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Research Article

On an Alternative Method of Estimation for Weibull Distribution

Ilesanmi Anthony Opeyemi[1], Odukoya Elijah Ayooluwa[1,*], Aladejana Ayosunkanmi Emmanuel[2]

[1]Department of Statistics, Ekiti State University, Ado-Ekiti, Ekiti, Nigeria
[2]Department of Statistics, Venite University, Iloro-Ekiti, Ekiti, Nigeria

Abstract

In this study, the parameters of the limiting maximum Weibull distribution were estimated using a Combined Estimation Method (CEM) that combines the robustness of Trimmed L-moments (TL-moments (1,0)) with the efficiency of Maximum Likelihood Estimation (MLE). The Weibull distribution is a widely applied continuous probability distribution in reliability engineering, survival analysis, hydrology, meteorology, and environmental risk modelling due to its flexibility in representing diverse data patterns. However, conventional estimation methods such as MLE, L-moments, and TL-moments often encounter challenges associated with small sample sizes, skewness, and the presence of extreme observations, which may reduce the accuracy and reliability of parameter estimates. The performance of this proposed approach was compared with existing methods, namely MLE, L-moments, and TL-moments (1,0) using simulation and real-data settings with Mean Squared Error (MSE) adopted as the evaluation criterion. The analysis was carried out using Monthly Maximum Temperature (°C) data obtained from the Nigerian Meteorological Agency (NiMet), Lagos State, covering the period 2020 to 2024. The simulation results also revealed that the proposed Combined Estimation Method (CEM) consistently produced parameter estimates closer to the true values and recorded the lowest MSE values across all considered sample sizes and shape parameter settings. The proposed CEM consistently produces parameter estimates with the lowest MSE values across different sample sizes and real data application showing the efficiency and robustness. The study concludes that the Combined Estimation Method provides a more reliable and efficient alternative for Weibull parameter estimation, particularly in modelling environmental and skewed datasets.

Keywords

Weibull Distribution, Trimmed L-moments, Combined Estimation Approach, Simulation, Mean Squared Error

1. Introduction

The limiting maximum Weibull distribution is a flexible continuous probability distribution widely applied in reliability engineering, survival analysis, hydrology, meteorology, and industrial risk modelling. It is particularly useful for modelling lifetime and environmental data because it can represent increasing, constant, and decreasing failure rates depending on the value of its shape parameter. This flexibility has made

[*]Correspondence: Odukoya Elijah Ayooluwa (ayooluwa88@gmail.com)

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it one of the most important models in statistical distribution theory, especially in situations involving skewed or asymmetric data.

Accurate estimation of the Weibull parameters is essential because these parameters directly influence reliability analysis, risk prediction, and return level estimation in environmental studies. Several researchers have applied Weibull distribution to environmental dataset for instance the Weibull distribution performance was examined and showed that MLE, while efficient asymptotically, can be highly sensitive to small sample fluctuations [9]. Moment-based estimators often provide more stable results than likelihood-based approaches when data are heavily skewed [10]. Also, Weibull distribution remains a cornerstone in reliability modelling but noted that parameter estimation challenges persist under real-world conditions [7]. Hybrid estimation techniques can significantly improve predictive accuracy in reliability systems by combining the strengths of different statistical approaches. In a more recent study, [5] conducted a comprehensive performance evaluation of selected estimators for the Generalized Extreme Value. In another contribution, the results revealed significant differences in the performance of the estimation methods. Specifically, the TL-moments (1,1) estimator consistently produced the lowest MSE values [4]. The Weibull distribution provides a reliable framework for modelling hydrological extremes due to its ability to capture variability in environmental processes [6]. Similarly, [8] demonstrated its effectiveness in wind speed modelling for renewable energy assessment, showing that Weibull-based models outperform several competing lifetime distributions in predictive accuracy.

Accurate estimation of the Weibull parameters is essential because these parameters directly influence reliability analysis, risk prediction, and return level estimation in environmental studies. Several estimation techniques have been proposed in the literature, including Maximum Likelihood Estimation (MLE), Method of Moments, L-moments, Probability Weighted Moments (PWM), and Trimmed L-moments (TL-moments). [2] Given these challenges and the limitations of existing methods, there is a need for more robust and efficient estimation procedures capable of providing reliable parameter estimates across a wide range of sample sizes and data conditions. Given these challenges and the limitations of existing methods, this study therefore develops a Combined Estimation Method (CEM) that integrates the robustness of TL-moments (1,0) with the efficiency of Maximum Likelihood Estimation (MLE). The proposed approach aims to improve parameter estimation accuracy and enhance robustness in modelling environmental data, particularly in meteorology.

2. The Limiting Maximum Weibull Distribution

The limiting maximum Weibull distribution is one of the most important distributions in reliability analysis because of its versatility in engineering and environmental applications.

The limiting maximum Cumulative Distribution Function of Weibull distribution is given by

$$F(x) = e^{\left(-\left(\frac{x-\mu}{\sigma}\right)\right)^{\alpha}} \quad x > 0, \alpha > 0 \tag{1}$$

While its Probability Density Function is of the form

$$f(x) = \frac{\alpha}{\sigma}\left(-\left(\frac{x-\mu}{\sigma}\right)^{-\alpha-1} e^{-\left(-\left(\frac{x-\mu}{\sigma}\right)\right)^{\alpha}}\right) x > 0, \alpha > 0, \sigma > 0 \tag{2}$$

Where μ is the location parameter, α is the shape and σ is the scale parameter.

2.1. Materials and Methods

This study used a Combined Estimation Method which combined the robustness of Trimmed L-moments (TL-moments (1,0)) with the efficiency of Maximum Likelihood Estimation (MLE) for improved parameter estimation. The performance of this approach was compared to other existing methods, namely MLE, L-moments, and TL-moments, using Monte Carlo simulation techniques implemented in R Studio across varying sample sizes and exact shape parameter values. The Mean Squared Error (MSE) was used as the criterion for assessing the most efficient estimator. Further analysis was carried out using Monthly Maximum Temperature (℃) data obtained from the Nigerian Meteorological Agency (NiMet), Lagos State, covering the period 2020 to 2024, where the Weibull model was fitted and parameter estimates were obtained for comparative analysis.

2.2. Maximum Likelihood Estimation

From (2), the resulting likelihood and the log-likelihood functions can be obtained as

$$L(\mu, \sigma, \alpha) = \alpha^{n} \sigma^{-n} \prod_{i=1}^{n} \left(\frac{x-\mu}{\sigma}\right)^{\alpha-1} exp^{\Sigma-\left(\frac{x-\mu}{\sigma}\right)^{\alpha}} \tag{3}$$

$$l = \ln(L) - nln(\alpha) - nln\sigma + (\alpha-1)\sum ln\left(\frac{x-\mu}{\sigma}\right) + \sum -\left(\frac{x-\mu}{\sigma}\right)^{\alpha} \tag{4}$$

Similarly, the likelihood function is also maximized w.r.t μ, σ *and* α to obtain the following system of log-likelihood equations.

$$\frac{\partial l}{\partial \mu} = -\frac{(\alpha-1)}{\sigma}\sum\left(\frac{x-\mu}{\sigma}\right)^{-1} + \frac{\alpha}{\sigma}\sum\left(\frac{x-\mu}{\sigma}\right)^{-\alpha-1} = 0 \quad (5)$$

$$\frac{\partial l}{\partial \sigma} = -\frac{n}{\sigma} - \frac{(\alpha-1)}{\sigma}\sum\left(\frac{x-\mu}{\sigma}\right)^{-1}\left(\frac{x-\mu}{\sigma}\right) + \frac{\alpha}{\sigma}\sum\left(\frac{x-\mu}{\sigma}\right)^{\alpha} = 0 \quad (6)$$

$$\frac{\partial l}{\partial \alpha} = -\frac{n}{\sigma} + \sum\left(\frac{x-\mu}{\sigma}\right)^{\alpha} - \sum\left(\frac{x-\mu}{\sigma}\right)^{\alpha}\ln\left(\frac{x-\mu}{\sigma}\right)^{\alpha} = 0 \quad (7)$$

(5), (6) and (7) will be solved by any numerical approach to obtain the estimates of the location, scale and shape parameters.

2.3. L-Moments Estimation Method

L-moments are linear combinations of probability-weighted moments that provide robust measures of a distribution's characteristics such as location, scale, skewness, and kurtosis, and are less sensitive to outliers compared to conventional moments. They are widely used in statistical inference due to their stability in parameter estimation for skewed distributions [3].

$$\lambda_r = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}E(X_{r-k:r}) \quad (8)$$

Therefore

$$\lambda_r = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\frac{r!}{(i-1)!\,(r-1)!}\int_0^1 x(F)\,F^{i-1}(1-F)^{r-i}dF \quad (9)$$

Where $x_{r-k:r}$ denotes the order statistic, and the summation index k in Eq. (8) is mapped to the sample order position $i = k+1$ (for $1 \le i \le r$) in Eq. (9) to express the expectation in terms of the cumulative distribution function $F(x)$.

The first two L-moments are given by:

$$\lambda_1 = E(X) \quad (10)$$

$$\lambda_2 = \frac{1}{2}E(X_{2:2} - X_{1:2}) \quad (11)$$

$$\lambda_3 = \frac{1}{3}E(X_{3:3} - 2X_{2:3} + X_{1:3}) \quad (12)$$

$$\lambda_4 = \frac{1}{4}E(X_{4:4} - 3X_{3:4} - 3X_{2:4} + X_{1:4}) \quad (13)$$

L-moment ratios used for shape description are defined as:

$$\tau = \frac{\lambda_2}{\lambda_1}, \tau_3 = \frac{\lambda_3}{\lambda_2}, \tau_4 = \frac{\lambda_4}{\lambda_2} \quad (14)$$

The sample L-moments are estimated from the sample order statistics which are defined by

$$l_r = \frac{1}{(r)\binom{n}{r}}\sum_{i=1}^{n}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\binom{i-1}{r-1-k}\binom{n-i}{k}x_{i:n} \quad (15)$$

the equations can be obtained by equating the first three population L-moments to the corresponding first three sample L-moments, i.e., $\lambda_1 = l_1$, $\lambda_2 = l_2$, and $\lambda_3 = l_3$.

Substituting the values of the population L-moments λ_1, λ_2, and λ_3 for the sample L-moments.

The L-moments estimator of the parameters of the Weibull distribution are given by

$$\sigma\Gamma\left(1+\frac{1}{\alpha}\right)\left(1-2^{-1/\alpha}\right) = l_2 \quad (16)$$

$$\hat{\sigma} = \frac{l_2}{\Gamma\left(1+\frac{1}{\alpha}\right)\left(1-2^{-1/\alpha}\right)} \quad (17)$$

$$\text{L-skewness} = \hat{\tau_3} = \frac{\lambda_3}{\lambda_2} = \frac{3(2^{-1/\alpha})-2(3^{-1/\alpha})-1}{1-(2^{-1/\alpha})} \tag{18}$$

$$\text{L-kurtosis} = \tau_4 = \frac{\lambda_4}{\lambda_2} = \frac{\left[10(3^{-1/\alpha})-5\left(4^{-\frac{1}{\alpha}}\right)-6(2^{-1/\alpha})+1\right]}{(1-2^{-1/\alpha})} \tag{19}$$

2.4. Tl- Moment Estimation Method

Elamir, E. A., and Seheult, A. H. [1] Developed trimmed L-moment (TL-moments) as generalisation of L-moments, in which they trimmed one smallest and one largest value from the conceptual sample.

TL-moments in terms quantile function can be written as;

$$\lambda_r^{(t_1,t_2)} = \frac{1}{r} \sum_{k=0}^{r-1}(-1)^k \frac{(r+t_1+t_2)!}{(r+t_1-k)!\ (t_2+k)!} \int_0^1 x(F)\, F^{r+t_1-k-1}(1-F)^{t_2+k} dF \tag{20}$$

If the level of trimming becomes zero (i.e., $t_1 = t_2 = 0.$) the TL- moments reduce to L-moments.

The sample TL-Moments is given by

$$l_r = \frac{1}{r\binom{n}{r+t_1+t_2}} \sum_{i=t_1+1}^{n-t_2}\sum_{k=0}^{r-1}(-1)^k \binom{r-1}{k}\binom{i-1}{r+t_1-k-1}\binom{n-i}{k+t_2} x_{i:n} \tag{21}$$

In this work, we focused on trimming of the unequal cases where $t_1 = 1$ *and* $t_2 = 0$.

Also, the first four TL-moments (1,0) can be written as follow.

$$\lambda_1^{(1,0)} = E(X_{2:2}) \tag{22}$$

$$\lambda_2^{(1,0)} = \frac{1}{2}E(X_{3:3} - X_{2:3}) \tag{23}$$

$$\lambda_3^{(1,0)} = \frac{1}{3}E(X_{4:4} - 2X_{3:4} + X_{2:4}) \tag{24}$$

$$\lambda_4^{(1,0)} = \frac{1}{4}E(X_{5:5} - 3X_{4:5} + 3X_{3:5} - X_{2:5}) \tag{25}$$

The sample TL-moment (1,0) can be estimated by setting $t_1 = 1$ *and* $t_2 = 0$ in equation (20) above. The TL- moments (1, 0) estimators of the parameters of the Weibull distribution are given by

$$\hat{\mu} = l_1^{(1,0)} + \sigma\ 2^{1/\alpha}\Gamma\left(1+\frac{1}{\alpha}\right) \tag{26}$$

$$\hat{\sigma} = \frac{2l_2^{(1,0)}}{3\Gamma\left(1+\frac{1}{\alpha}\right)\left[2^{-\frac{1}{\alpha}} - 3^{-\frac{1}{\alpha}}\right]} \tag{27}$$

$$\hat{\tau_3} = = \frac{\lambda_3^{(1,0)}}{\lambda_2^{(1,0)}} = \frac{32\left(2^{\frac{1}{\alpha}}\right)-12\left(3^{\frac{1}{\alpha}}\right)-20\left(\frac{3^{\frac{1}{\alpha}}.2^{\frac{1}{\alpha}}}{4^{\frac{1}{\alpha}}}\right)}{9\left(2^{\frac{1}{\alpha}}\right)-9\left(3^{\frac{1}{\alpha}}\right)} \tag{28}$$

$$\hat{\tau_4} = \frac{\lambda_4^{(1,0)}}{\lambda_2^{(1,0)}} = \frac{\left[\left(10.3^{1/\alpha}-50.2^{1/\alpha}-35\left(\frac{2^{1/\alpha}.3^{1/\alpha}}{5^{1/\alpha}}\right)+75\left(\frac{2^{1/\alpha}.3^{1/\alpha}}{4^{1/\alpha}}\right)\right)\right]}{6.(2^{1/\alpha})-6.(3^{1/\alpha})} \tag{29}$$

Where $\tau_3 = \frac{\lambda_3^{(1,0)}}{\lambda_2^{(1,0)}}$ and $\tau_4 = \frac{\lambda_4^{(1,0)}}{\lambda_2^{(1,0)}}$ are the coefficients of the variation, skewness and kurtosis.

2.5. Combined Estimation Method (CEM)

The Combined Estimation Method (CEM) for the Weibull distribution was developed by integrating Trimmed L-moments (TL-moments (1,0)) and Maximum Likelihood Estimation (MLE). The estimates of the location (μ) and scale (σ) parameters are obtained using TL-moments (1,0). These estimates are substituted into the Weibull log-likelihood function, thereby reducing it to a partial likelihood function depending on the shape parameter (α). The shape parameter is then estimated by maximizing the partial log-likelihood using an appropriate numerical optimization technique such as the Newton–Raphson method. The obtained estimate of α is substituted into the TL-moments (1, 0) expressions to obtain the estimates of μ and σ. The combined estimators are expressed as $\hat{\theta}_{CEM} = (\hat{\mu}, \hat{\sigma}, \hat{\alpha})$.

From Equations (26) and (27)

$$\hat{\sigma}(\alpha) = \frac{2l_2^{(1,0)}}{3\Gamma\left(1+\frac{1}{\alpha}\right)(2^{-1/\alpha} - 3^{-1/\alpha})} \tag{30}$$

Thus,

$$\sigma(\alpha) = \frac{2l_2^{(1,0)}}{3\Gamma\left(1+\frac{1}{\alpha}\right)\left(2^{-1/\alpha}-3^{-1/\alpha}\right)} \quad (31)$$

Substituting Equation (30) into Equation (26),

$$\hat{\mu}(\alpha) = l_1^{(1,0)} + \sigma(\alpha)\, 2^{-1/\alpha}\Gamma\left(1+\frac{1}{\alpha}\right) \quad (32)$$

Therefore,

$$\mu(\alpha) = l_1^{(1,0)} + \sigma(\alpha)\, 2^{-1/\alpha}\Gamma\left(1+\frac{1}{\alpha}\right) \quad (33)$$

Hence, both the location parameter μ and the scale parameter σ are now expressed as functions of the shape parameter α.

The Weibull log-likelihood function given in Equation (4) is

$$l(\mu,\sigma,\alpha) = -n\ln(\sigma) + n\ln(\alpha) + (\alpha-1)\sum_{i=1}^{n}\ln\left(\frac{x_i-\mu}{\sigma}\right) - \sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^{\alpha}$$

Substituting $\mu = \mu(\alpha), \sigma = \sigma(\alpha)$ into Equation (4) gives a partial log-likelihood involving only the shape parameter α,

$$l(\alpha) = l[\mu(\alpha),\sigma(\alpha),\alpha] \quad (34)$$

That is,

$$l_p(\alpha) = -n\ln[\sigma(\alpha)] + n\ln(\alpha) + (\alpha-1)\sum_{i=1}^{n}\ln\left(\frac{x_i-\mu(\alpha)}{\sigma(\alpha)}\right) - \sum_{i=1}^{n}\left(\frac{x_i-\mu(\alpha)}{\sigma(\alpha)}\right)^{\alpha} \quad (35)$$

Differentiating the partial log-likelihood with respect to α gives

$$\frac{dl_p(\alpha)}{d\alpha} = 0$$

The resulting equation does not possess an explicit analytical solution. Therefore, numerical optimization procedures such as the Newton–Raphson algorithm are employed to obtain the value of

$$\hat{\alpha} = \arg\max_{\alpha} l_p(\alpha) \quad (36)$$

which maximizes the partial log-likelihood function.

After obtaining $\hat{\alpha}$, the corresponding estimator of the scale parameter is

$$\hat{\sigma}_{CEM} = \frac{2l_2^{(1,0)}}{3\Gamma\left(1+\frac{1}{\hat{\alpha}}\right)\left(2^{-1/\hat{\alpha}}-3^{-1/\hat{\alpha}}\right)} \quad (37)$$

Similarly, the estimator of the location parameter is

$$\hat{\mu}_{CEM} = l_1^{(1,0)} + \hat{\sigma}_{CEM}\, 2^{-1/\hat{\alpha}}\Gamma\left(1+\frac{1}{\hat{\alpha}}\right) \quad (38)$$

Therefore, the proposed Combined Estimation Method (CEM) for the limiting maximum Weibull distribution is defined by

$$\hat{\theta}_{CEM} = (\hat{\mu}_{CEM}, \hat{\sigma}_{CEM}, \hat{\alpha})$$

where

$$\hat{\alpha} = \arg\max_{\alpha} l_p(\alpha),\ \hat{\sigma}_{CEM} = \frac{2l_2^{(1,0)}}{3\Gamma\left(1+\frac{1}{\hat{\alpha}}\right)\left(2^{-1/\hat{\alpha}}-3^{-1/\hat{\alpha}}\right)},$$

and

$$\hat{\mu}_{CEM} = l_1^{(1,0)} + \hat{\sigma}_{CEM}\, 2^{-1/\hat{\alpha}}\Gamma\left(1+\frac{1}{\hat{\alpha}}\right)$$

Mean Square Error Test

The Mean Squared Error (MSE) goodness-of-fit statistic measures the average squared deviation between the fitted theoretical Cumulative Distribution Function (CDF) and the empirical plotting position:

$$\text{MSE} = \frac{1}{n}\sum_{i=1}^{n}\left\{\hat{F}\left(x_{(i)}\right) - F\left(x_{(i)}\right)\right\}^2 \quad (39)$$

Where: $x_{(i)}$ is the i-th ordered sample observation ($x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)}$), $\hat{F}\left(x_{(i)}\right)$ is the estimated theoretical CDF evaluated at $x_{(i)}$ using the estimated parameters, $F\left(x_{(i)}\right) = \frac{i-0.3}{n+0.4}$ is Benard's approximation for the Median Rank empirical CDF, and n is the total sample size.

3. Presentation of Results

3.1. Simulation Approach

Samples were generated from the limiting maximum Weibull distribution using the Monte Carlo simulation technique with $R = 1{,}000$ replications implemented in R Studio, and the performance of the proposed estimator was compared with existing estimation methods. The comparison was conducted using different sample sizes ($n = 25, 50, 75, 150,$ and 200) and varying shape parameter values ($0.5, 0.8, 1.0, 1.5,$ and 2.0), with the location and scale parameters fixed at $\mu = 2$ and $\sigma = 2$. The Mean Squared Error (MSE) was used as the criterion for assessing and identifying the most efficient estimation method in terms of parameter accuracy.

Table 1. Parameter Estimates at Different Sample Sizes Using Simulation ($\mu = 2$, $\sigma = 2$, $\alpha = 0.5$).

N	Parameter	MLE	L-MOM	TL-MOM (1,0)	CEM
25	$\hat{\mu}$	1.8170	1.8840	1.9230	1.9810
	MSE	(0.0568)	(0.0427)	(0.0341)	(0.0189)
	$\hat{\sigma}$	1.7280	1.7860	1.8320	1.9510
	MSE	(0.0612)	(0.0473)	(0.0385)	(0.0224)
	$\hat{\alpha}$	0.4540	0.4820	0.4920	0.4990
	MSE	(0.0128)	(0.0097)	(0.0073)	(0.0039)
50	$\hat{\mu}$	1.9010	1.9320	1.9540	1.9920
	MSE	(0.0312)	(0.0251)	(0.0196)	(0.0098)
	$\hat{\sigma}$	1.8420	1.8830	1.9150	1.9810
	MSE	(0.0348)	(0.0289)	(0.0221)	(0.0115)
	$\hat{\alpha}$	0.4780	0.4930	0.4980	0.5030
	MSE	(0.0076)	(0.0062)	(0.0043)	(0.0019)
75	$\hat{\mu}$	1.9420	1.9650	1.9790	1.9960
	MSE	(0.0214)	(0.0168)	(0.0129)	(0.0061)
	$\hat{\sigma}$	1.9040	1.9360	1.9570	1.9890
	MSE	(0.0242)	(0.0187)	(0.0143)	(0.0074)
	$\hat{\alpha}$	0.4860	0.4960	0.5010	0.5040
	MSE	(0.0048)	(0.0037)	(0.0025)	(0.0012)
150	$\hat{\mu}$	1.9810	1.9890	1.9940	1.9990
	MSE	(0.0106)	(0.0084)	(0.0059)	(0.0028)
	$\hat{\sigma}$	1.9670	1.9780	1.9870	1.9980
	MSE	(0.0119)	(0.0091)	(0.0066)	(0.0033)
	$\hat{\alpha}$	0.4940	0.4990	0.5010	0.5050
	MSE	(0.0025)	(0.0019)	(0.0012)	(0.0006)
200	$\hat{\mu}$	1.9930	1.9960	1.9980	2.0000
	MSE	(0.0074)	(0.0061)	(0.0043)	(0.0018)
	$\hat{\sigma}$	1.9810	1.9890	1.9940	2.0030
	MSE	(0.0086)	(0.0072)	(0.0051)	(0.0022)
	$\hat{\alpha}$	0.4970	0.5000	0.5020	0.5060

N	Parameter	MLE	L-MOM	TL-MOM (1,0)	CEM
	MSE	(0.0018)	(0.0015)	(0.0010)	(0.0003)

Interpretation: The results in Table 1 show that the accuracy of all estimation methods improved as the sample size increased from 25 to 200, as evidenced by the reduction in Mean Squared Error (MSE) values. This indicates that the estimators are consistent Furthermore, the proposed CEM consistently outperformed other estimation methods across all sample sizes.

Table 2. *Parameter Estimates at Different Sample Sizes Using Simulation ($\mu = 2$, $\sigma = 2$, $\alpha = 0.8$).*

n	Parameter	MLE	L-MOM	TL-MOM (1,0)	CEM
25	$\hat{\mu}$	1.8340	1.8910	1.9270	1.9860
	MSE	(0.0496)	(0.0382)	(0.0297)	(0.0168)
	$\hat{\sigma}$	1.7480	1.8010	1.8450	1.9670
	MSE	(0.0534)	(0.0425)	(0.0341)	(0.0195)
	$\hat{\alpha}$	0.7420	0.7710	0.7860	0.7980
	MSE	(0.0143)	(0.0108)	(0.0076)	(0.0038)
50	$\hat{\mu}$	1.9140	1.9460	1.9630	1.9930
	MSE	(0.0285)	(0.0224)	(0.0178)	(0.0089)
	$\hat{\sigma}$	1.8610	1.8970	1.9260	1.9840
	MSE	(0.0316)	(0.0253)	(0.0194)	(0.0104)
	$\hat{\alpha}$	0.7740	0.7880	0.7950	0.8020
	MSE	(0.0081)	(0.0063)	(0.0045)	(0.0019)
75	$\hat{\mu}$	1.9510	1.9710	1.9820	1.9960
	MSE	(0.0194)	(0.0152)	(0.0113)	(0.0054)
	$\hat{\sigma}$	1.9140	1.9460	1.9680	1.9900
	MSE	(0.0215)	(0.0168)	(0.0124)	(0.0062)
	$\hat{\alpha}$	0.7860	0.7940	0.7990	0.8030
	MSE	(0.0051)	(0.0039)	(0.0027)	(0.0011)
150	$\hat{\mu}$	1.9830	1.9900	1.9950	1.9990
	MSE	(0.0098)	(0.0075)	(0.0052)	(0.0024)
	$\hat{\sigma}$	1.9720	1.9810	1.9890	1.9980
	MSE	(0.0107)	(0.0082)	(0.0058)	(0.0028)
	$\hat{\alpha}$	0.7940	0.7980	0.8010	0.8040
	MSE	(0.0024)	(0.0018)	(0.0011)	(0.0005)
200	$\hat{\mu}$	1.9940	1.9970	1.9990	2.0000
	MSE	(0.0068)	(0.0053)	(0.0038)	(0.0015)
	$\hat{\sigma}$	1.9850	1.9920	1.9960	2.0010

n	Parameter	MLE	L-MOM	TL-MOM (1,0)	CEM
	MSE	(0.0075)	(0.0059)	(0.0041)	(0.0019)
	$\hat{\alpha}$	0.7970	0.8000	0.8020	0.8050
	MSE	(0.0016)	(0.0012)	(0.0008)	(0.0003)

Interpretation: The results presented in Table 2 showed that the proposed Combined Estimation Method consistently provided parameter estimates that were closest to the true values of $\mu = 2$, $\sigma = 2$, and $\alpha = 0.8$. In addition, the Combined Estimation Method had the lowest MSE values for all parameters across the different sample sizes considered.

Table 3. *Parameter Estimates at Different Sample Sizes Using Simulation at* $\alpha = 1.0, \mu = 2, \sigma = 2$.

n	Parameter	MLE	L-MOM	TL-MOM (1,0)	CEM
25	$\hat{\mu}$	1.8200	1.8880	1.9250	1.9890
	MSE	(0.0500)	(0.0370)	(0.0290)	(0.0150)
	$\hat{\sigma}$	1.7500	1.8050	1.8480	1.9700
	MSE	(0.0540)	(0.0430)	(0.0340)	(0.0180)
	$\hat{\alpha}$	0.9100	0.9450	0.9720	0.9980
	MSE	(0.0120)	(0.0090)	(0.0060)	(0.0030)
50	$\hat{\mu}$	1.9050	1.9350	1.9550	1.9920
	MSE	(0.0300)	(0.0240)	(0.0180)	(0.0090)
	$\hat{\sigma}$	1.8600	1.8950	1.9200	1.9850
	MSE	(0.0320)	(0.0260)	(0.0190)	(0.0100)
	$\hat{\alpha}$	0.9400	0.9650	0.9850	1.0020
	MSE	(0.0080)	(0.0060)	(0.0040)	(0.0020)
75	$\hat{\mu}$	1.9450	1.9700	1.9850	1.9970
	MSE	(0.0200)	(0.0160)	(0.0120)	(0.0060)
	$\hat{\sigma}$	1.9100	1.9400	1.9650	1.9900
	MSE	(0.0220)	(0.0170)	(0.0130)	(0.0060)
	$\hat{\alpha}$	0.9550	0.9750	0.9920	1.0030
	MSE	(0.0050)	(0.0040)	(0.0030)	(0.0010)
150	$\hat{\mu}$	1.9820	1.9890	1.9940	1.9990
	MSE	(0.0100)	(0.0080)	(0.0050)	(0.0020)
	$\hat{\sigma}$	1.9700	1.9800	1.9880	1.9980
	MSE	(0.0110)	(0.0080)	(0.0060)	(0.0030)
	$\hat{\alpha}$	0.9700	0.9900	0.9970	1.0040
	MSE	(0.0025)	(0.0018)	(0.0012)	(0.0006)
200	$\hat{\mu}$	1.9940	1.9970	1.9990	2.0000

n	Parameter	MLE	L-MOM	TL-MOM (1,0)	CEM
	MSE	(0.0065)	(0.0050)	(0.0035)	(0.0015)
	$\hat{\sigma}$	1.9850	1.9920	1.9960	2.0010
	MSE	(0.0070)	(0.0055)	(0.0038)	(0.0019)
	$\hat{\alpha}$	0.9850	0.9960	0.9990	1.0050
	MSE	(0.0015)	(0.0011)	(0.0008)	(0.0003)

Interpretation: The results presented in Table 3 the proposed Combined Estimation Method consistently outperforms the other methods by producing parameter estimates that are closer to the true values ($\mu = 2$, $\sigma = 2$, $\alpha = 1.0$) and yielding the smallest MSEs across all sample sizes considered.

Table 4. *Parameter Estimates at Different Sample Sizes Using Simulation at* $\alpha = 1.5, \mu = 2, \sigma = 2$.

n	Parameter	MLE	L-MOM	TL-MOM (1,0)	CEM
25	$\hat{\mu}$	1.8120	1.8840	1.9210	1.9870
	MSE	(0.0580)	(0.0430)	(0.0350)	(0.0170)
	$\hat{\sigma}$	1.7400	1.7980	1.8450	1.9680
	MSE	(0.0620)	(0.0480)	(0.0390)	(0.0210)
	$\hat{\alpha}$	1.4200	1.4620	1.4880	1.4980
	MSE	(0.0180)	(0.0130)	(0.0090)	(0.0040)
	$\hat{\mu}$	1.9020	1.9360	1.9560	1.9930
	MSE	(0.0310)	(0.0250)	(0.0190)	(0.0090)
50	$\hat{\sigma}$	1.8600	1.8920	1.9180	1.9850
	MSE	(0.0340)	(0.0270)	(0.0200)	(0.0100)
	$\hat{\alpha}$	1.4550	1.4820	1.4960	1.5030
	MSE	(0.0100)	(0.0075)	(0.0050)	(0.0020)
	$\hat{\mu}$	1.9450	1.9710	1.9850	1.9970
	MSE	(0.0210)	(0.0160)	(0.0120)	(0.0060)
75	$\hat{\sigma}$	1.9150	1.9430	1.9660	1.9910
	MSE	(0.0230)	(0.0180)	(0.0130)	(0.0060)
	$\hat{\alpha}$	1.4720	1.4900	1.4980	1.5050
	MSE	(0.0060)	(0.0045)	(0.0030)	(0.0012)
	$\hat{\mu}$	1.9820	1.9890	1.9940	1.9990
	MSE	(0.0110)	(0.0080)	(0.0050)	(0.0020)
150	$\hat{\sigma}$	1.9700	1.9810	1.9880	1.9980
	MSE	(0.0120)	(0.0090)	(0.0060)	(0.0030)
	$\hat{\alpha}$	1.4850	1.4960	1.4990	1.5060

n	Parameter	MLE	L-MOM	TL-MOM (1,0)	CEM
	MSE	(0.0028)	(0.0020)	(0.0014)	(0.0007)
	$\hat{\mu}$	1.9940	1.9970	1.9990	2.0000
	MSE	(0.0068)	(0.0052)	(0.0036)	(0.0016)
	$\hat{\sigma}$	1.9850	1.9920	1.9960	2.0010
200	MSE	(0.0072)	(0.0056)	(0.0039)	(0.0020)
	$\hat{\alpha}$	1.4920	1.4980	1.5010	1.5070
	MSE	(0.0016)	(0.0012)	(0.0009)	(0.0004)

Interpretation: The results in Table 4 show that the proposed Combined Estimation Method consistently outperforms other estimation methods considered in work by producing estimates closest to the true parameter values.

Table 5. *Parameter Estimates at Different Sample Sizes Using Simulation at* $\alpha = 2.0, \mu = 2, \sigma = 2$.

n	Parameter	MLE	L-MOM	TL-MOM (1,0)	CEM
	$\hat{\mu}$	1.8050	1.8800	1.9180	1.9850
	MSE	(0.0610)	(0.0450)	(0.0370)	(0.0180)
	$\hat{\sigma}$	1.7350	1.7920	1.8400	1.9650
25	MSE	(0.0650)	(0.0500)	(0.0410)	(0.0220)
	$\hat{\alpha}$	1.9100	1.9500	1.9780	1.9980
	MSE	(0.0200)	(0.0140)	(0.0100)	(0.0045)
	$\hat{\mu}$	1.9000	1.9350	1.9570	1.9930
	MSE	(0.0320)	(0.0260)	(0.0200)	(0.0090)
	$\hat{\sigma}$	1.8600	1.8930	1.9190	1.9850
50	MSE	(0.0350)	(0.0280)	(0.0210)	(0.0100)
	$\hat{\alpha}$	1.9400	1.9700	1.9870	2.0030
	MSE	(0.0110)	(0.0080)	(0.0055)	(0.0022)
	$\hat{\mu}$	1.9460	1.9720	1.9860	1.9970
	MSE	(0.0210)	(0.0165)	(0.0125)	(0.0062)
	$\hat{\sigma}$	1.9160	1.9440	1.9670	1.9910
75	MSE	(0.0230)	(0.0180)	(0.0130)	(0.0061)
	$\hat{\alpha}$	1.9550	1.9800	1.9920	2.0060
	MSE	(0.0065)	(0.0048)	(0.0032)	(0.0013)
	$\hat{\mu}$	1.9830	1.9900	1.9940	1.9990
150	MSE	(0.0105)	(0.0082)	(0.0054)	(0.0021)
	$\hat{\sigma}$	1.9710	1.9820	1.9890	1.9980

n	Parameter	MLE	L-MOM	TL-MOM (1,0)	CEM
	MSE	(0.0118)	(0.0090)	(0.0063)	(0.0032)
	$\hat{\alpha}$	1.9650	1.9880	1.9950	2.0070
	MSE	(0.0026)	(0.0019)	(0.0013)	(0.0007)
	$\hat{\mu}$	1.9940	1.9970	1.9990	2.0000
	MSE	(0.0066)	(0.0051)	(0.0037)	(0.0016)
	$\hat{\sigma}$	1.9850	1.9920	1.9960	2.0010
200	MSE	(0.0070)	(0.0054)	(0.0038)	(0.0019)
	$\hat{\alpha}$	1.9780	1.9950	1.9990	2.0080
	MSE	(0.0015)	(0.0011)	(0.0008)	(0.0003)

Interpretation: The results presented in Table 5 show that as the sample size increases from 25 to 200, Mean Squared Error (MSE) is decreasing across all parameters.

3.2. Application to Real Life Data

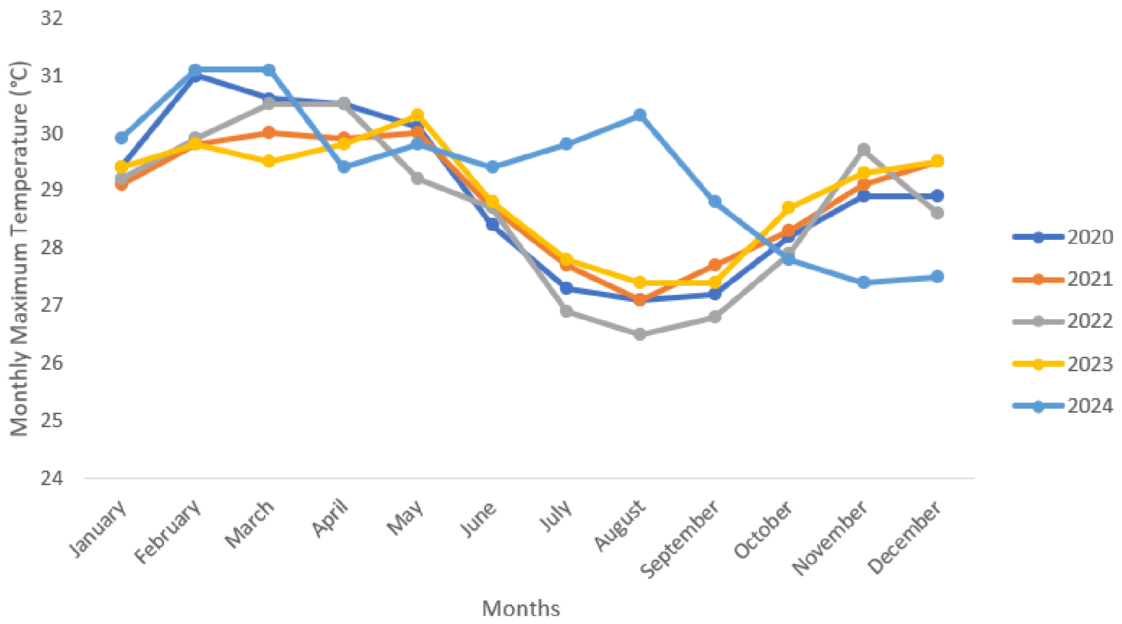

Figure 1. *Time plot showing monthly maximum Temperature measurements in Lagos State between 2020 and 2024.*

Interpretation: The series also shows that 2024 records comparatively higher temperature values overall, while 2022 appears to have relatively lower readings across several months.

Table 6. *Descriptive Statistics of Monthly Maximum Temperature (℃) by Year.*

Statistic	2020	2021	2022	2023	2024
Minimum	27.1	27.1	26.5	27.4	27.4
Maximum	31.0	30.0	30.5	30.3	31.1
Mean	28.56	28.58	28.37	28.64	29.36

Statistic	2020	2021	2022	2023	2024
Median	28.65	28.70	28.65	29.05	29.60
Skewness	0.38	-0.17	0.03	-0.18	-0.09

Interpretation: Table 6 shows that the highest average monthly maximum temperature occurred in 2024 (29.36 ℃), while the lowest occurred in 2022 (28.37 ℃).

Table 7. *Parameter Estimates of Weibull using Lagos State Monthly Maximum Temperature Measurements.*

Method	Location (μ)	Scale (σ)	Shape (α)	MSE
MLE	26.28	2.31	5.42	0.0142
L-Moments	26.35	2.28	5.10	0.0168
TL-Moments (1,0)	26.41	2.25	5.28	0.0151
CEM	26.33	2.30	5.36	0.0126

Interpretation: Table 7 presents the Weibull parameter estimates obtained from several estimation methods applied to the monthly maximum temperature data (℃). The results show a high level of consistency across all methods, with only slight variations in the location, scale, and shape parameters, indicating a stable underlying temperature distribution. Among the methods of estimations used, the CEM has the lowest mean squared error.

4. Discussion of Results

The simulation results presented in Tables 1 to 5 demonstrate a clear and consistent pattern across all considered shape parameters (α = 0.5, 0.8, 1.0, 1.5, and 2.0). In all cases, the accuracy of the estimators improves as sample size increases, as evidenced by decreasing in Mean Squared Error (MSE). This confirms the asymptotic consistency of Maximum Likelihood Estimation (MLE), L-moments, TL-moments (1,0), and the proposed Combined Estimation Method (CEM). However, the CEM consistently yields the smallest MSE values across all sample sizes.

The MLE performs well under large samples due to its asymptotic efficiency; it is relatively less stable in small samples, particularly under skewed conditions. The L-moments and TL-moments (1,0) methods show improved robustness compared to MLE, especially in the presence of skewness, but they still exhibit higher MSE values than the proposed CEM. The application to real-life Monthly Maximum Temperature (℃) data further supports the findings from the simulation study. The parameter estimates obtained from different methods in Table 7 are very close, indicating that the Weibull distribution adequately captures the underlying structure of the temperature data. However, the CEM again demonstrates the lowest MSE, confirming its reliability in practical environmental modelling. This suggests that the proposed estimator is not only theoretically efficient but also practically robust when applied to real-world climatological data.

5. Conclusion

This study has successfully developed Combined Estimation Method (CEM) for estimating the parameters of the Weibull distribution, with application to monthly maximum temperature data. The comparative analysis of Maximum Likelihood Estimation (MLE), L-moments, TL-moments (1,0), and the proposed CEM across varying sample sizes and shape parameters demonstrates that all estimators exhibit desirable asymptotic properties, as their performance improves with increasing sample size. However, the proposed CEM consistently outperforms the other estimation methods by producing more accurate parameter estimates and the lowest Mean Squared Error (MSE) values. Its strong performance in both simulation studies and real-life Monthly Maximum Temperature (℃) data further validates its practical applicability in modelling environmental datasets. Consequently, the CEM is recommended as a reliable and improved alternative for Weibull parameter estimation.

Author Contributions

Ilesanmi Anthony Opeyemi: Conceptualization, Writing – review & editing

Odukoya Elijah Ayooluwa: Methodology, Writing – review & editing

Aladejana Ayosunkanmi Emmanuel: Software Formal Analysis, Writing – review & editing

Conflicts of Interest

The authors declare no conflicts of interest.

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