relatively large.

Kolmogorov-Smirnov Test (K-S Test): The K-S test is a nonparametric test that compares the empirical cumulative distribution function of the observed data to the theoretical CDF of the fitted distribution [31, 44]. The maximum absolute difference between the ECDF and the theoretical CDF is calculated. The K-S statistic is compared with a critical value to see whether or not the difference is statistically significant. A small K-S statistic suggests a good fit. The K-S test, although a formally hypothesis testing procedure for goodness of fit, is affected by deviations that occur near the center of the distribution and has a low power for small sizes of samples.

Anderson-Darling Test (A-D Test): The A-D test enhances the goodness-of-fit tests, just like the K-S test, by comparing the ECDF against the theoretical CDF [8]. Because of this, the tails take more weight into the consideration in A-D tests, making them more sensitive to deviations in the tails themselves compared to the K-S tests. This makes it especially suitable in examining the fit of distributions to heavy-tailed data. A smaller A-D statistic indicates a better fit.

Visual Assessment (Probability Plots, Q-Q Plots): In addition to formal tests, graphical assessment contributes significantly to the assessment of goodness of fit. Probability plots (P-P plots) and quantile-quantile plots (Q-Q plots) are methods for graphically relating the quantiles of observed data to the quantiles of the fitted distribution. If the model fits well, most points in the plot would fall approximately in a straight line; departure from this line indicates a lack of fit. Q-Q plots are valuable for visually assessing the fit in the tails of the distribution.

Chi-Squared Goodness-of-Fit Test: This test examines how well the observed data fit some hypothesized distribution. The data are partitioned into intervals, and observed frequencies in each interval are compared with the expected frequencies, calculated on the basis of some hypothesized distribution. In general, the larger the calculated chi-squared statistic, the worse the fit. This test is quite sensitive to how intervals are defined and is best suited for discrete data or for continuous data that have been appropriately grouped. Its use with continuous data has become much less frequent, as K-S or A-D statistics are often more appropriate for the latter.

The aforementioned methods will provide for a complete evaluation of the goodness of fit of the EIUTD to survival data; the AIC and BIC will be used for model comparison and the K-S and A-D tests will furnish formal tests for fit. An assessment of the model fit will be conducted by means of a visual examination using the probability plots and Q-Q plots, in addition to the numerical results described above. This study aims at giving robust evidence of the EIUTD's capacity to model heavy-tailed survival data by a multi-approached defense.

3. Results and Discussion

3.1. Introduction

This chapter includes the results for the Exponentiated Inverse Unit Teissier Distribution (EIUTD), including: mathematical properties, parameter estimation via Maximum Likelihood Estimation, simulation studies, and goodness-of-fit tests.

3.2. Probabilistic Functions of the EIUTD

In this part, we present the main probabilistic functions of the Exponentiated Inverse Unit Teissier Distribution (EIUTD) such as the probability density function (PDF), a survival function, and the hazard function. These functions are important for understanding the characteristics of the distribution and for statistical modeling applications. All functions are defined for $x > 1$, with parameters $\alpha > 0$ and $\beta > 0$.

The PDF of the EIUTD, derived by exponentiating the baseline IUTD (Equation 6), is:

$$g(x;\alpha,\beta) = \alpha\beta e^{1-x^{\beta}} x^{\beta-1}(x^{\beta}-1)\left[1-e^{1-x^{\beta}}x^{\beta}\right]^{\alpha-1}, \quad x>1, \alpha>0, \beta>0$$

The survival function $S(x;\alpha,\beta)$ of the EIUTD, representing the probability that a variable exceeds x, is derived from the CDF (Equation 5):

$$S(x;\alpha,\beta) = 1-\left[1-e^{1-x^{\beta}}x^{\beta}\right]^{\alpha}, \quad x>1 \qquad (19)$$

The corresponding hazard rate function (HRF), which characterizes the instantaneous risk of failure, is given by:

$$h(x;\alpha,\beta) = \frac{g(x;\alpha,\beta)}{S(x;\alpha,\beta)} = \frac{\alpha\beta e^{1-x^{\beta}} x^{\beta-1}(x^{\beta}-1)\left[1-e^{1-x^{\beta}}x^{\beta}\right]^{\alpha-1}}{1-\left[1-e^{1-x^{\beta}}x^{\beta}\right]^{\alpha}} \qquad (20)$$

3.2.1. Shape Characteristics of the PDF

Figure 1 shows the Exponentiated Inverse Unit Teissier Distribution (EIUTD) probability density functions (PDF), which highlight shape regimes that result from interaction effects between the shape parameter α and curvature parameter β. For a variety of PDFs, α generally controls tail weight, while β generally alters peak sharpness. These PDFs also suggest bifurcation between parameters where α is the appropriate parameter for extreme-value modeling, while β affects central tendency characteristics, which is an important distinction for survival analysis applications.

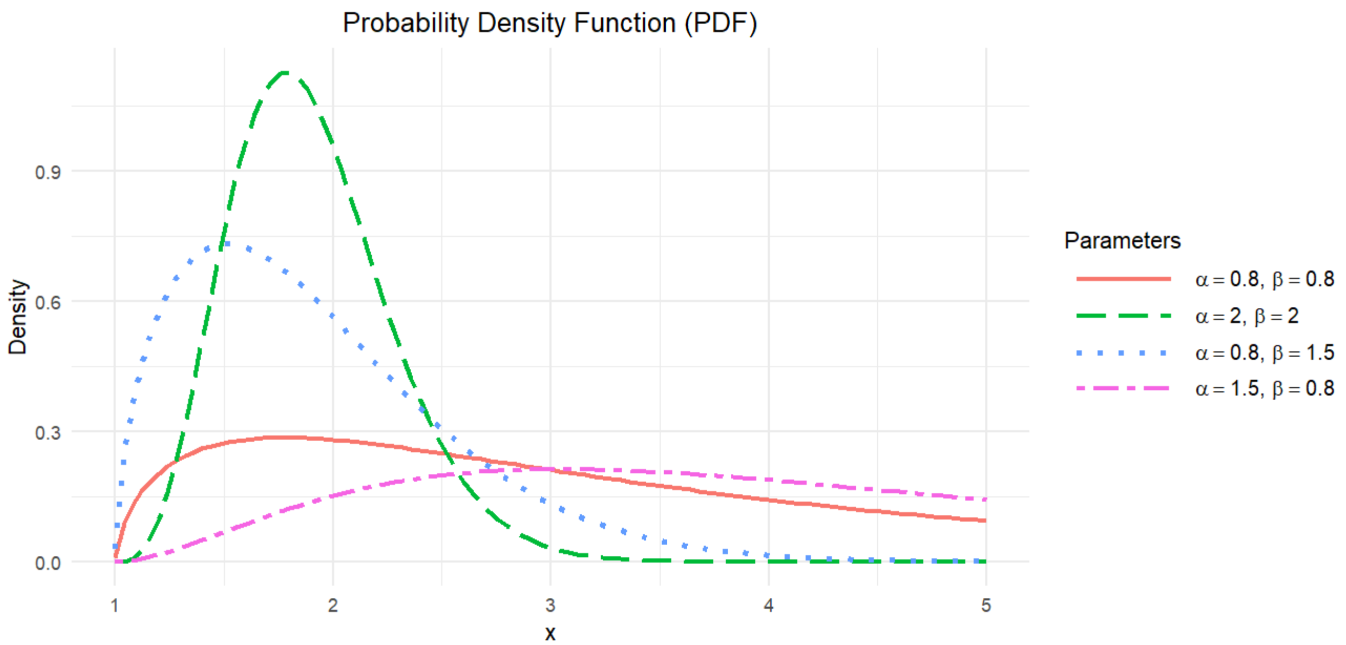


Figure 1. *Probability Density Function (PDF) of the EIUTD showing four parameter combinations.*

Proposition 3.1 (*HRF Behavior*). The hazard rate function admits three characteristic regimes:

1) Decreasing Hazard Rate (DHR) ($\alpha < 1, \beta < 1$): Models systems with high initial failure risk
2) Increasing Hazard Rate (IHR) ($\alpha > 1, \beta > 1$): Characterizes aging systems with wear-out failures
3) Bathtub Hazard Rate (BHR) ($\alpha \gtrless 1, \beta \lessgtr 1$): Exhibits decreasing-then-increasing pattern

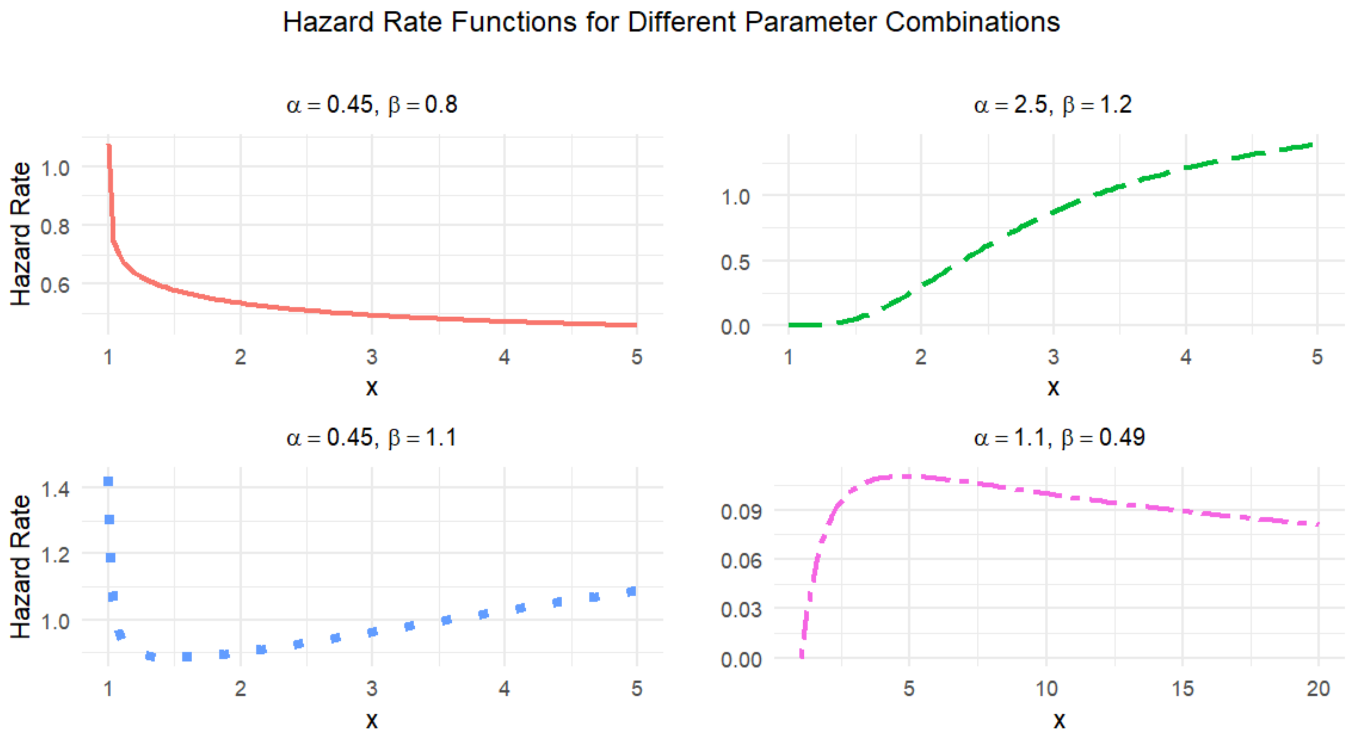


Figure 2. *Hazard Rate Functions (HRF) of the EIUTD showing four parameter combinations. Each subplot displays the hazard rate curve for specific values of shape parameters α and β.*

Proof Sketch. The HRF shape emerges from competition between the numerator $g(x)$ and denominator $S(x)$. When $\alpha < 1$, $S(x)$ decays slowly dominating the model yielding a decreasing hazard. Circumstances with $\alpha > 1$ include a rapid decline in the survival function $S(x)$ dominates the numerator $g(x)$ and rises the hazard rate. Bathtub behavior occurs when $g(x)$ temporarily dominates (decreasing phase) followed by the survival function $S(x)$ dominating (increasing phase), this would be characteristic of any heterogeneous populations with multiple failure modes.

It is this flexibility that makes the EIUTD a well matched model for survival analysis that has both monotonic and non-monotonic hazard patterns. The α-β parameter plane offers serious users a continuum of models from quickly decaying to a semi-stable failure process.

3.3. Mathematical Properties

3.3.1. Quantile Function

Quantile function, which is used for random variate generation and percentile calculations, is essential for the EIUTD. We report an explicit form of the quantile function that utilizes the Lambert W function, so that distribution quantiles can be efficiently calculated.

Proposition 3.2 (Quantile Expression). The quantile function $Q(p)$ of the EIUTD is given by:

$$Q(p) = \left[-W_{-1}\left(-(1-p^{1/\alpha})e^{-1}\right)\right]^{1/\beta}, \quad 0 < p < 1$$

where W_{-1} denotes the negative branch of the Lambert W function.

Proof.
The quantile function $Q(p)$ of the EIUTD is obtained by inverting the CDF $G(x) = \left[1 - e^{1-x^\beta}x^\beta\right]^\alpha$. Setting $G(x) = p$ and solving for x:

First, take the α-th root: $p^{1/\alpha} = 1 - e^{1-x^\beta}x^\beta$.
Next, substitute $z = x^\beta$: $ze^{1-z} = 1 - p^{1/\alpha}$.
Then rearrange: $-ze^{-z} = -(1-p^{1/\alpha})e^{-1}$.
Finally, apply the Lambert W function: $z = -W_{-1}\left(-(1-p^{1/\alpha})e^{-1}\right)$.

The quantile function is:

$$Q(p) = \left[-W_{-1}\left(-(1-p^{1/\alpha})e^{-1}\right)\right]^{1/\beta}, \quad 0 < p < 1 \tag{21}$$

where W_{-1} is the negative branch of the Lambert W function.

3.3.2. Moments

The moments of a probability distribution are fundamental indicators of the population's central tendency, dispersion, and shape characteristics. In the case of the EIUTD, we determined closed-form expressions for the moments and examined their mathematical properties.

Proposition 3.3 (General Moment Expression). The r-th raw moment of the EIUTD is given by:

$$E[X^r] = \alpha \sum_{k=0}^{\infty} (-1)^k \binom{\alpha-1}{k} e^{k+1} \left[\frac{\Gamma\left(\frac{r}{\beta}+k+2, k+1\right)}{(k+1)^{\frac{r}{\beta}+k+2}} - \frac{\Gamma\left(\frac{r}{\beta}+k+1, k+1\right)}{(k+1)^{\frac{r}{\beta}+k+1}} \right]$$

where $\Gamma(s,z) = \int_z^{\infty} t^{s-1} e^{-t} dt$ denotes the upper incomplete gamma function.

Proof. Starting from the moment definition:

$$E[X^r] = \int_1^{\infty} x^r g(x;\alpha,\beta)\, dx,$$

where $g(x;\alpha,\beta)$ is the PDF of the EIUTD:

$$g(x;\alpha,\beta) = \alpha\beta e^{1-x^\beta} x^{\beta-1}(x^\beta - 1)\left[1 - e^{1-x^\beta} x^\beta\right]^{\alpha-1}.$$

Using the substitution $t = x^\beta$, $x = t^{\frac{1}{\beta}}$, and $dx = \frac{1}{\beta} t^{\frac{1}{\beta}-1} dt$, the integral becomes:

$$E[X^r] = \alpha \int_1^{\infty} t^{\frac{r}{\beta}} e^{1-t}(t-1)\left[1 - e^{1-t} t\right]^{\alpha-1} dt.$$

Expanding $\left[1 - e^{1-t} t\right]^{\alpha-1}$ using the binomial series:

$$\left[1 - e^{1-t} t\right]^{\alpha-1} = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha-1}{k} e^{k(1-t)} t^k,$$

the moment expression becomes:

$$E[X^r] = \alpha \sum_{k=0}^{\infty} (-1)^k \binom{\alpha-1}{k} e^{k+1} \int_1^{\infty} t^{\frac{r}{\beta}+k} e^{-(k+1)t}(t-1) dt.$$

Splitting the integral:

$$\int_1^{\infty} t^{\frac{r}{\beta}+k} e^{-(k+1)t}(t-1) dt = \int_1^{\infty} t^{\frac{r}{\beta}+k+1} e^{-(k+1)t} dt - \int_1^{\infty} t^{\frac{r}{\beta}+k} e^{-(k+1)t} dt.$$

Expressing the integrals in terms of the upper incomplete gamma function $\Gamma(a,z)$:

$$\int_1^{\infty} t^{\frac{r}{\beta}+k+1} e^{-(k+1)t} dt = \frac{\Gamma\left(\frac{r}{\beta}+k+2, k+1\right)}{(k+1)^{\frac{r}{\beta}+k+2}},$$

$$\int_1^{\infty} t^{\frac{r}{\beta}+k} e^{-(k+1)t} dt = \frac{\Gamma\left(\frac{r}{\beta}+k+1, k+1\right)}{(k+1)^{\frac{r}{\beta}+k+1}}.$$

Thus, the r-th moment is:

$$E[X^r] = \alpha \sum_{k=0}^{\infty} (-1)^k \binom{\alpha-1}{k} e^{k+1} \left[\frac{\Gamma\left(\frac{r}{\beta}+k+2, k+1\right)}{(k+1)^{\frac{r}{\beta}+k+2}} - \frac{\Gamma\left(\frac{r}{\beta}+k+1, k+1\right)}{(k+1)^{\frac{r}{\beta}+k+1}} \right] \quad (22)$$

Table 1. *Moments and Statistics for the Exponentiated Inverse Unit Teissier Distribution (EIUTD).*

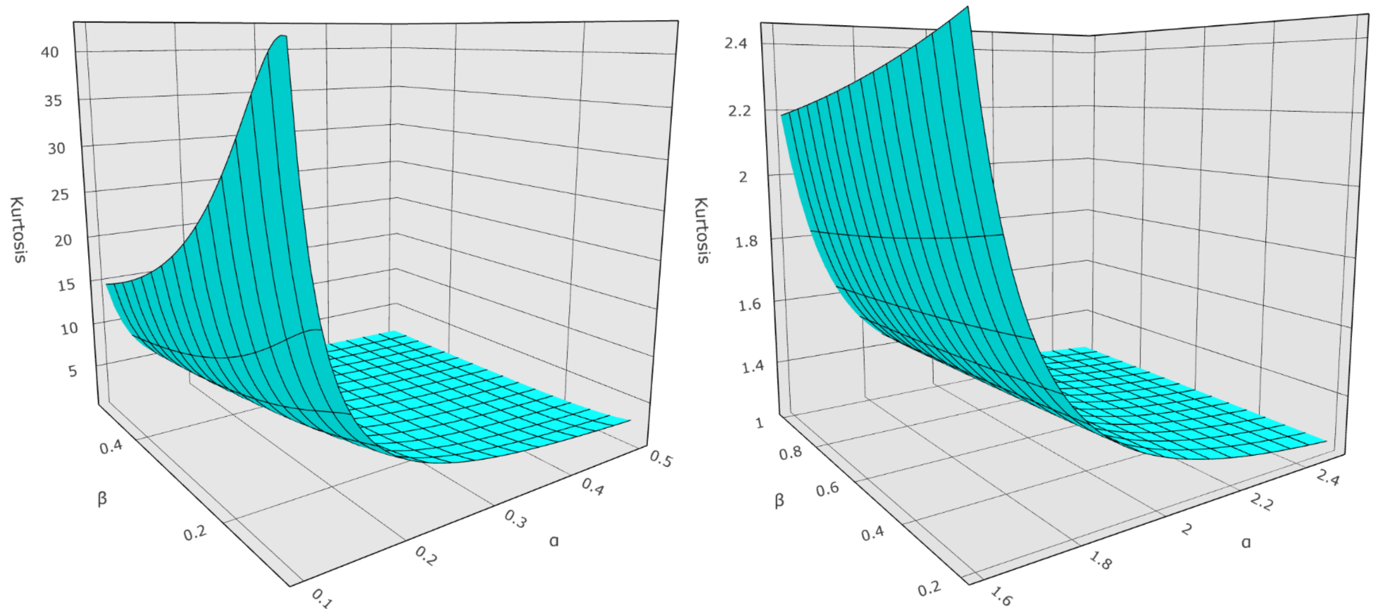
Parameters		Moments and Statistics							
0.10	0.60	2.02	11.47	177.99	4835.50	2.72	1.35	6.20	66.18
0.40	1.20	1.87	4.22	11.67	39.00	0.86	0.46	1.68	6.66
0.80	2.00	1.62	2.78	5.02	9.61	0.38	0.24	0.83	3.62
1.00	1.00	3.00	11.00	49.00	261.00	1.41	0.47	1.41	6.00
1.20	1.80	1.86	3.67	7.66	16.92	0.45	0.24	0.77	3.63
1.80	0.90	4.23	21.60	132.51	967.77	1.92	0.45	1.38	6.12
2.50	2.50	1.71	3.00	5.37	9.79	0.25	0.14	0.49	3.25

3.3.3. Bowley's Skewness and Moors Kurtosis

The tail behavior of the EIUTD distribution was analyzed using Moors kurtosis, defined as:

$$K_M = \frac{(Q_{0.875} - Q_{0.625}) + (Q_{0.375} - Q_{0.125})}{Q_{0.75} - Q_{0.25}} \quad (23)$$

where Q_p represents the p-th quantile of the distribution.



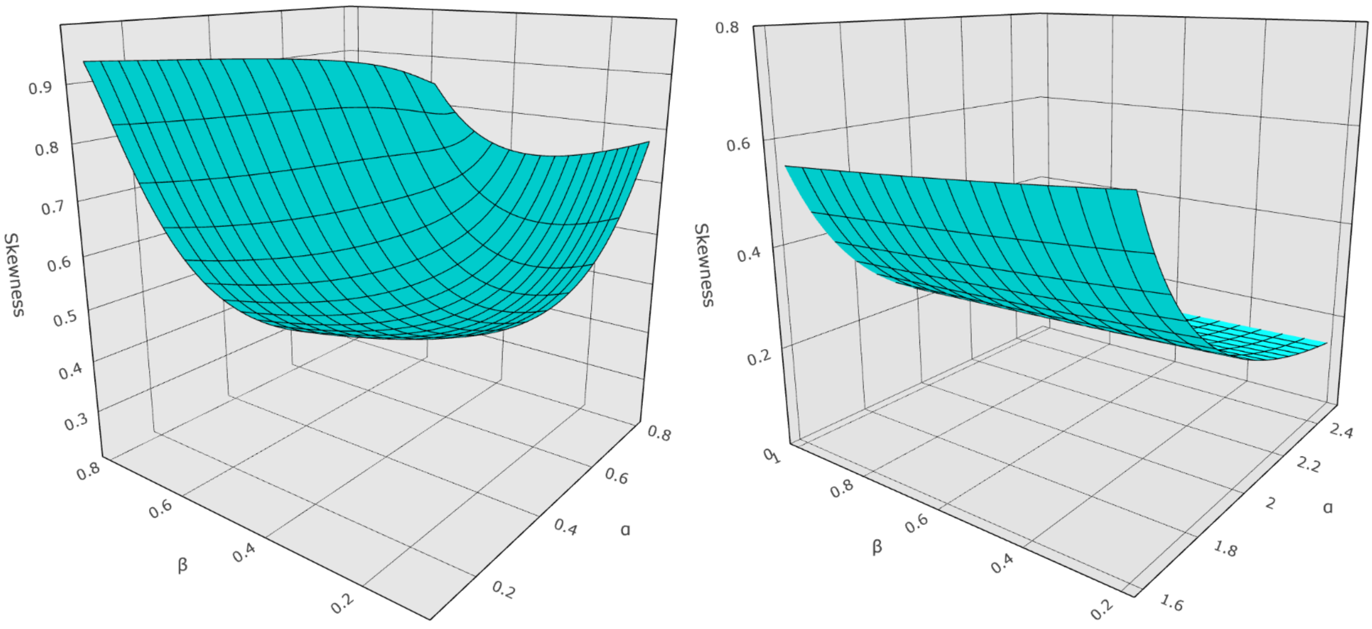
(a) *(b)*

Figure 3. *Visualization of Moors kurtosis for the EIUTD distribution. The plots demonstrate how different combinations of α and β parameters affect the tail behavior of the distribution.*

The asymmetry of the EIUTD distribution was analyzed using Bowley's skewness measure, defined as:

$$S_B = \frac{Q_{0.75} + Q_{0.25} - 2Q_{0.5}}{Q_{0.75} - Q_{0.25}} \tag{24}$$

where Q_p represents the p-th quantile of the distribution.



(a) *(b)*

Figure 4. *Visualization of Bowley's skewness coefficient across α-β parameter space.*

3.3.4. Entropy

The Shannon entropy is a measure of uncertainty in the EIUTD's probability distribution. For example, given a continuous random variable X with a PDF of $g(x)$ it is defined as:

$$H(X) = -\int_1^\infty g(x) \ln g(x)\, dx$$

For the EIUTD we obtain an analytical expression by systematically separating log-density function.

The Rényi entropy, which was proposed by Alfréd Rényi in 1961, provides a generalization of Shannon entropy by providing a parameter ϑ that allows the sensitivity to different parts of the distribution to be adjusted. The Rényi entropy is defined as:

$$H_\vartheta(X) = \frac{1}{1-\vartheta} \log \left(\int_1^\infty g(x)^\vartheta \, dx \right) \tag{25}$$

for $\vartheta > 0, \vartheta \neq 1$.

As $\vartheta \to 1$, Rényi entropy approaches Shannon entropy; however, the different choice of ϑ emphasize varying aspects of the distribution's tail behavior. For example, $\vartheta < 1$ puts more weight on low-probability events than $\vartheta > 1$ emphasizes high-probability regions. The flexibility of Rényi entropy makes it particularly well-suited to understand the EIUTD, because the tail behavior is most relevant in modeling survival data.

Proposition 3.4 (Shannon Entropy Expression). The Shannon entropy of the EIUTD is given by:

$$H(X) = -\ln(\alpha\beta) - \frac{1}{\alpha} + \mathbb{E}[X^\beta] - (\beta-1)\mathbb{E}[\ln X] - \mathbb{E}[\ln(X^\beta - 1)] \tag{26}$$

where all expectations are taken with respect to the EIUTD distribution.

Proof. Below, the Shannon entropy is computed through the following steps: First, we decompose the logarithmic term:

$$\ln g(x) = \ln(\alpha\beta) + (1 - x^\beta) + (\beta - 1)\ln x + \ln(x^\beta - 1) + (\alpha - 1)\ln\left[1 - e^{1-x^\beta} x^\beta\right].$$

Substituting into the entropy formula gives:

$$\begin{aligned} H(X) &= -\ln(\alpha\beta) - \left(1 - \mathbb{E}[X^\beta]\right) - (\beta-1)\mathbb{E}[\ln X] - \mathbb{E}[\ln(X^\beta - 1)] \\ &\quad - (\alpha - 1)\mathbb{E}\left[\ln\left(1 - e^{1-X^\beta} X^\beta\right)\right]. \end{aligned}$$

To evaluate the last expectation, let $u = 1 - e^{1-x^\beta} x^\beta$, which transforms the integral:

$$\mathbb{E}\left[\ln\left(1 - e^{1-X^\beta} X^\beta\right)\right] = \int_0^1 \ln u \cdot \alpha u^{\alpha-1} du = -\frac{1}{\alpha},$$

using the standard result $\int_0^1 u^{\alpha-1} \ln u \, du = -\frac{1}{\alpha^2}$.

Combining all terms yields the final entropy expression:

$$H(X) = -\ln(\alpha\beta) - \frac{1}{\alpha} + \mathbb{E}[X^\beta] - (\beta-1)\mathbb{E}[\ln X] - \mathbb{E}[\ln(X^\beta - 1)].$$

Note: The expectations can be computed numerically or through series expansions of the EIUTD moments.

Table 2. *Shannon and Rényi Entropy Values for EIUTD.*

α	β	Rényi ($\vartheta = 0.1$)	Rényi ($\vartheta = 0.5$)	Rényi ($\vartheta = 2$)	Shannon
0.3	0.4	7.432	4.026	0.998	2.585
0.3	0.8	3.419	1.893	0.188	1.216
0.3	1.0	2.652	1.421	-0.066	0.857
0.3	1.6	1.454	0.600	-0.588	0.184
0.6	0.4	7.509	4.484	2.823	3.513
0.6	0.8	3.488	2.210	1.523	1.818
0.6	1.0	2.717	1.706	1.158	1.394
0.6	1.6	1.511	0.836	0.461	0.624
1.0	0.4	7.564	4.772	3.479	3.994
1.0	0.8	3.535	2.372	1.821	2.050
1.0	1.0	2.759	1.838	1.386	1.577
1.0	1.6	1.544	0.921	0.586	0.732
1.7	0.4	7.621	5.030	3.969	4.381
1.7	0.8	3.578	2.487	1.987	2.195
1.7	1.0	2.795	1.920	1.491	1.673
1.7	1.6	1.566	0.949	0.602	0.755

3.3.5. Order Statistics

Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the order statistics from a random sample of size n following the EIUTD with CDF $G(x)$ and PDF $g(x)$. The PDF of the k-th order statistic is given by:

$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!}[G(x)]^{k-1}[1-G(x)]^{n-k}g(x)$$

Substituting the EIUTD CDF and PDF from Equations (5) and (6) yields:

$$f_{X_{(k)}}(x) = \frac{n!\alpha\beta}{(k-1)!(n-k)!}e^{1-x^\beta}x^{\beta-1}(x^\beta-1)\left[1-e^{1-x^\beta}x^\beta\right]^{\alpha k-1} \times \left[1-\left(1-e^{1-x^\beta}x^\beta\right)^\alpha\right]^{n-k} \tag{27}$$

where $x > 1$, $\alpha > 0$, $\beta > 0$, and $k = 1, 2, \ldots, n$.

The maximum and minimum order statistics, corresponding to $X_{(n)}$ and $X_{(1)}$ respectively, are of particular interest for analyzing extreme values in survival data. The PDF of the maximum order statistic $X_{(n)}$ is obtained by setting $k = n$:

$$f_{X_{(n)}}(x) = n[G(x;\alpha,\beta)]^{n-1}g(x;\alpha,\beta). \tag{28}$$

Substituting the CDF and PDF of the EIUTD:

$$\begin{aligned} f_{X_{(n)}}(x) &= n\left(\left[1-e^{1-x^\beta}x^\beta\right]^\alpha\right)^{n-1} \\ &\quad\times\left(\alpha\beta e^{1-x^\beta}x^{\beta-1}(x^\beta-1)\left[1-e^{1-x^\beta}x^\beta\right]^{\alpha-1}\right) \\ &= n\alpha\beta e^{1-x^\beta}x^{\beta-1}(x^\beta-1)\left[1-e^{1-x^\beta}x^\beta\right]^{\alpha(n-1)+(\alpha-1)} \\ &= n\alpha\beta e^{1-x^\beta}x^{\beta-1}(x^\beta-1)\left[1-e^{1-x^\beta}x^\beta\right]^{\alpha n-1} \end{aligned} \tag{29}$$

for $x > 1$, $\alpha > 0$, $\beta > 0$. This expression describes the distribution of the largest observation, which is useful for modeling extreme events.

The PDF of the minimum order statistic $X_{(1)}$ is obtained by setting $k = 1$:

$$f_{X_{(1)}}(x) = n[1-G(x;\alpha,\beta)]^{n-1}g(x;\alpha,\beta). \tag{30}$$

Substituting the CDF and PDF of the EIUTD:

$$\begin{aligned} f_{X_{(1)}}(x) &= n\left(1-\left[1-e^{1-x^\beta}x^\beta\right]^\alpha\right)^{n-1} \\ &\quad\times\left(\alpha\beta e^{1-x^\beta}x^{\beta-1}(x^\beta-1)\left[1-e^{1-x^\beta}x^\beta\right]^{\alpha-1}\right) \\ &= n\alpha\beta e^{1-x^\beta}x^{\beta-1}(x^\beta-1)\left[1-e^{1-x^\beta}x^\beta\right]^{\alpha-1} \\ &\quad\times\left(1-\left[1-e^{1-x^\beta}x^\beta\right]^\alpha\right)^{n-1}, \end{aligned} \tag{31}$$

for $x > 1$, $\alpha > 0$, $\beta > 0$. This distribution characterizes the smallest observation, relevant for assessing the earliest failure times in survival analysis.

3.4. Parameter Estimation

This subsection provides details regarding the parameter estimation of the Exponentiated Inverse Unit Teissier Distribution (EIUTD) and is primarily focused on Maximum Likelihood Estimation (MLE) to estimate the parameters α and β. It will derive the likelihood function, and the log-likelihood function for a random sample from the EIUTD, and presents the partial derivatives required to maximize the log-likelihood function. The subsection presents a simulation study to investigate performance of the MLE estimates with samples generated from EIUTD's quantile function. The root mean square error (RMSE) and Average Bias (AB) of the estimators were used to assess MLE performance across varying sample sizes. The analysis produced figures that illustrate that estimator performance increased as the sample size increased, and that MLE estimates were consistent.

3.4.1. Maximum Likelihood Estimation for EIUTD

Let $X_1, X_2, \ldots, X_n$ be an independent random sample from the EIUTD with PDF given in Equation (6). The likelihood function is:

$$L(\alpha,\beta) = \prod_{i=1}^{n}\alpha\beta e^{1-x_i^\beta}x_i^{\beta-1}(x_i^\beta-1)\left[1-e^{1-x_i^\beta}x_i^\beta\right]^{\alpha-1}$$

The log-likelihood function becomes:

$$\ell(\alpha,\beta) = n\ln\alpha + n\ln\beta + \sum_{i=1}^{n}(1-x_i^\beta) + (\beta-1)\sum_{i=1}^{n}\ln x_i + \sum_{i=1}^{n}\ln(x_i^\beta-1) + (\alpha-1)\sum_{i=1}^{n}\ln\left[1-e^{1-x_i^\beta}x_i^\beta\right]$$

The partial derivatives with respect to α and β are:

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\ln\left[1-e^{1-x_i^\beta}x_i^\beta\right]$$

$$\frac{\partial\ell}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\ln x_i + \sum_{i=1}^{n}\frac{x_i^\beta\ln x_i}{x_i^\beta-1} - \sum_{i=1}^{n}x_i^\beta\ln x_i - (\alpha-1)\sum_{i=1}^{n}\frac{e^{1-x_i^\beta}x_i^\beta(1-x_i^\beta)\ln x_i}{1-e^{1-x_i^\beta}x_i^\beta}$$

Setting $\frac{\partial\ell}{\partial\alpha} = 0$ gives the closed-form solution for α:

$$\hat{\alpha} = -\frac{n}{\sum_{i=1}^{n}\ln\left[1-e^{1-x_i^\beta}x_i^\beta\right]} \tag{32}$$

For β, we solve numerically using nonlinear optimization methods:

$$\frac{n}{\beta} + \sum_{i=1}^{n}\ln x_i + \sum_{i=1}^{n}\frac{x_i^\beta\ln x_i}{x_i^\beta-1} - \sum_{i=1}^{n}x_i^\beta\ln x_i - (\alpha-1)\sum_{i=1}^{n}\frac{e^{1-x_i^\beta}x_i^\beta(1-x_i^\beta)\ln x_i}{1-e^{1-x_i^\beta}x_i^\beta} = 0 \tag{33}$$

The MLE estimates $(\hat{\alpha}, \hat{\beta})$ are obtained by simultaneously solving (32) and (33) using numerical methods like Newton-Raphson or BFGS.

3.4.2. Simulation Study Design

We perform a simulation study to assess the MLE performance for the EIUTD parameters (α, β). The study consists of:

1) Generating random samples from the EIUTD quantile function.
2) Estimating parameters using MLE for a range of sample sizes.
3) Computing Root Mean Squared Error (RMSE) and Mean Relative Estimate (MRE).

3.4.3. Data Generation

The quantile function of the EIUTD is:

$$Q(p) = \left[-W_{-1}\left(-(1-p^{1/\alpha})e^{-1}\right)\right]^{1/\beta},$$

where W_{-1} is the Lambert W function (negative branch). We generate $N = 1000$ samples for each sample size $n \in \{50, 100, 200, 500\}$.

3.4.4. Performance Metrics

To comprehensively assess the performance of the maximum likelihood estimators (MLE) of the Exponentiated Inverse Unit Teissier Distribution (EIUTD) two complementary measures are used: Root Mean Squared Error (RMSE) and Average Bias (AB). The measures allow us to evaluate first the accuracy and second the systematic bias of the parameters.

The Root Mean Squared Error (RMSE) quantifies the overall accuracy by measuring the magnitude of estimation errors and is defined as:

$$\text{RMSE}(\theta) = \sqrt{\frac{1}{N}\sum_{j=1}^{N}(\hat{\theta}_j - \theta^{\text{true}})^2} \tag{34}$$

where:

1) $\theta \in \{\alpha, \beta\}$ represents the parameters.
2) $\hat{\theta}_j$ is the estimated value in the j-th simulation.
3) θ^{true} are the true parameter values (Set I: $\alpha^{\text{true}} = 0.4$, $\beta^{\text{true}} = 0.9$; Set II: $\alpha^{\text{true}} = 0.5$, $\beta^{\text{true}} = 1.7$; Set III: $\alpha^{\text{true}} = 1.2$, $\beta^{\text{true}} = 0.4$; Set IV: $\alpha^{\text{true}} = 1.9$, $\beta^{\text{true}} = 1.4$)
4) $N = 1000$ is the number of Monte Carlo replications.

The Average Bias (AB) measures the systematic distance between estimated parameters and true parameters and indicates both the direction and size of bias:

$$\text{AB}(\theta) = \frac{1}{N}\sum_{j=1}^{N}(\hat{\theta}_j - \theta^{\text{true}}) \tag{35}$$

Interpretation guidelines:

1) AB = 0: Indicates no systematic bias, suggesting unbiased estimation on average.
2) AB > 0: Indicates systematic overestimation of the parameter.
3) AB < 0: Indicates systematic underestimation of the parameter.

The distribution of MLE estimates $\hat{\alpha}$ and $\hat{\beta}$ for both parameter sets is visualized across different sample sizes, demonstrating the variability and convergence of the estimates.

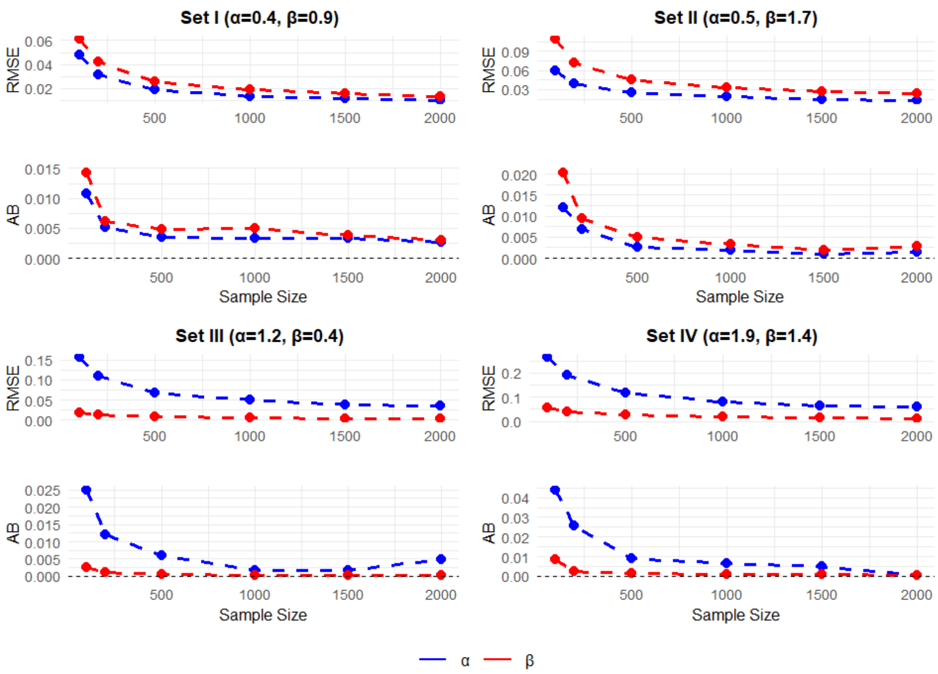


Figure 5. *RMSE and AB for α and β as functions of sample size for various parameter Sets, demonstrating decreasing RMSE and AB approaching zero, indicative of estimator consistency.*

Figure 5 demonstrates that larger sample sizes improved RMSE and converged the AB towards zero. This agrees with the theoretical property of estimator consistency. The numerical optimization successfully recovered the parameters, with increased computational complexity for larger sample sizes.

3.5. Application to Real Data

This section provides examples of how the Exponentiated Inverse Unit Teissier Distribution (EIUTD) can be applied to two types of real-world data. The first is from the Kenya Demographic and Health Survey (DHS) Child Mortality data from 2022 and the second is data regarding the length of time it took to recover from COVID-19 in Kenya. These two applications highlight the ability of the EIUTD to model survival data in each of these areas, Demographic Health Research and Infectious Disease Epidemiology, where an accurate representation of the time until an event occurs is necessary for planning for policy and intervention.

3.5.1. Dataset I: DHS Child Mortality Data

The first dataset comprises child mortality times extracted from the 2022 Kenya DHS Children's Recode file (Kenya National Bureau of Statistics, 2022). The data were analysed specifically for deaths that occurred to children aged between 12 and 59 months, ($n = 77$) and represent complete observations of the events of child mortality. This age group has been selected in order to capture the unique characteristics of the critical period of vulnerability for children outside of infancy

Table 3. Descriptive Statistics of Child Mortality Times (Months).

Statistic	Months
Minimum	12.00
Maximum	48.00
Mean	21.51
Median	18.00
Skewness	1.39
Kurtosis	4.43

The descriptive statistics contained in (Table 3) reveal several important characteristics of child mortality patterns in Kenya. The mean child mortality time was found to be 21.51 months (range 12 - 48 months) with a median of 18.00 indicating right-skewed distribution. The skewness coefficient (1.39) indicates a high degree of asymmetry whilst kurtosis (4.43) indicates a leptokurtic pattern with more extreme cases than would be expected under the normal distribution. The presence of extreme values cause modeling difficulties when using conventional distributions.

The Total Time on Test (TTT) graph illustrates a markedly concave pattern (Figure 6). The TTT plot consistently lies above the uniform reference line. This suggests that the mortality risk for this cohort of children would be accelerating toward the highest age limit. Thus, child survivors are subject to an increasing hazard profile as time passes. This type of IHR is considered to be epidemiologically plausible for children who survive the post-infancy period and experience a decline in physiological resistance to infections and nutritional deficiencies as they age within high-mortality settings. This may reflect the accumulating impact of ongoing environmental exposure or the reduction in the level of maternal antibodies.

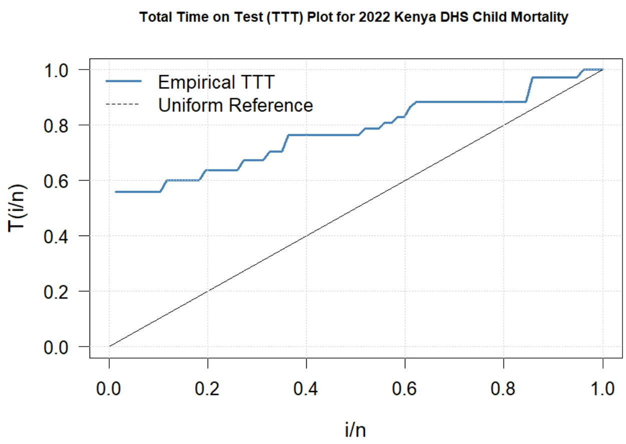


Figure 6. Total Time on Test (TTT) Plot for DHS Child Mortality Data (2022).

Based on model comparison statistics (Table 4), the EIUTD outperformed all other models across each of the assessed criteria. The EIUTD has the lowest AIC (529.19) and BIC (533.88) compared to baseline IUTD (AIC = 653.00) and the other conventional models. Results from the Kolmogorov-Smirnov test found a $p = 0.1289$ for the EIUTD, showing the EIUTD does not significantly differ from observed data whereas the IUTD, Weibull, and Exponential models are significantly different than actual data at conventional significance level ($p < 0.001$). The Anderson-Darling (AD = 1.4808) and CramÃ©r-von Mises (CvM = 0.2098) statistics were both substantially lower than the corresponding statistics for competing models, confirming the superiority of the tail behavior of the EIUTD.

Table 4. *Model Comparison Statistics: DHS Child Mortality Data.*

Model	LL	AIC	BIC	KS	AD	CvM
IUTD	-325.5022	653.0044	655.3482	$<$0.001	1.8511	0.2643
EIUTD	-262.5967	529.1933	533.8809	0.1289	1.4808	0.2098
Weibull	-275.9843	555.9686	560.6562	$<$0.001	3.6137	0.5694
Lognormal	-264.8143	533.6286	538.3162	0.0436	1.8244	0.2599
Gamma	-268.4817	540.9634	545.6510	0.0102	2.4323	0.3584
Exponential	-313.2633	628.5267	630.8705	$<$0.001	2.4246	0.3572

Maximum likelihood estimates (see Table 5) indicate that the EIUTD has $\hat{\alpha} = 19.1977$ (SE = 5.0391) and $\hat{\beta} = 0.6119$ (SE = 0.0193). The large value of $\hat{\alpha}$, along with $\hat{\beta} < 1$, results in a transitional regime where the exponentiation parameter dominates, allowing hazards to behave flexibly producing patterns that follow the increasing risk indicated by the TTT plot. The large standard error of α indicates how complicated estimation of tail behavior can be. The baseline IUTD provides $\hat{\beta} = 0.3900$, but because it has a single-parameter structure, it does not allow for the shape flexibility necessary for this data as evidenced by its poor fit statistics.

Table 5. *Maximum Likelihood Estimates: DHS Child Mortality Data.*

Model	Parameter	MLE	SE
IUTD	beta	0.3900	0.0173
EIUTD	alpha	19.1977	5.0391
	beta	0.6119	0.0193
Weibull	shape	2.4505	0.1986
	scale	24.3138	1.2031
Lognormal	meanlog	2.9915	0.0431
	sdlog	0.3786	0.0305
Gamma	shape	6.6678	1.0487
	rate	0.3100	0.0506
Exponential	rate	0.0465	0.0053

These findings demonstrate that the EIUTD is statistically valid and provides a meaningful demographic basis for evaluating child mortality outcomes and is able to identify the variations in patterns of survival that traditional models cannot.

3.5.2. Dataset II: COVID-19 Recovery Times

The COVID-19 recovery time dataset ($n = 107$) includes only those who were fully vaccinated, were under 35 years of age at the time of recovery, and met all minimum study requirement criteria. Because this recovery time subset includes all patients from a single homogeneous cohort, we will be able to accurately characterize COVID-19 recovery dynamics without being impacted by confounding from vaccination status, gender and advanced age effects.

Table 6. *Descriptive Statistics of COVID-19 Recovery Times (Days).*

Statistic	Days
Minimum	2.00
Maximum	14.00
Mean	6.10
Median	6.00
Skewness	0.91
Kurtosis	4.12

The recovery times (Table 6) range from 2 to 14 days, and the means and medians are virtually the same (6.10 days versus 6.00 days), indicating nearly symmetric data. In addition, skewness is equal to 0.91, which indicates moderate positive skewness. On the other hand, kurtosis is equal to 4.12, which demonstrates heavier tails than the normal distribution. Therefore, while most patients are expected to have a full recovery from COVID-19 within a six-day window (based on the means), some patients will experience prolonged recovery from COVID-19. This information is important to consider when planning for hospital capacity in the future.

The TTT plot in Figure 7 demonstrates a concave shape and lies above the uniform reference line (i.e., has an increasing hazard rate (IHR)). This indicates that as the length of time from infection progresses, so does the likelihood of recovery occurring. From an epidemiologic and clinical standpoint, this finding is consistent with the expected pattern of illness progression over time due to a viral infection. Specifically, as each day of an individual's immune response matures after being infected, the likelihood of an individual transitioning from being symptomatic to being fully recovered from COVID-19 increases rapidly after the initial days of infection.

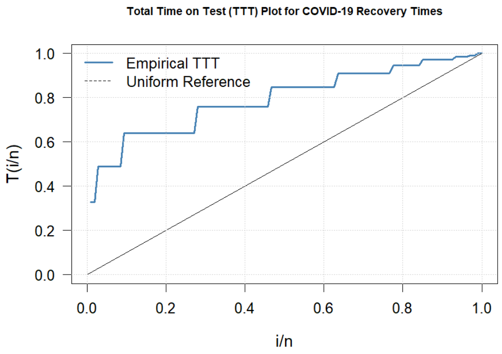


Figure 7. *Total Time on Test (TTT) Plot for COVID-19 Recovery Times (Days).*

The EIUTD has shown competitive performance (Table 7), with AIC = 471.96 for the EIUTD versus 470.74 for the Lognormal model and 470.50 for the gamma model (the Kolmogorov-Smirnov test also supported the EIUTD - $p = 0.1825$) while rejecting IUTD and the Exponential baselines very strongly ($p < 0.001$). This suggests that the EIUTD can provide a good alternative to traditional log-skewed distributions in terms of analyzing COVID-19 recovery data, especially given that it has a unified α-β parameterization that offers greater clarity in terms of interpreting hazard dynamics.

Table 7. *Model Comparison Statistics: COVID-19 Recovery Data.*

Model	LL	AIC	BIC	KS	AD	CvM
IUTD	-268.8993	539.7985	542.4714	$<$0.001	1.1905	0.2027
EIUTD	-233.9783	471.9566	477.3023	0.1825	1.2512	0.2112
Weibull	-238.6942	481.3883	486.7340	0.0603	1.7698	0.3051
Lognormal	-233.3698	470.7396	476.0853	0.2636	1.1874	0.2041
Gamma	-233.4094	470.8188	476.1645	0.1299	1.1870	0.2080
Exponential	-300.5361	603.0721	605.7449	$<$0.001	1.1851	0.2077

Maximum likelihood estimates for the EIUTD (Table 8) yield $\hat{\alpha} = 3.8833$ (SE = 0.5783) and $\hat{\beta} = 0.8379$ (SE = 0.0266). The shape parameter $\hat{\alpha} > 1$ dominates the model behavior, placing the distribution in a flexible regime that accommodates the increasing hazard pattern identified in the TTT plot.

Table 8. *Maximum Likelihood Estimates: COVID-19 Recovery Data.*

Model	Parameter	MLE	SE
IUTD	beta	0.6510	0.0245
EIUTD	alpha	3.8833	0.5783
	beta	0.8379	0.0266
Weibull	shape	2.7905	0.1962
	scale	6.8548	0.2517
Lognormal	meanlog	1.7395	0.0364
	sdlog	0.3763	0.0257
Gamma	shape	7.3830	0.9874
	rate	1.2098	0.1674
Exponential	rate	0.1639	0.0158

These applications support the EIUTD's real-world usefulness for survival analysis in many areas. The distribution results in successful development of hazard conditions that are increasing with respect to demographic health studies and infectious disease models, at the same time providing statistical support (based on good fitting indices and goodness-of-fit statistics) which gives investigators a high confidence level in their findings.

4. Conclusions and Recommendations

4.1. Introduction

In this last chapter of the work, we summarize the key conclusions resulting from our investigation into and analysis of the Exponentiated Inverse Unit Teissier Distribution (EIUTD). We provide concluding remarks regarding the mathematical properties of the distribution, the accuracy of estimating it, and the general applicability of the EIUTD in various areas. Finally, we provide concise recommendations for future research directions that outline how to continue to develop and use the EIUTD in the realm of statistical modeling and survival analysis.

4.2. Conclusions

The study successfully developed the Exponentiated Inverse Unit Teissier Distribution (EIUTD) to generalize the Inverse Unit Teissier Distribution (IUTD) by adding a shape parameter, α, thus removing the IUTD's single-parameter limitation. The EIUTD can model complex survival data with heavy-tailed characteristics and increasing hazard trends.

The EIUTD's statistical properties were derived and discussed in complete detail, providing a solid mathematical basis upon which to build. The cumulative distribution function, probability density function, moments, quantile function (using Lambert W), entropy, and order statistics were each derived mathematically and completely.

Maximum Likelihood Estimation showed that the EIUTD had strong consistency and efficiency in simulation studies, with decreasing root mean squared error, and decreasing bias as sample size increased. Goodness-of-fit testing validated that the EIUTD performed better than the traditional distributions (Weibull, Lognormal, Gamma, Exponential) when modeling child mortality data from the 2022 Kenya DHS, as well as showing it had competitive performance when modeling COVID-19 recovery times.

In demographic applications to the 2022 Kenya DHS child mortality data, the EIUTD had a superior fit (AIC = 529.19; KS $p = 0.1289$); the EIUTD outperformed the IUTD baseline (AIC = 653.00). For COVID-19 recovery times in Kenya, the EIUTD was competitive (AIC = 471.96; KS $p = 0.1825$), demonstrating its versatility across multiple domains of public health. The distribution's ability to capture increasing hazard patterns aligns with epidemiological expectations for child mortality risk progression and COVID-19 recovery dynamics.

4.3. Recommendations

More work should be conducted in the future to extend the EIUTD work within a Bayesian inference framework for improved small-sample parameter estimates for censored biomedical data. Regression formulations with covariates in the accelerated failure time or proportional hazards framework will also improve the application of the EIUTD and allow generalizable applications to clinical trials and reliability engineering. Development of multivariate formats of the EIUTD with copula methods and open-source software would allow users to consider dependent failure modes in complicated systems. Ultimately, these directions will strengthen the EIUTD as a general tool for modern survival analysis, resulting in a more powerful method that eliminates important barriers between statistical theory and real applications where proper extreme modeling can lead to improved risk mitigation and saving lives.

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Abbreviations

AB	Average Bias
AIC	Akaike Information Criterion
AD	Anderson-darling
BIC	Bayesian Information Criterion
CV	Coefficient of Variation
CvM	Cramer-von Mises
EIUTD	Exponentiated Inverse Unit Teissier Distribution
HRF	Hazard Rate Function
IUTD	Inverse Unit Teissier Distribution
KS	Kolmogorov-smirnov
LL	Log-likelihood
MRE	Mean Relative Estimate
RMSE	Root Mean Squared Error
TTT	Total Time on Test
UTD	Unit Teissier Distribution

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Author Contributions

John Kimani: Conceptualization, Data curation, Formal Analysis, Funding acquisition, Investigation, Methodology, Software, Validation, Visualization, Writing - original draft, Writing - review & editing

Nicholas Makumi: Project administration, Supervision

Kilai Mutua: Project administration, Supervision

Data Availability Statement

The authors confirm that all data used in this study are as indicated in the text and tables and where necessary, appropriate citations have been made.

Conflicts of Interest

The authors declare that they have no conflict of interest.

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