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Bivariate Radialsymmetric Kernel Density Estimation of Well-being Distribution and Poverty Index

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Abstract: In two previous papers, we proposed two bivariate product kernel estimators for the bi-dimensional extension of the Foster, Greer and Thorbecke (FGT) index based respectively on classical and adaptive kernel. The Foster, Greer and Thorbecke (FGT) index was introduced in the literature for the purpose of a dominance approach to multidimensional poverty. The poverty measure utilized in this dominance method is fundamentally a generalization, from one to two dimensions, of this Foster, Greer and Thorbecke index with separate poverty aversion parameters for each dimension. This statistical method (the product kernel estimator) has the following main disadvantes: the curse of dimensionality, the complex selection of the bandwidth, and high computational costs. In this study, we focus on a bivariate radialsymmetric kernel estimator for the bi-dimensional extension of the Foster, Greer and Thorbecke index. Our new bivariate kernel estimator is developed with a classical bivariate kernel of Parzen-Rosenblatt of a probability density function (pdf) utilizing Riemann sums. We next provide complete asymptotic behaviour by establishing both almost-sure uniform and uniform mean square consistencies for the bivariate radialsymmetric kernel estimator. A simulation study indicates that the proposed bivariate kernel estimator performs favorably for small samples comparatively to the bivariate multiplicative kernel estimator.

Keywords: Bidimensional Poverty Index, Foster-Greer-Thorbecke (FGT) Measure, Bivariate Radialsymmetric Kernel Density Estimation, Nonparametric Estimation, Uniform Almost-sure Consistency, Riemann Sums, Well-being Distribution, Rate of Convergence

1. Introduction and Motivations

Kernel density estimation has become a popular tool for visualizing the distribution of univariate data [see Ref. [20], [21], [16], [22], for example, for an overview]. Univariate kernel density estimation has received considerable attention in the literature, partly because of its practical utility, and partly because of providing a simple testing ground for learning about nonparametric smoothing.

The objective of bivariate nonparametric density estimation is to approximate the probability density function (pdf) $f(x) = f(x_1, x_2)$ of the random vector $X = (X_1, X_2)^T$. Standard treatments of this topic were provided by Silverman [20], Scott [17], and Wand and Jones [22], who discussed both multiplicative and radialsymmetric kernel constructions, as well as bandwidth selection strategies (see also Sheather and Jones [19] for practical data-driven methods). In the two-dimensional case, the bivariate kernel density estimator is expressed as

$$\hat{f}_h = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{h_1 \times h_2}\mathcal{K}\left(\frac{X_{i1}-x_1}{h_1}, \frac{X_{i2}-x_2}{h_2}\right) \qquad (1)$$

where $\mathcal{K} : \mathbb{R}^2 \to \mathbb{R}$ denoting a bivariate kernel function. Note,

that (1) assume that the bandwidth h is a vector of bandwidths $h = (h_1, h_2)^T$.
The bi-dimensional kernel function $\mathcal{K}(u) = \mathcal{K}(u_1, u_2)$ takes on two forms.
The easiest solution is to use a multiplicative kernel

$$\mathcal{K}(u) = K(u_1) \times K(u_2) \tag{2}$$

with K denoting an univariate kernel function. For univariate kernels supported on $[-1, 1]$, such as the Epanechnikov kernel $K(u) = 0.75(1 - u^2)(I(|u| \leq 1))$, observations within a rectangular neighborhood around x are used to estimate the density at the point x.
Alternatively, one may use a genuine bivariate kernel function $\mathcal{K}(u)$, such as the bivariate Epanechninokov kernel:

$$\mathcal{K}(u) \simeq (1 - u_1^2 - u_2^2)(I(u_1^2 + u_2^2 \leq 1)).$$

Such bivariate kernels can be derived from univariate kernels by defining

$$\mathcal{K}(u) \simeq K(||u||), \tag{3}$$

where $||u|| = \sqrt{u_1^2 + u_2^2}$ denotes the Euclidean norm of the vector u. Kernels of the form (3) utilize observations within a circular neighborhood around x for density estimation. These kernels are commonly termed spherical or radialsymmetric, as $\mathcal{K}(u)$ assumes identical values for all u points lying on a sphere centered at zero. Figure 1 displays contour plots comparing the bivariate product with (left panel) with the bivariate radialsymmetric kernel (right panel).

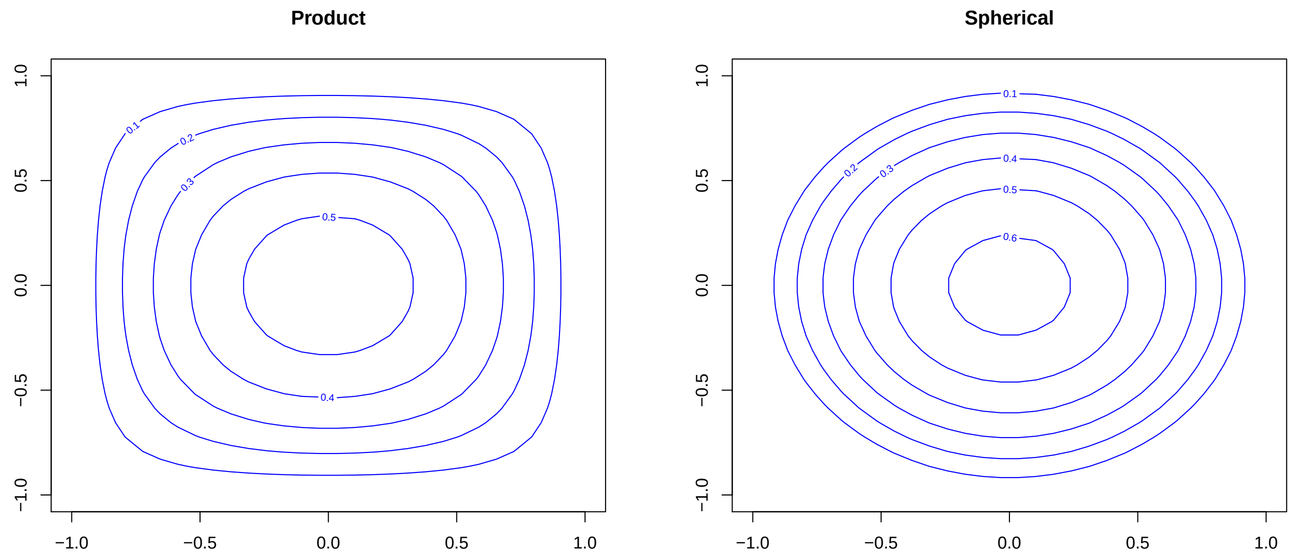

Figure 1. *Contours from bivariate product (left) and bivariate radialsymmetric (right) Epanechnikoc kernel.*

The weighs of the kernels in Figure 1 correspond to equal bandwidth in both directions, i.e. $h = (h_1, h_2)^T = (1, 1)^T$. When employing different bandwidths, observations around x in the density estimate $\hat{f}_h(x)$ will be used with different weighs across both dimensions. For more detailed we refer to Härdle and Müller [14].

More generally, the product kernel estimator presents a number of drawbacks. Firstly it has the problem of curse of dimensionality meaning that as the dimension increases, the volume of the data space grows exponentially. The available data becomes extremely sparse. So the estimator requires a massive amount of data to remain reliable.

Secondly there is the problem of the bandwith selection. As we know it the bandwidth determines how smooth the estimated curve. A too small bandwidth includes an undersmoothing estimate. A too large bandwidth hides real patterns (oversmoothing). Then getting the optimal bandwidth for each dimension is complex and often requires heavy trial-and-error or automated selection algorithms. Thirdly, with the product kernel estimator, the algorithm must compute distances and kernel functions between all single data point and all evaluation point. Therefore the treatment of large datasets becomes highly time-consuming and requires significant memory resources. Fourthly, the product kernel estimator relies on data points surrounding the evaluation region. At the extreme edges of the data, there are no points available on one side. Thus the estimation becomes biased and much less accurate near the boundaries of the data support. Finally, the product kernel treats the smoothing process as if the dimensions were independent. Then if the variables have complex, non-linear correlations, the kernel may fail to capture the true joint structure exactly.

However the radialsymmetric kernel measures distance rather than absolute direction. It has a number of practical advantages. First, it utilizes mathematical tricks to move simple two-dimensional data into an infinite number of dimensions. This processus makes mixed up data easy to separate with a straight line. Second, the radialsymmetric kernel is meshless. It does not force the data to fit into a strict grid. This allows it to easily adapts to strange, real-world shapes. Third, the radialsymmetric kernel only cares about how close data points are to each other, not their specific location on a map. This makes it very resistants to outside noise. Fourth, by relying completely on distance, it avoids complex calculations. This allows for smooth, accurate modeling even in very high dimensions.

To focus on the robustness analysis of the selection of poverty lines and poverty measures, Duclos et al. [7] considered a multidimensional extension of the FGT class of measures. They used the dominance approach for poverty comparisons, as initially developed in Atkinson [3], in Foster and Shorrocks [10], Foster and Shorrocks [11] and in Foster and Shorrocks [12]. A main benefit of this method is its capability in producing poverty orderings that are reliable with respect to the establishing of poverty lines. Then the delicacy of most of poverty measures to the poverty line leads to this more essential approach. Furthermore, it also makes sure robustness regarding the choice of a multidimensional poverty index over broad classes of them, as well reliability over the way in which multidimensional indicators interface between them, when characterizing total individual well-being. Duclos et al. [8] as well as considered the bivariate stochastic dominance methods to analyze the incidence of poverty, evaluated in relation to household expenditures per capita and child height-for-age indicators. Such important properties of this measure motivated Ciss et al. [23] to introduce an asymptotic theory based on estimators constructed on random samples that would provide accurate approximations for small sizes, see Ciss et al. [23] and Diallo et al. [1].

To construct more understandable notions, x and y remain two indicators of individual well-being among, income, expenditures, caloric consumption, life expectancy, height, body weight, extent of personal safety and freedom, etc., see Ciss et al. [23] and Diallo et al. [1]. All through this paper (X, Y) denotes the value of (x, y) for a randomly chosen individual of the population. Then (X, Y) is a random pair of nonnegative real numbers defined of a given probability space $(\Omega, \mathcal{A}, \mathbb{P})$ whose cumulative distribution function (cdf) is indicated by $F(.,.)$ and we assumse that it admits a pdf $f(.,.)$. Hereafter all the requirements are done with regard to this probability space.

The use of kernel smoothing for poverty index estimation was pioneered by Dia [13], who established fundamental asymptotic properties for the univariate case. The bidimensional extension of the FGT [9] (Foster, Greer, Thorbecke) poverty index, as formulated by Duclos et al. [7] is denoted by $P(z, \alpha)$ and is defined as follows, for $\alpha = (\alpha_1 \geq 0, \alpha_2 \geq 0)$:

$$P(z,\alpha) = \begin{cases} \int_0^{z_1}\int_0^{z_2} \left(\frac{z_1 - x_1}{z_1}\right)^{\alpha_1}\left(\frac{z_2 - x_2}{z_2}\right)^{\alpha_2} f(x)\,dx_1 dx_2 & \text{if } z_i > 0 \text{ for } i = 1,2 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

where $z = (z_1, z_2)$ denotes the poverty line vector; with z_i representing the poverty threshold for dimension x_i.
Axiomatic approaches to multidimensional poverty measurement including the bidimensional FGT class have been extensively studied by Sen [18], Atkinson [3], Bourguignon and Chakravarty [5], and more recently by Alkire and Foster [2] in the context of counting-based measures.

Now allow us consider, for $n \geq 1$, a random sample

$$(X_{11}, X_{12}), (X_{21}, X_{22}), \cdots, (X_{n1}, X_{n2})$$

drawn from (X_1, X_2), on the aforementioned the probability space. The empirical plug-in estimator of (4) is specified by

$$\widehat{P}_n(z,\alpha) = \frac{1}{n}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\Big(1 - \frac{X_{i_1}1}{z_1}\Big)_+^{\alpha_1}\Big(1 - \frac{X_{i_2}2}{z_2}\Big)_+^{\alpha_2} \qquad \text{where} \qquad x_+ = \max(0,x) \tag{5}$$

Combining (1) and (2), and employing Riemann sum approximations, Ciss et al. [23] proposed the undermentioned kernel estimator for the bidimensional poverty index (4) : $P_n(z_1, z_2, \alpha_1, \alpha_2)$

$$= \frac{1}{n}\sum_{k=1}^{n}\sum_{i_1=0}^{[\frac{z_1}{h_1}]}\sum_{i_2=0}^{[\frac{z_2}{h_2}]}\Big(1 - \frac{i_1 h_1}{z_1}\Big)^{\alpha_1}\Big(1 - \frac{i_2 h_2}{z_2}\Big)^{\alpha_2} K\Big(\frac{X_{k1} - i_1 h_1}{h_1}\Big)K\Big(\frac{X_{k2} - i_2 h_2}{h_2}\Big) \tag{6}$$

where $[\frac{\cdot}{h}]$ design the integer part of $\frac{\cdot}{h}$ and for $\alpha_i = 0$ or $\alpha_i \geq 1$.
Their findings are also expansions of those of Dia [13] and of Ciss et al. [24] in one dimension. By (1) and (2), and moreover using Riemann sum approximations, Diallo et al. [1] proposed a adaptive kernel estimator version of the

bidimensional poverty index 6. Their results gave extensions of those of Zakaria et al. [25].

They both [23] and [1] assign lower weights to data pairs further from the reference point, rather than applying equal weights to all points nearby observations. However, the product of two univariate kernels does not necessarily assign identical weights to all points located equal Euclidean distance from the reference point. To address this limitation, a bivariate radialsymmetric kernel must be employed. Nonparametric smoothing techniques for poverty and inequality measures have also been advocated by Davidson and Flachaire [6] and Biewen [4], who emphasized the benefits of kernel-based approaches over purely empirical estimators, particularly in the context of bootstrap inference. We therefore propose the undermentioned kernel estimator of the bidimensional poverty index (4):

$$= \frac{1}{n}\sum_{k=1}^{n}\sum_{i_1=0}^{[\frac{z_1}{h_1}]}\sum_{i_2=0}^{[\frac{z_2}{h_2}]}\Big(1-\frac{i_1h_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{i_2h_2}{z_2}\Big)^{\alpha_2}\mathcal{K}\Big(\frac{X_{k1}-i_1h_1}{h_1},\frac{X_{k2}-i_2h_2}{h_2}\Big) \tag{7}$$

$\mathcal{K}$ is a Borel function verifying the next hypotheses

$$(H_1)\ \sup_{-\infty<x,y<+\infty}|\mathcal{K}(x,y)|<+\infty,$$
$$(H_2)\ \int_{\mathbb{R}^2}\mathcal{K}(x,y)\,dxdy=1,$$
$$(H_3)\ \lim_{||(x,y)||\to\pm\infty}||(x,y)|||\mathcal{K}(x,y)|=0$$

The estimator in (7) uses a genuine bivariate radialsymmetric kernel, which assigns equal weights to all points located at the same Euclidean distance from the reference point (z_1, z_2).

2. Construction of the Bivariate Radialsymmetric Kernel Estimator Using the Riemann Sum

For $z_1 > 0, z_2 > 0$ and

$$\Delta_{h_1,i}=[h_1 i, h_1(i+1)] \qquad i=0,\cdots,[\frac{z_1}{h_1}],\quad \Delta_{h_2,j}=[h_2 j, h_2(j+1)] \qquad j=0,\cdots,[\frac{z_2}{h_2}].$$

We obtain the succeeding Riemann sum over the two-dimensional rectangle $[0, z_1]\times[0, z_2]$:

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n}\sum_{i=0}^{[\frac{z_1}{h_1}]-1}\sum_{j=0}^{[\frac{z_2}{h_2}]-1}\Big(1-\frac{ih_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{jh_1}{z_1}\Big)^{\alpha_2}\mathcal{K}\Big(\frac{X_{k1}-ih_1}{h_1},\frac{X_{k2}-ih_2}{h_2}\Big)\\
&+\Big(z_1-h_1[\frac{z_1}{h_1}]\Big)\Big(z_2-h_2[\frac{z_2}{h_2}]\Big)\Big(1-\frac{h_1[\frac{z_1}{h_1}]}{z_1}\Big)^{\alpha_1}\Big(1-\frac{h_2[\frac{z_2}{h_2}]}{z_2}\Big)^{\alpha_2}\\
&\qquad\times\frac{1}{n}\sum_{k=1}^{n}\frac{1}{h_1h_2}\mathcal{K}\Big(\frac{X_{k1}-[\frac{z_1}{h_1}]h_1}{h_1},\frac{X_{k2}-[\frac{z_1}{h_1}]h_2}{h_2}\Big)\\
&+\frac{1}{nh_1}\sum_{k=1}^{n}\sum_{j=0}^{[\frac{z_2}{h_2}]-1}\Big(z_1-h_1[\frac{z_1}{h_1}]\Big)\Big(1-\frac{h_1[\frac{z_1}{h_1}]}{z_1}\Big)^{\alpha_1}\Big(1-\frac{jh_2}{z_2}\Big)^{\alpha_2}\\
&\qquad\times\mathcal{K}\Big(\frac{X_{k1}-[\frac{z_1}{h_1}]h_1}{h_1},\frac{X_{k2}-jh_2}{h_2}\Big)\\
&+\frac{1}{nh_2}\sum_{k=1}^{n}\sum_{i=0}^{[\frac{z_1}{h_1}]-1}\Big(z_2-h_2[\frac{z_2}{h_2}]\Big)\Big(1-\frac{h_2[\frac{z_2}{h_2}]}{z_2}\Big)^{\alpha_2}\Big(1-\frac{ih_1}{z_1}\Big)^{\alpha_1}\\
&\qquad\times\mathcal{K}\Big(\frac{X_{k1}-ih_1}{h_1},\frac{X_{k2}-[\frac{z_2}{h_2}h_2}{h_2}\Big)
\end{aligned}$$

corresponding to the integral

$$\int_{[0,z_1]\times[0,z_2]}\Big(\frac{z_1-x}{z_1}\Big)^{\alpha_1}\Big(\frac{z_2-y}{z_2}\Big)^{\alpha_2}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{h_1h_2}\mathcal{K}\Big(\frac{x-X_{i1}}{h_1},\frac{x-X_{i2}}{h_2}\Big)\,dxdy.$$

This sum can be written as:

$$\frac{1}{n}\sum_{k=1}^{n}\sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]}\Big(1\;-\;\frac{ih_2}{z_2}\Big)^{\alpha_1}\Big(1\;-\;\frac{jh_2}{z_2}\Big)^{\alpha_2}\mathcal{K}\Big(\frac{X_{k1}-ih_1}{h_1},\frac{X_{k2}-ih_2}{h_2}\Big)\;+\;\mathcal{V}_n(z_1,z_2)\;+\;\mathcal{W}_n(z_1,z_2)\;+\;\mathcal{U}_n(z_1,z_2)$$

with

$$\begin{aligned}\mathcal{V}_n(z_1,z_2)=\frac{1}{n}\frac{\Big(z_1-h_1[\frac{z_1}{h_1}]\Big)\Big(z_2-h_2[\frac{z_2}{h_2}]\Big)-h_1h_2}{h_1h_2}\Big(1-\frac{h_1[\frac{z_1}{h_1}]}{z_1}\Big)^{\alpha_1}\Big(1-\frac{h_2[\frac{z_2}{h_2}]}{z_2}\Big)^{\alpha_2}\\ \times\sum_{k=1}^{n}\mathcal{K}\Big(\frac{X_{k1}-[\frac{z_1}{h_1}]h_1}{h_1},\frac{X_{k2}-[\frac{z_1}{h_1}]h_2}{h_2}\Big).\end{aligned}$$

$$\begin{aligned}\mathcal{W}_n(z_1,z_2)=\frac{1}{n}\sum_{k=1}^{n}\sum_{j=0}^{[\frac{z_2}{h_2}]-1}\Big(1-\frac{h_1[\frac{z_1}{h_1}]}{z_1}\Big)^{\alpha_1}\Big(1-\frac{jh_2}{z_2}\Big)^{\alpha_2}\\ \times\mathcal{K}\Big(\frac{X_{k1}-[\frac{z_1}{h_1}]h_1}{h_1},\frac{X_{k2}-jh_2}{h_2}\Big)\Big(\frac{1}{h_1}\Big(z_1-h_1[\frac{z_1}{h_1}]\Big)-1\Big)\end{aligned}$$

$$\begin{aligned}\mathcal{U}_n(z_1,z_2)=\frac{1}{n}\sum_{k=1}^{n}\sum_{0=1}^{[\frac{z_1}{h_1}]-1}\Big(1-\frac{h_2[\frac{z_2}{h_2}]}{z_2}\Big)^{\alpha_2}\Big(1-\frac{ih_1}{z_1}\Big)^{\alpha_1}\\ \times\mathcal{K}\Big(\frac{X_{k2}-[\frac{z_2}{h_2}]h_2}{h_2},\frac{X_{k1}-ih_1}{h_1}\Big)\Big(\frac{1}{h_2}\Big(z_2-h_2[\frac{z_2}{h_2}]\Big)-1\Big).\end{aligned}$$

Now it comes up that

$$\mathcal{V}_n\longrightarrow 0,\qquad \mathcal{W}_n\longrightarrow 0,\qquad \mathcal{U}_n\longrightarrow 0,\qquad \text{almost sure}\quad\text{as}\qquad n\to+\infty.$$

We have:

$$|\mathcal{V}_n(z_1,z_2)|\le\begin{cases}\Big(1-\frac{h_1[\frac{z_1}{h_1}]}{z_1}\Big)^{\alpha_1}\Big(1-\frac{h_y[\frac{z_2}{h_2}]}{z_2}\Big)^{\alpha_2}\sup_{(x,y)\in\mathbb{R}^2}|\mathcal{K}(x,y)| & \text{if}\quad(\alpha_1>0,\alpha_2>0)\\ \frac{2}{n}\sum_{k=1}^{n}\mathcal{K}\Big(\frac{X_{k1}-[\frac{z_1}{h_1}]h_1}{h_1},\frac{X_{k2}-[\frac{z_1}{h_1}]h_2}{h_2}\Big) & \text{if}\quad(\alpha_1=0,\alpha_2=0)\\ \Big(1-\frac{h_1[\frac{z_1}{h_1}]}{z_1}\Big)^{\alpha_1}\sup_{(x,y)\in\mathbb{R}^2}|\mathcal{K}(x,y)| & \text{if}\quad(\alpha_1>0,\alpha_2=0)\\ \Big(1-\frac{h_2[\frac{z_2}{h_2}]}{z_2}\Big)^{\alpha_2}\sup_{(x,y)\in\mathbb{R}^2}|\mathcal{K}(x,y)| & \text{if}\quad(\alpha_1=0,\alpha_2>0).\end{cases}$$

We have:

$$\begin{aligned}\Big|\frac{\Big(z_1-h_1[\frac{z_1}{h_1}]\Big)\Big(z_2-h_2[\frac{z_2}{h_2}]\Big)-h_1h_2}{h_1h_2}\Big|&=\Big|\frac{z_1z_2-z_1h_2[\frac{z_2}{h_1}]-z_2h_1[\frac{z_1}{h_1}]+h_1h_2[\frac{z_1}{h_1}][\frac{z_2}{h_2}]-h_1h_2}{h_1h_2}\Big|\\ &=\Big|\frac{z_1z_2}{h_1h_2}+[\frac{z_1}{h_1}][\frac{z_2}{h_2}]-\frac{z_1}{h_1}[\frac{z_2}{h_2}]-\frac{z_2}{h_2}[\frac{z_1}{h_1}]-1\Big|\\ &=\Big|\frac{z_2}{h_2}(\frac{z_2}{h_1}-[\frac{z_1}{h_1}])+[\frac{z_2}{h_2}]([\frac{z_1}{h_1}]-\frac{z_1}{h_1})-1\Big|\\ &=\Big|([\frac{z_2}{h_2}]-\frac{z_2}{h_2})([\frac{z_1}{h_1}]-\frac{z_1}{h_1})-1\Big|\le 2.\end{aligned}$$

We have according to the next cases: $(\alpha_1>0,\alpha_2>0),(\alpha_1>0,\alpha_2=0)$ and $(\alpha_1=0,\alpha_2>0)$,

$$\mathcal{V}_n(z_1,z_2)\longrightarrow 0\quad p.s.\qquad\text{as}\quad n\to+\infty.$$

For $(\alpha_1 = 0, \alpha_2 = 0)$, we obtain:

$$\begin{aligned}
&\bullet \Big|\mathbb{E}\Big[\frac{1}{n}\sum_{k=1}^{n}\mathcal{K}\Big(\frac{X_{k1}-[\frac{z_1}{h_1}]h_1}{h_1}, \frac{X_{k2}-[\frac{z_1}{h_1}]h_2}{h_2}\Big)\Big]\Big| \\
&= \Big|h_1 h_2 \int_{\mathbb{R}^2} \mathcal{K}(u,v) f(x,y)\, dudv\Big| \\
&\leq h_1 h_2 \sup_{(u,v)\in\mathbb{R}^2} \int_{\mathbb{R}^2} |\mathcal{K}(u,v)|\, dudv = O(h_1 h_2) \\
&\bullet \mathbb{V}ar(X) = \mathbb{E}(X^2) - \mathbb{E}^2(X) \leq \mathbb{E}(X^2).
\end{aligned}$$

Yet

$$\mathbb{E}\Big[\mathcal{K}^2\Big(\frac{X_{k1}-\big[\frac{z_1}{h_1}\big]h_1}{h_1}, \frac{X_{k2}-\big[\frac{z_1}{h_1}\big]h_2}{h_2}\Big)\Big] \leq h_1 h_2 \sup_{(x,y)\in\mathbb{R}^2} \int_{\mathbb{R}^2} |\mathcal{K}^2(u,v)|\, dudv = O(h_1 h_2).$$

Because of the strong Kolmogorov's theorem of the large numbers, we obtain

$$\frac{2}{n}\sum_{i=1}^{n}\mathcal{K}\Big(\frac{X_{k1}-\big[\frac{z_1}{h_1}\big]h_1}{h_1}, \frac{X_{k2}-\big[\frac{z_1}{h_1}\big]h_2}{h_2}\Big) \longrightarrow 0 \quad \text{as} \quad n \to +\infty.$$

$\mathcal{K}$ is bounded by assumption, there exists $M \in \mathbb{R}_+$ such that $|\mathcal{K}| \leq M$. Hence we get:

$$|\mathcal{W}_n(z_1,z_2)| \leq \begin{cases} \Big(1 - \frac{h_1\big[\frac{z_1}{h_1}\big]}{z_1}\Big)^{\alpha_1} \sup_{(x,y)\in\mathbb{R}^2} |\mathcal{K}(x,y)| & \text{if} \quad (\alpha_1 > 0, \alpha_2 > 0) \\ \frac{1}{n} M \big[\frac{z_2}{h_2}\big] & \text{if} \quad (\alpha_1 = 0, \alpha_2 = 0) \\ \big[\frac{z_2}{h_2}\big] \sup_{(x,y)\in\mathbb{R}^2} |\mathcal{K}(x,y)| & \text{if} \quad (\alpha_1 > 0, \alpha_2 = 0) \\ \big[\frac{z_2}{h_2}\big] \sup_{(x,y)\in\mathbb{R}^2} |\mathcal{K}(x,y)| & \text{if} \quad (\alpha_1 = 0, \alpha_2 > 0). \end{cases}$$

Then

$$\mathcal{W}_n(z_1,z_2) \longrightarrow 0 \quad p.s. \qquad \text{as} \quad n \to +\infty.$$

For the case of $\mathcal{U}_n(z_1,z_2)$ it is enough from $\mathcal{W}_n(z_1,z_2)$ to make a change of position between:

$$(X \text{ and } Y); \quad (x \text{ and } y); \quad (i \text{ and } j).$$

Remark that

$$\left|\frac{z_1 - h_1\big[\frac{z_1}{h_1}\big] - h_1}{h_1}\right| = \Big|\frac{z_1}{h_1} - \Big[\frac{z_1}{h_1}\Big] - 1\Big| \leq 1.$$

Additional Hypotheses About the Bivariate Kernel $\mathcal{K}$

(H_4) $\mathcal{K}$ is a bounded variation function $V_{-\infty}^{u}K$ on $\mathbb{R}^2$. Suppose $V(\mathbb{R}^2)$ be its total variation.

(H_5) $\int_{\mathbb{R}^2} ||(u,v)|| |\mathcal{K}(u,v)| < +\infty$.

(H_6) There exists a nonincreasing function λ such that $\forall$ $x = (x_1,x_2), y = (y_1,y_2)$ we have

$$|\mathcal{K}(x) - \mathcal{K}(y)| \leq \lambda \|x - y\| \quad \text{and} \quad \lambda(u,v) \longrightarrow 0, (u,v) \to (0,0), u \geq 0, v \geq 0.$$

$\|\cdot\|$ represents the Euclidean norm.

Moreover $\lambda(\frac{u}{h_1}, \frac{v}{h_2}) = O(h_1 h_2)$ with $h = (h_1,h_2)$ on any bounded two-dimensional rectangle.

3. Convergence Properties of the Estimator

Our main results rely on the following additional assumptions regarding the function f:

C_1**:** f is uniformly continuous.

C_2**:** f admits almost everywhere derivative $f' \in L_1(\mathbb{R} \times \mathbb{R})$.

3.1. Uniform Almost Sure Consistency and the Bias Behavior of the Estimator

Theorem 1. Suppose that the assumptions H_4 and C_1 hold. Then for all $b > 0$ $(b = ((b_1, b_2))$, the estimator $P_n^{\mathcal{K}}(z_1, z_2, \alpha_1, \alpha_2)$ converges uniformly almost sure on $[0, b_1] \times [0, b_2]$ to $P(z_1, z_2, \alpha_1, \alpha_2)$ as $n \to +\infty$ $i.e.$

$$\mathbb{P}(\lim_{n\to+\infty} \sup_{(z_1,z_2)\in[0,b_1]\times[0,b_2]} |P_n^{\mathcal{K}}(z_1, z_2, \alpha_1, \alpha_2) - P(z_1, z_2, \alpha_1, \alpha_2)| = 0) = 1$$

provided $nh_1^2 h_2^2 (LogLogn)^{-1} \to +\infty$ as $n \to +\infty$.

Theorem 2. Assume that the assumptions H_4, H_5 and C_2 hold for every $b > 0$ $(b = ((b_1, b_2))$, the estimator $P_n^{\mathcal{K}}(z, \alpha)$ converges uniformly almost sure on $[0, b_1] \times [0, b_2]$ to $P(z, \alpha)$ as $n \to +\infty$ $i.e.$

$$\mathbb{P}(\lim_{n\to+\infty} \sup_{z=(z_1,z_2)\in[0,b_1]\times[0,b_2]} |P_n^{\mathcal{K}}(z, \alpha) - P(z, \alpha)| = 0) = 1$$

provided $nh_1^2 h_2^2 (LogLogn)^{-1} \to +\infty$ as $n \to +\infty$.

Lemma 1. If the assumptions C_1 holds, then for every $b > 0$, we obtain:

$$\lim_{n\to+\infty} \sup_{z=(z_1,z_2)\in[0,b_1]\times[0,b_2]} |\mathbb{E}(P_n^{\mathcal{K}}(z, \alpha)) - P(z, \alpha)| \to 0, \quad n \to +\infty.$$

Lemma 2. If the assumptions H_5 and C_2 hold, then:

$$\begin{aligned}\sup_{z=(z_1,z_2)\in\mathbb{R}^2} |\mathbb{E}(P_n^{\mathcal{K}}(z, \alpha)) - P(z, \alpha)| \leq h_1 h_2 \Big(\big(\int_{\mathbb{R}^2} |f^{'}(x, y)|\, dxdy \big) \\ \times \big(\int_{\mathbb{R}^2} (|u| + 1)(|v| + 1)|\mathcal{K}(u, v)|\, dudv \big) \\ + 4(\alpha_1\alpha_2 M + Ah_1h_2) \int_{-\infty}^{+\infty} |\mathcal{K}(u, v)|\, dudv \Big)\end{aligned}$$

where

$$M = \sup_{(z_1,z_2)\in\mathbb{R}^2} \frac{F(z_1, z_2)}{z_1 z_2} \qquad A = \sup_{(x,y)\in\mathbb{R}^2} f(x, y).$$

Remark 1. If $\mathcal{K}$ satisfies the assumption H_5, then by H_1, the kernel $\hat{\mathcal{K}} = \frac{\mathcal{K}^2}{\int_{\mathbb{R}^2} \mathcal{K}^2(y_1,y_2)\, dy_1 dy_2}$ also satisfies it.

The two preceding lemmas conduct to the next corollaries:

Corollary 1. Under the assumptions of Lemma 1, we obtain uniformly on $[0, b_1] \times [0, b_2]$ (resp $\mathbb{R}^2$)

$$\begin{aligned}\lim_{n\to+\infty} \mathbb{E}\Big(\sum_{i=1}^{[\frac{z_1}{h_1}]} \sum_{j=1}^{[\frac{z_2}{h_2}]} \Big(1 - \frac{ih_1}{z_1}\Big)^{2\alpha_1} \Big(1 - \frac{jh_2}{z_2}\Big)^{2\alpha_2} \mathcal{K}^2\Big(\frac{X_{k1} - ih_1}{h_1}, \frac{X_{k2} - ih_2}{h_2}\Big)\Big) \\ = \Big(\int_{\mathbb{R}^2} \mathcal{K}^2(y_1, y_2)\, dy_1 dy_2 \Big) P(z_1, z_2, 2\alpha_1, 2\alpha_2).\end{aligned}$$

Corollary 2. If the assumptions of theorem 2 hold and if $h_1 h_2 = O(n^{-1} \log\log n)^{1/4}$, then for every $b > 0$, we get almost surely

$$\sup_{z=(z_1,z_2)\in[0,b_1]\times[0,b_2]} |P_n^{\mathcal{K}}(z, \alpha) - P(z, \alpha)| = O(n^{-1} \log\log n)^{1/4}.$$

3.2. Uniform Mean Square Consistency of the Estimator

Theorem 3. If the assumptions H_6 and C_1 hold. Then

1)

$$\lim_{n\to+\infty} n\mathbb{V}ar(P_n^{\mathcal{K}}(z, \alpha)) = \Big(\int_{\mathbb{R}^2} \mathcal{K}^2(y_1, y_2)\, dy_1 dy_2 \Big) P(z, 2\alpha_1, 2\alpha_2) - \Big(P(z, \alpha_1, \alpha_2)\Big)^2.$$

2) for all $b > 0$,

$$\lim_{n\to+\infty} \sup_{z=(z_1,z_2)\in[0,b_1]\times[0,b_2]} \mathbb{E}\Big(P_n^{\mathcal{K}}(z, \alpha_1, \alpha_2) - P(z, \alpha_1, \alpha_2)\Big)^2 = 0.$$

Theorem 4. Assume that H_6 and C_2 hold. Then:

1)

$$\lim_{n\to+\infty} n\mathbb{V}ar(P_n^{\mathcal{K}}(z,\alpha_1,\alpha_2)) = \Big(\int_{\mathbb{R}^2}\mathcal{K}^2(y_1,y_2)\,dy_1dy_2\Big)P(z,2\alpha_1,2\alpha_2) - \Big(P(z,\alpha_1,\alpha_2)\Big)^2.$$

2) moreover, if the assumption H_5 holds we have for every $b>0$,

$$\lim_{n\to+\infty}\sup_{z=(z_1,z_2)\in[0,b_1]\times[0,b_2]}\mathbb{E}\Big(P_n^{\mathcal{K}}(z,\alpha_1,\alpha_2)-P(z,\alpha_1,\alpha_2)\Big)^2=0.$$

The proof of this theorem assumes either C_1 or C_2 holds and relies on theorem 5 below, which is established using the **Lemma** 3. Let $0\le\theta_i^j\le 1, i,j=1,2$. Then for every $x=(x_1,x_2), y=(y_1,y_2)$ and $x\neq y$ we obtain

$$\lim_{n\to+\infty}\sup_{(z_1,z_2)\in[0,1]\times[0,1]}\Big((h_1h_2)^{-2}\int_{\mathbb{R}^2}|\mathcal{K}(\frac{u_1-x_1+\theta_1^1}{h_1},\frac{u_2-x_2+\theta_1^2}{h_2}) \times\mathcal{K}(\frac{u_1-y_1+\theta_2^1}{h_1},\frac{u_2-y_2+\theta_2^2}{h_2})|f(u_1,u_2)\,du_1du_2\Big)=0.$$

Theorem 5. Assume that the assumption H_6 holds. Then for every $b>0$,

$$\lim_{n\to+\infty}\sup_{(z_1,z_2)\in[0,b_1]\times[0,b_2]}\sum_{0\le i_1\neq j_1\le[\frac{z_1}{h_1}]}\sum_{0\le i_2\neq j_2\le[\frac{z_2}{h_2}]}\Big(1-\frac{i_1h_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{j_1h_1}{z_1}\Big)^{\alpha_1} \times\Big(1-\frac{i_2h_2}{z_2}\Big)^{\alpha_2}\Big(1-\frac{j_2h_2}{z_2}\Big)^{\alpha_2} \times\int_{\mathbb{R}^2}\mathcal{K}\Big(\frac{u_1-i_1h_1}{h_1},\frac{u_1-j_1h_1}{h_1}\Big)\mathcal{K}\Big(\frac{u_2-i_2h_2}{h_2},\frac{u_2-j_2h_2}{h_2}\Big)f(u_1,u_2)\,du_1du_2=0.$$

Remark 2. The bivariate radialsymmetric kernel estimator $P_n^{\mathcal{K}}(z,\alpha_1,\alpha_2)$ has asymptotic efficiency regarding to $\widehat{P}_n^{\mathcal{K}}(z,\alpha_1,\alpha_2)$,

$$e(z_1,z_2,\alpha_1,\alpha_2) = \frac{\Big((\int_{R^2}\mathcal{K}^2(y_1,y_2)\,dy_1dy_2)P(z,2\alpha_1,2\alpha_2)-(P(z,\alpha_1,\alpha_2))^2\Big)}{\big(P(z,2\alpha_1,2\alpha_2)-(P(z,\alpha_1,\alpha_2))^2\big)}.$$

The integral $\int_{R^2}\mathcal{K}^2(y_1,y_2)\,dy_1dy_2$ is strictly inferior to 1 for $\mathcal{K}$ bivariate radialsymmetric kernel [20]. Then we have in this case $e(z,\alpha_1,\alpha_2)<1$. In Theorem 4, the speeed of convergence in mean square is estimated to $O(\frac{1}{n})$ if h_1h_2 is estimated to $O(\frac{1}{\sqrt{n}})$.

4. Simulations

4.1. Proceedings

We carry out a simulation study to compare the finite-sample performance of the proposed radialsymmetric estimator with the multiplicative kernel estimator. We performed 75 replications of samples of size $n = 50$. The data were generated from a bivariate distribution with independent coordinates: X following a Pareto distribution on $[0,1]$ with parameters $x_0 = 0.02$ and $b = 0.2$, and Y following an exponential distribution with parameter $\lambda = 1$. The Epanechnikov kernel, which satisfies hypotheses H_1 through H_6, was employed. For fixed poverty lines (z_1,z_2), we computed our estimator $P_n^{\mathcal{K}}(z,\alpha_1,\alpha_2)$ based on the radialsymmetric kernel density, and the multiplicative kernel estimator $P_n(z,\alpha_1,\alpha_2)$ with bandwidths $h_j=1/\sqrt{n\log n}$ for $j=1,2$. We consider the respective sequences:

$$(P_{n,1}(z,\alpha_1,\alpha_2),P_{n,2}(z,\alpha_1,\alpha_2),\dots,P_{n,75}(z,\alpha_1,\alpha_2))$$

and

$$(P_{n,1}^{\mathcal{K}}(z,\alpha_1,\alpha_2),P_{n,2}^{\mathcal{K}}(z,\alpha_1,\alpha_2),\dots,P_{n,75}^{\mathcal{K}}(z,\alpha_1,\alpha_2)).$$

For each estimator, we compute the mean value mv_i, the mean squared error MSE_i, and the standard deviation σ_i across the 75 replications, with $i=1$ corespond to the estimator based on the Parzen bivariate product kernel density and $i=2$ to our new radialsymmetric kernel estimator.

For our new bivariate radialsymmetric kernel estimator, we have:

$$mv_2=\overline{P_n^{\mathcal{K}}(z,\alpha_1,\alpha_2)}=\frac{1}{75}\sum_{i=1}^{75}P_{n,i}^{\mathcal{K}}(z,\alpha_1,\alpha_2)$$

$$MSE_2 = \frac{1}{75}\sum_{i=1}^{75}(P^{\mathcal{K}}_{n,i}(z,\alpha_1,\alpha_2) - P^{\mathcal{K}}(z,\alpha_1,\alpha_2))^2$$

$$\sigma_2 = \frac{1}{74}\sum_{i=1}^{75}(P^{\mathcal{K}}_{n,i}(z,\alpha_1,\alpha_2) - \overline{P^{\mathcal{K}}_{n}(z,\alpha_1,\alpha_2)})^2.$$

For the bivariate product kernel estimator, we define similarly mv_1, MSE_1, σ_1.

Next, we choose $(\alpha_1,\alpha_2) = (0,0)$, $(\alpha_1,\alpha_2) = (1,1)$ and $(\alpha_1,\alpha_2) = (2,2)$. The outcomes are summerized in the table bellow.

4.2. Table of Outcomes

The simulation study below is carried out on 75 replications of samples of size $n = 50$.

Table 1. *Comparative table of the simulation study* $n = 50$.

(z_1, z_2)	(0.1,0.1)	(0.1,0.4)	(0.1,0.8)	(0.4,0.4)	(0.4,0.8)
(α_1,α_2)=(0,0), n=50					
MSE_1	0.0004961909	0.00196089	0.002708704	0.002293248	0.004116847
MSE_2	0.0004977057	0.001959314	0.002708338	0.002294269	0.004115129
σ_1	0.02242535	0.04458013	0.05239569	0.04821035	0.06459474
σ_2	0.02245955	0.04456221	0.05239215	0.04822108	0.06458126
(α_1,α_2)=(1,1),n=50					
MSE_1	$5.76579e-05$	0.000270894	0.0003731209	0.0005470031	0.00114277
MSE_2	$5.76579e-05$	0.0002709673	0.0003734548	0.0005467546	0.001141166
σ_1	0.007644414	0.01656969	0.01944642	0.02354559	0.03403253
σ_2	0.007644414	0.01657194	0.01945511	0.02354025	0.03400863
(α_1,α_2)=(2,2),n=50					
MSE_1	$1.305752e-05$	$7.640142e-05$	0.0001393062	0.000361386	0.0006796745
MSE_2	$1.308341e-05$	$7.619261e-05$	0.0001390598	0.0003599706	0.0006791613
σ_1	0.003637853	0.008799652	0.01188229	0.01913817	0.02624613
σ_2	0.003641458	0.008787619	0.01187177	0.01910066	0.02623622

4.3. Commentary

The studied cases are $P(z,0,0)$, $P(z,1,1)$ and $P(z,2,2)$ that are usually and in that order called the poverty rate intersection, the depth of poverty intersection and the severity of poverty intersection in two-dimensional [7].

Simulation results indicate that the radialsymmetric estimator yields smaller mean squared errors for small sample sizes, especially when the true density exhibits spherical contours. We can conclude that our new bivariate radialsymmetric kernel estimator behaves much better.

5. Details of the Main Proofs

Lemma 4. Let $0 \le \theta_i < 1$, $i = 1,2$. If f is uniformly continuous and bounded, we get uniformly comparatively to $(\theta_i, y_i) \in [0,1]\times\mathbb{R}$

$$\lim_{n\to+\infty}\int_{\mathbb{R}^2}(h_1h_2)^{-1}\mathcal{K}(\frac{u_1}{h_1},\frac{u_2}{h_2})f(u_1+y_1-\theta_1h_1,u_2+y_2-\theta_2h_2)\,du_1du_2 = f(y_1,y_2)\int_{\mathbb{R}^2}\mathcal{K}(u_1,u_2)\,du_1du_2.$$

Proof : Let $\varepsilon > 0$, there remains $\eta > 0$ such that $\|\theta h\| < \eta$, we have:

$$\begin{aligned}&\Big|\int_{\mathbb{R}^2}(h_1h_2)^{-1}\mathcal{K}\Big(\frac{u_1}{h_1},\frac{u_2}{h_2}\Big)f(u_1+y_1-\theta_1h_1,u_2+y_2-\theta_2h_2)\,du_1du_2-\int_{\mathbb{R}^2}(h_1h_2)^{-1}\mathcal{K}\Big(\frac{u_1}{h_1},\frac{u_2}{h_2}\Big)f(u_1+y_1,u_2+y_2)\,du_1du_2\Big|\\&\le\int_{\mathbb{R}^2}(h_1h_2)^{-1}\Big|\mathcal{K}\Big(\frac{u_1}{h_1},\frac{u_2}{h_2}\Big)\Big||f(u_1+y_1-\theta_1h_1,u_2+y_2-\theta_2h_2)-f(u_1+y_1,u_2+y_2)|\,du_1du_2\\&\le\varepsilon\int_{\mathbb{R}^2}(h_1h_2)^{-1}\Big|\mathcal{K}\Big(\frac{u_1}{h_1},\frac{u_2}{h_2}\Big)\Big|\,du_1du_2.\end{aligned}$$

For all vector (y_1,y_2); take then n great enough like that $\|h\| < \eta$ to obtain the following inequality for every

$(\theta_i, y_i), \quad i = 1, 2$. It crops up that

$$\Big| \int_{\mathbb{R}^2} (h_1h_2)^{-1} \mathcal{K}\Big(\frac{u_1}{h_1}, \frac{u_2}{h_2}\Big) f(u_1 + y_1 - \theta_1 h_1, u_2 + y_2 - \theta_2 h_2)\, du_1 du_2 - f(y_1, y_2) \int_{\mathbb{R}^2} \mathcal{K}(u_1, u_2)\, du_1 du_2 \Big|$$
$$\leq \Big| \int_{\mathbb{R}^2} (h_1h_2)^{-1} \mathcal{K}\Big(\frac{u_1}{h_1}, \frac{u_2}{h_2}\Big) f(u_1 + y_1 - \theta_1 h_1, u_2 + y_2 - \theta_2 h_2)\, du_1 du_2$$
$$- \int_{\mathbb{R}^2} (h_1h_2)^{-1} \mathcal{K}\Big(\frac{u_1}{h_1}, \frac{u_2}{h_2}\Big) f(u_1 + y_1, u_2 + y_2)\, du_1 du_2 \Big|.$$

Let $g(y_1, y_2)$ verifying

$$\int_{\mathbb{R}^2} |g(y_1, y_2)|\, dy_1 dy_2 < \infty.$$

Consider the vector of a positive constant sequence, $h_n = (h_{1,n}, h_{2,n})$ verifying

$$\lim_{n \to +\infty} h_n = 0.$$

Define

$$g_n(x_1, x_2) = \frac{1}{h_1 h_2} \int_{\mathbb{R}^2} \mathcal{K}\Big(\frac{y_1}{h_1}, \frac{y_2}{h_2}\Big) g(x_1 - y_1, x_2 - y_2)\, dy_1 dy_2.$$

Then at all couple of point (x_1, x_2) of continuity of $g(\cdot, \cdot)$ we have:

$$g_n(x_1, x_2) - g(x_1, x_2) \int_{\mathbb{R}^2} \mathcal{K}(y_1, y_2)\, dy_1 dy_2 = \int_{\mathbb{R}^2} (h_1h_2)^{-1} \mathcal{K}\Big(\frac{y_1}{h_1}, \frac{y_2}{h_2}\Big) g(x_1 - y_1, x_2 - y_2)\, dy_1 dy_2.$$

Let $\delta > 0$, and share the area of integration into two areas: $\|y\| \leq \delta$ and $\|y\| > \delta$. Then we have

$$\begin{aligned}
G &= |g_n(x_1, x_2) - g(x_1, x_2) \int_{\mathbb{R}^2} \mathcal{K}(y_1, y_2)\, dy_1 dy_2| \\
&\leq \max_{\|y\| \leq \delta} |g(x_1 - y_1, x_2 - y_2) - g(x_1, x_2) \int_{\|z\| \leq \frac{\delta}{\|h\|}} |\mathcal{K}(z_1, z_2)|\, dz_1 dz_2 \\
&+ \int_{\|y\| \geq \delta} \frac{|g(x_1 - y_1, x_2 - y_2)|}{\|y\|} \frac{\|y\|}{h_1 h_2} \mathcal{K}\Big(\frac{y_1}{h_1}, \frac{y_2}{h_2}\Big)\, dy_1 dy_2 \\
&+ |g(x_1, x_2)| \int_{\|y\| \geq \delta} (h_1h_2)^{-1} \mathcal{K}\Big(\frac{u_1}{h_1}, \frac{u_2}{h_2}\Big)\, dy_1 dy_2 \\
&\leq \max_{\|y\| \leq \delta} |g(x_1 - y_1, x_2 - y_2) - g(x_1, x_2)| \int_{\mathbb{R}^2} |\mathcal{K}(z_1, z_2)|\, dz_1 dz_2 \\
&+ \frac{1}{\delta} \sup_{\|z\| \geq \frac{\delta}{h_1 h_2}} \|(z_1, z_2)\|\, |\mathcal{K}(z_1, z_2)| \int_{\mathbb{R}^2} |g(y_1, y_2)|\, dy_1 dy_2 \\
&+ |g(x_1, x_2)| \int \int_{\|z\| \geq \frac{\delta}{h_1 h_2}} |\mathcal{K}(z_1, z_2)|\, dz_1 dz_2.
\end{aligned}$$

The latter term of this inequality converges to 0 as n tends to ∞, and then lets δ tends to 0. Then at all couple of point (x_1, x_2) of continuity of $g(\cdot, \cdot)$,

$$\lim_{n \to +\infty} g_n(x_1, x_2) = g(x_1, x_2) \int_{\mathbb{R}^2} \mathcal{K}(y_1, y_2)\, dy_1 dy_2.$$

Therefore

$$\lim_{n \to +\infty} \int_{\mathbb{R}^2} (h_1h_2)^{-1} \mathcal{K}\Big(\frac{u_1}{h_1}, \frac{u_2}{h_2}\Big) f(u_1 + y_1, u_2 + y_2)\, du_1 du_2 = f(y_1, y_2) \int_{\mathbb{R}^2} \mathcal{K}(u_1, u_2)\, du_1 du_2$$

uniformly. The demonstration of the lemma is complete. ■

5.1. Convergence Properties of Estimator

5.1.1. Proof of the Lemma 1

Remark first that

$$\lim_{z=(z_1,z_2)\to(0,0)} \frac{F(z)-F(z_1,0)-F(0,z_2)+F(0,0)}{z_1z_2} = f(0,0) \quad \text{for} \quad (x_0,y_0)=(0,0).$$

Indeed using the mean value theorem, we get:

$$F(z)-F(z_1,0) = (z_2-0)\frac{\partial F(z_1,s_1)}{\partial z_2} \quad \text{with} \quad s_1 \in]0,z_2[$$

and

$$F(0,0)-F(0,z_2) = -(z_2-0)\frac{\partial F(0,s_1)}{\partial z_2} \quad \text{with} \quad s_1 \in]0,z_2[.$$

Hence

$$\begin{aligned} F(z)-F(z_1,0)-F(0,z_2)+F(0,0) &= z_2\Big(\frac{\partial F(z_1,s_1)}{\partial z_2} - \frac{\partial F(0,s_1)}{\partial z_2}\Big) \\ &= z_2(z_1-0)\frac{\partial F(s_1,s_2)}{\partial z_1\partial z_2} \quad \text{with} \quad s_1\in]0,z_1[\\ &= z_1z_2\frac{\partial^2 F(s_1,s_2)}{\partial z_1\partial z_2}. \end{aligned}$$

Then

$$\frac{F(z)-F(z_1,0)-F(0,z_2)+F(0,0)}{z_1z_2} = \frac{\partial^2 F(s_1,s_2)}{\partial z_1\partial z_2}$$

and

$$\frac{\partial^2 F(s_1,s_2)}{\partial z_1\partial z_2} \to \frac{\partial^2 F(0,0)}{\partial z_1\partial z_2} = f(0,0) \quad \text{as} \quad (z_1,z_2)\to(0,0).$$

Therefore the ratio $\frac{F(z_1,z_2)}{z_1z_2}$ is limited.

Consider $\bar{\Delta}_{h_1,i} = \Delta_{h_1,i}\cap[0,z_1]$; $\bar{\Delta}_{h_2,j} = \Delta_{h_2,j}\cap[0,z_2]$ and χ_B the indicator function of the set $B_1\times B_2$. Let $(z_1,z_2)\in[0,b_1]\times[0,b_2]$. We have:

$$\mathbb{E}(P_n^{\mathcal{K}}(z,\alpha_1,\alpha_2)) = \sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]}\Big(1-\frac{ih_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{jh_2}{z_2}\Big)^{\alpha_2}\int_{\mathbb{R}^2}\mathcal{K}(u_1,u_2)\times f(u_1h_1-ih_1,u_2h_2-jh_2)\,du_1du_2$$

which we can rewrite like the next form:

$$\begin{aligned} &\int_{[0,z_1]\times[0,z_2]}\sum_{i=0}^{[\frac{z_1}{h_1}]}\chi_{\bar{\Delta}_{h_1,i}}(x)\sum_{j=0}^{[\frac{z_2}{h_2}]}\chi_{\bar{\Delta}_{h_2,j}}(y)\Big(1-\frac{ih_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{jh_2}{z_2}\Big)^{\alpha_2} \\ &\qquad\times\int_{\mathbb{R}^2}\mathcal{K}(u_1,u_2)f(u_1h_1-ih_1,u_2h_2-jh_2)\,du_1du_2dxdy \\ &+\big(h_1([\frac{z_1}{h_1}]+1)-z_1\big)\big(h_2([\frac{z_2}{h_2}]+1)-z_2\big)\left(1-\frac{h_1[\frac{z_1}{h_1}]}{z_1}\right)^{\alpha_1}\left(1-\frac{h_2[\frac{z_2}{h_2}]}{z_2}\right)^{\alpha_2} \\ &\qquad\times\int_{\mathbb{R}^2}\mathcal{K}(u_1,u_2)f\left(u_1h_1+\frac{h_1[\frac{z_1}{h_1}]}{z_1},u_2h_2-\frac{h_2[\frac{z_2}{h_2}]}{z_2}\right)du_1du_2. \end{aligned} \tag{8}$$

We have:

$$\begin{aligned} &\sup_{z=(z_1,z_2)\in\mathbb{R}^2}|(h_1([\frac{z_1}{h_1}]+1)-z_1)(h_2([\frac{z_2}{h_2}]+1)-z_2)\Big(1-\frac{h_1[\frac{z_1}{h_1}]}{z_1}\Big)^{\alpha_1}\Big(1-\frac{h_2[\frac{z_2}{h_2}]}{z_2}\Big)^{\alpha_2} \\ &\qquad\times\int_{\mathbb{R}^2}\mathcal{K}(u_1,u_2)f\Big(u_1h_1+\frac{h_1[\frac{z_1}{h_1}]}{z_1},u_2h_2-\frac{h_2[\frac{z_2}{h_2}]}{z_2}\Big)du_1du_2| \\ &\qquad\le h_1h_2\sup_{(x,y)\in\mathbb{R}^2}f(x,y)\int_{\mathbb{R}^2}|\mathcal{K}(u_1,u_2)|\,du_1du_2. \end{aligned} \tag{9}$$

Since we have $|h_i([\frac{z_i}{h_i}]+1)-z_i|\le h_i \quad i=1,2.$
By hypothese H_2 we have:

$$P(z,\alpha_1,\alpha_2)=\int_{[0,z_1]\times[0,z_2]}\Big(1-\frac{x}{z_1}\Big)^{\alpha_1}\Big(1-\frac{y}{z_2}\Big)^{\alpha_2}\int_{\mathbb{R}^2}\mathcal{K}(u_1,u_2)\,du_1du_2 f(x,y)\,dxdy. \tag{10}$$

Let $(x,y)\in\bar{\Delta}_{h_1,i}\times\bar{\Delta}_{h_2,j}$. According to the terms in (8) and (10) we have:

$$\begin{aligned}
&\Big|\Big(1-\frac{ih_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{jh_2}{z_2}\Big)^{\alpha_2}\int_{\mathbb{R}^2}\mathcal{K}(u_1,u_2)f(u_1h_1-ih_1,u_1h_2-jh_2)\,du_1du_2\\
&\qquad\qquad -\Big(1-\frac{x}{z_1}\Big)^{\alpha_1}\Big(1-\frac{y}{z_2}\Big)^{\alpha_2}\mathcal{K}(u_1,u_2)\,du_1du_2 f(x,y)\Big|\\
&\qquad =\Big|\int_{\mathbb{R}^2}\Big[\Big(1-\frac{ih_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{jh_2}{z_2}\Big)^{\alpha_2}f(u_1h_1-ih_1,u_2h_2-jh_2)\\
&\qquad\qquad -\Big(1-\frac{x}{z_1}\Big)^{\alpha_1}\Big(1-\frac{y}{z_2}\Big)^{\alpha_2}f(x,y)\Big]\mathcal{K}(u_1,u_2)\,du_1du_2\Big|\\
&\le\int_{\mathbb{R}^2}\Big|\Big(1-\frac{ih_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{jh_2}{z_2}\Big)^{\alpha_2}-\Big(1-\frac{x}{z_1}\Big)^{\alpha_1}\Big(1-\frac{y}{z_2}\Big)^{\alpha_2}\Big|f(x,y)|\mathcal{K}(u_1,u_2)|\,du_1du_2\\
&\quad+\int_{\mathbb{R}^2}\Big|\Big(1-\frac{ih_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{jh_2}{z_2}\Big)^{\alpha_2}\Big||f(u_1h_1-ih_1,u_1h_2-jh_2)-f(x,y)||\mathcal{K}(u_1,u_2)|\,du_1du_2.
\end{aligned}\tag{11}$$

Suppose $(x,y)\in\Delta_{h_1,i}\times\Delta_{h_2,j}$. Because of the first order Taylor formula applied to the function, we have

$$g(x,y)=\Big(1-\frac{x}{z_1}\Big)^{\alpha_1}\Big(1-\frac{y}{z_2}\Big)^{\alpha_2},$$

for $c_1\in]h_1i,x[$ and $c_2\in]h_2j,y[$

$$\begin{aligned}
&\Big|\Big(1-\frac{ih_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{jh_2}{z_2}\Big)^{\alpha_2}-\Big(1-\frac{x}{z_1}\Big)^{\alpha_1}\Big(1-\frac{y}{z_2}\Big)^{\alpha_2}\Big|\\
&=\Big|\Big(1-\frac{c_1}{z_1}\Big)^{\alpha_1-1}\frac{\alpha_1}{z_1}\Big(1-\frac{c_2}{z_2}\Big)^{\alpha_2}\Big(ih_1-x\Big)+\Big(1-\frac{c_1}{z_1}\Big)^{\alpha_1}\frac{\alpha_2}{z_2}\Big(1-\frac{c_2}{z_2}\Big)^{\alpha_2-1}\Big(jh_2-y\Big)\Big|\\
&\le 2\Big(\frac{\alpha_1h_1}{z_1}+\frac{\alpha_2h_2}{z_2}\Big)=2\Big(\frac{z_2\alpha_1h_1+z_1\alpha_2h_2}{z_1z_2}\Big).
\end{aligned}$$

Then, representing by $I_1^{i,j}(x,y)$ the first integral term of the expression in right hand-side of (11) and

$$I_1(x,y)=\sum_{i=0}^{[\frac{z_1}{h_1}]}\chi_{\bar{\Delta}_{h_1,i}}(x)\sum_{j=0}^{[\frac{z_2}{h_2}]}\chi_{\bar{\Delta}_{h_2,j}}(y)I_1^{i,j}(x,y).$$

We have:

$$\begin{aligned}
\int_0^{z_1}\int_0^{z_2}I_1(x,y)\,dxdy&\le\frac{2(\alpha_1z_2+\alpha_2z_1)\Big\|\Big(h_1,h_2\Big)\Big\|}{z_1z_2}\int_{[0,z_1]\times[0,z_2]}\Big(\int_{\mathbb{R}^2}f(x,y)|\mathcal{K}(u_1,u_2)|\,du_1du_2\Big)\,dxdy\\
&=2(\alpha_1z_2+\alpha_2z_1)\Big\|\Big(h_1,h_2\Big)\Big\|\Big(\int_{\mathbb{R}^2}|\mathcal{K}(u_1,u_2)|\,du_1du_2\Big)\frac{F(z)}{z_1z_2}.
\end{aligned}\tag{12}$$

Denoting by $I_2^{i,j}(x,y)$ the second integral term of the formula in right hand-side of (11) and

$$I_2(x,y)=\sum_{i=0}^{[\frac{z_1}{h_1}]}\chi_{\bar{\Delta}_{h_1,i}}(x)\sum_{j=0}^{[\frac{z_2}{h_2}]}\chi_{\bar{\Delta}_{h_2,j}}(y)I_2^{i,j}(x,y).$$

We obtain:

$$\begin{aligned}
I_2^{i,j}(x,y)&\le\int_{\mathbb{R}^2}|f(u_1h_1-ih_1,u_2h_2-jh_2)-f(ih_1,jh_2)||\mathcal{K}(u_1,u_2)|\,du_1du_2\\
&\quad+\int_{\mathbb{R}^2}|f(ih_1,jh_2)-f(x,y)||\mathcal{K}(u_1,u_2)|\,du_1du_2.
\end{aligned}\tag{13}$$

Suppose $\varepsilon > 0$, as f, is continuous uniformly, there exists $\eta_0 = \eta_0(z_1, z_2) > 0$ such that $\|(ih_1, jh_2) - (x, y)\| < \eta_0$ hence if $\|h\| < \eta_0$, we get $|f(u_1h_1 - ih_1, u_2h_2 - jh_2) - f(x, y)| < \frac{\varepsilon}{\|b\|}$. Then

$$\int_{[0,z_1]\times[0,z_2]} I_2(x,y)\,dxdy \leq \sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]} h_1h_2 \int_{\mathbb{R}^2} |f(u_1h_1 - ih_1, u_2h_2 - jh_2) - f(ih_1, jh_2)| \times |\mathcal{K}(u_1,u_2)|\,du_1du_2 + \varepsilon \int_{\mathbb{R}^2} |\mathcal{K}(u_1,u_2)|\,du_1du_2dxdy.$$

We get because of the uniform continuity of f:

$$\exists \quad \eta_1 = \eta_1(z_1, z_2) > 0, \|uh\| < \eta_1 \Rightarrow |f(u_1h_1 - ih_1, u_2h_2 - jh_2) - f(ih_1, jh_2)| < \frac{\varepsilon}{\|b\|}.$$

Therefore

$$\begin{aligned}
&\sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]} h_1h_2 \int_{\mathbb{R}^2} |f(u_1h_1 - ih_1, u_2h_2 - jh_2) - f(ih_1, jh_2)||\mathcal{K}(u_1,u_2)|\,du_1du_2 \\
&\leq \sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]} h_1h_2 \int_{\|uh\|<\eta_1} \frac{\varepsilon}{\|b\|}|\mathcal{K}(u_1,u_2)|\,du_1du_2 \\
&+\sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]} h_1h_2 \int_{\|uh\|\geq\eta_1} |f(u_1h_1 - ih_1, u_2h_2 - jh_2) - f(ih_1, jh_2)||\mathcal{K}(u_1,u_2)|\,du_1du_2 \\
&\leq \varepsilon \int_{\mathbb{R}^2} |\mathcal{K}(u_1,u_2)|\,du_1du_2 + \sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]} h_1h_2 \int_{\|uh\|\geq\eta_1} |f(u_1h_1 - ih_1, u_2h_2 - jh_2)| \\
&\times|\mathcal{K}(u_1,u_2)|\,du_1du_2 + \sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]} h_1h_2 \int_{\|uh\|\geq\eta_1} |f(ih_1, jh_2)||\mathcal{K}(u_1,u_2)|\,du_1du_2.
\end{aligned}$$

Consider f, continuous, hence Riemann-integrable; therefore

$$\sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]} h_1h_2|f(ih_1, jh_2)| \to \int_{[0,b_1]\times[0,b_2]} f(x,y)\,dxdy \qquad n \to +\infty.$$

Since

$$(h_1([\frac{b_1}{h_1}] + 1) - b_1)(h_2([\frac{b_2}{h_2}] + 1) - b_2)f(h_1[\frac{b_1}{h_1}], h_2[\frac{b_2}{h_2}]) \to 0, \quad n \to +\infty.$$

Then the sum

$$\sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]} h_1h_2|f(ih_1, jh_2)| \to \int_{[0,b_1]\times[0,b_2]} f(x,y)\,dxdy$$

is limited. Suppose A be this limit.
A change of variables $v_i = u_ih_i \quad i = 1, 2,$ gives

$$\begin{aligned}
&\sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]} h_1h_2 \int_{\|uh\|\geq\eta_1} |f(u_1h_1 - ih_1, u_2h_2 - jh_2)||\mathcal{K}(u_1,u_2)|\,du_1du_2 \\
&+\sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]} h_1h_2 \int_{\|uh\|\geq\eta_1} |f(ih_1, jh_2)||\mathcal{K}(u_1,u_2)|\,du_1du_2 \\
&\leq \sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]} h_1h_2 \int_{\|v\|\geq\eta_1} |f(v_1 - ih_1, v_2 - jh_2)|\frac{1}{h_1h_2}\Big|\mathcal{K}\Big(\frac{v_1}{h_1}, \frac{v_2}{h_2}\Big)\Big|\,dv_1dv_2 \\
&+A\int_{\|uh\|\geq\eta_1} |\mathcal{K}(u_1,u_2)|\,du_1du_2.
\end{aligned}$$

According to (H_3) one can find a fixed value $C > 0$ such that $|\frac{v_1}{h_1}\frac{v_2}{h_2}| \geq C$, we obtain

$$|\frac{v_1}{h_1}\frac{v_2}{h_2}|\Big|\mathcal{K}\Big(\frac{v_1}{h_1},\frac{v_2}{h_2}\Big)\Big| \leq \frac{\eta_1\varepsilon}{\|b\|}.$$

Consider $\eta = \inf(\eta_1, Ch_1h_2),\quad i = 1, 2$ to be small enough, hence

$$\frac{1}{\eta_1}\int_{\|v\|\geq\eta_1}\Big|f(v_1 - ih_1, v_2 - jh_2)\frac{v_1}{h_1}\frac{v_2}{h_2}\mathcal{K}\Big(\frac{v_1}{h_1},\frac{v_2}{h_2}\Big)\Big|\,dv_1dv_2$$
$$\leq \frac{\varepsilon}{\|b\|}\int_{\|v\|\geq\eta_1}|f(v_1 - ih_1, v_2 - jh_2)|\,dv_1dv_2 \leq \frac{\varepsilon}{\|b\|}\int_{\mathbb{R}^2}|f(x,y)|\,dxdy = \frac{\varepsilon}{\|b\|}$$

where

$$\sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]}h_1h_2\int_{\|v\|\geq\eta_1}|f(v_1 - ih_1, v_2 - jh_2)|\frac{1}{h_1h_2}\Big|\mathcal{K}\Big(\frac{v_1}{h_1},\frac{v_2}{h_2}\Big)\Big|\,dv_1dv_2 \leq \frac{z_1z_2\varepsilon}{\|b\|} \leq \varepsilon.$$

Since $A\int_{\|uh\|\geq\eta_1}|\mathcal{K}(u_1,u_2)|\,du_1du_2 \to 0, n \to +\infty$, we have together with (12) and (9)

$$\lim_{n\to+\infty}\sup_{z=(z_1,z_2)\in[0,b_1]\times[0,b_2]}|\mathbb{E}(P_n(z,\alpha_1,\alpha_2)) - P(z,\alpha_1,\alpha_2)| \leq 2\varepsilon\int\int_{-\infty}^{+\infty}\mathcal{K}(u_1,u_2)\,du_1du_2 + \varepsilon.$$

This completes the demontration of the lemma.

5.1.2. Demonstration of Lemma 2

Consider $(x, y) \in \bar{\Delta}_{h_1,i} \times \bar{\Delta}_{h_2,j}\qquad i = 1,\cdots,[\frac{z_1}{h_1}]; j = 1,\cdots,[\frac{z_2}{h_2}]$ we have:

$$\int_{\mathbb{R}^2}\Big|\Big(1-\frac{ih_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{jh_2}{z_2}\Big)^{\alpha_2}\Big||f(u_1h_1 - ih_1, u_1h_2 - jh_2)||\mathcal{K}(u_1,u_2)|\,du_1du_2$$
$$\leq \Big(\int_{\mathbb{R}^2}\int_{[x,x+h_1(|u_1|+1)]\times[y,y+h_2(|u_2|+1)]}|f^{'}(t_1,t_2)||\mathcal{K}(u_1,u_2)|\,du_1du_2\Big).$$

Hence $I_2(x, y)$ defining as previously in Lemma 1, the next inequality holds:

$$\int_{[0,z_1]\times[0,z_2]}I_2(x,y)\,dxdy$$
$$\leq \int_{[0,z_1]\times[0,z_2]}\Big(\int_{\mathbb{R}^2}\int_{[x,x+h_1(|u_1|+1)]\times[y,y+h_2(|u_2|+1)]}|f^{'}(t_1,t_2)||\mathcal{K}(u_1,u_2)|\,du_1du_2\Big)\,dxdy. \quad (14)$$

A change of variables $t_1 = x + h_1(|u_1| + 1)v_1; t_2 = y + h_2(|u_2| + 1)v_2$ and using Fubini's theorem give

$$\int_{[0,z_1]\times[0,z_2]}I_2(x,y)\,dxdy \leq h_1h_2\int_{\mathbb{R}^2}(|u_1|+1)(|u_2|+1)|\mathcal{K}(u_1,u_2)|$$
$$\times(\int_{\mathbb{R}^2})|f^{'}(x + h_1(|u_1|+1)v_1, y + h_2(|u_2|+1)v_2)|\,dxdy\,du_1du_2\int_{[0,1]\times[0,1]}dv_1dv_2$$
$$= \int_{\mathbb{R}^2}|f^{'}(x,y)|\,dxdy.$$

The proof is complete by combinating this last inequality with inequality (9) and (12).

5.1.3. Demonstration of Theorem 1 and Theorem 2

Consider $\hat{F}_n$ as the empirical distribution of $(X_1, Y_1), (X_2, Y_2),\cdot,\cdot,\cdot,(X_n, Y_n)$ given by

$$F_n(l_1,l_2) = n^{-1}\sum_{i=1}^{n}\sum_{j=1}^{m}(\chi_{X_i<l_1}\chi_{Y_j<l_2})$$

where χ_A represents the indicator function of A. We have

$$P_n^{\mathcal{K}}(z,\alpha_1,\alpha_2) = \int_{\mathbb{R}^2}\sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]}\Big(1-\frac{ih_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{jh_2}{z_2}\Big)^{\alpha_2}\mathcal{K}\Big(\frac{l_1 - ih_1}{h_1},\frac{l_2 - jh_2}{h_2}\Big)\,d\hat{F}_n(l_1,l_2)$$

and

$$\mathbb{E}(P_n^{\mathcal{K}}(z,\alpha_1,\alpha_2)) = \sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]}\int_{\mathbb{R}^2}\Big(1-\frac{ih_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{jh_2}{z_2}\Big)^{\alpha_2}\mathcal{K}\Big(\frac{l_1-ih_1}{h_1},\frac{l_2-jh_2}{h_2}\Big)\,dF(l_1,l_2).$$

We have

$$\begin{aligned}
&|P_n^{\mathcal{K}}(z,\alpha_1,\alpha_2)-\mathbb{E}(P_n^{\mathcal{K}}(z,\alpha_1,\alpha_2))|\\
&\qquad = \Big|\sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]}\int_{\mathbb{R}^2}\Big(1-\frac{ih_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{jh_2}{z_2}\Big)^{\alpha_2}\mathcal{K}\Big(\frac{l_1-ih_1}{h_1},\frac{l_2-jh_2}{h_2}\Big)\,(d\hat{F}_n(l_1,l_2)-dF(l_1,l_2))\Big|.
\end{aligned}$$

We first apply the Fubini's theorem and after integrate by parts with considering each variable.

We have

$$\begin{aligned}
&\int_{\mathbb{R}^2}\mathcal{K}\Big(\frac{l_1-ih_1}{h_1},\frac{l_2-jh_2}{h_2}\Big)\,(d\hat{F}_n(l_1,l_2)-dF(l_1,l_2))\\
&=\int_{\mathbb{R}^2}\mathcal{K}\Big(\frac{l_1-ih_1}{h_1},\frac{l_2-jh_2}{h_2}\Big)\,(d\hat{F}_n(l_1,l_2)-dF(l_1,l_2))\\
&=\int_{\mathbb{R}}\Big[\int_{\mathbb{R}}\mathcal{K}\Big(\frac{l_1-ih_1}{h_1},\frac{l_2-jh_2}{h_2}\Big)\frac{\partial^2(\hat{F}_n(l_1,l_2)-F(l_1,l_2))}{\partial l_1\partial l_2}dl_2\Big]dl_1\\
&=\int_{\mathbb{R}}\Big\{\Big[\mathcal{K}\Big(\frac{l_1-ih_1}{h_1},\frac{l_2-jh_2}{h_2}\Big)\frac{\partial(\hat{F}_n(l_1,l_2)-F(l_1,l_2))}{\partial l_1}dl_1\Big]_{-\infty}^{+\infty}\\
&-\int_{\mathbb{R}}\frac{\partial\mathcal{K}\Big(\frac{l_1-ih_1}{h_1},\frac{l_2-jh_2}{h_2}\Big)}{\partial l_2}dl_2\frac{\partial(\hat{F}_n(l_1,l_2)-F(l_1,l_2))}{\partial l_1}\Big\}dl_1\\
&=-\Big\{\int_{\mathbb{R}}\Big[\frac{\partial(\hat{F}_n(l_1,l_2)-F(l_1,l_2))}{\partial l_1}\int_{\mathbb{R}}\frac{\partial\mathcal{K}\Big(\frac{l_1-ih_1}{h_1},\frac{l_2-jh_2}{h_2}\Big)}{\partial l_2}dl_2\Big]dl_1\Big\}\\
&=-\Big\{\Big[\int_{\mathbb{R}}\frac{\partial\mathcal{K}\Big(\frac{l_1-ih_1}{h_1},\frac{l_2-jh_2}{h_2}\Big)}{\partial l_2}dl_2(\hat{F}_n(l_1,l_2)-F(l_1,l_2))\Big]_{-\infty}^{+\infty}\\
&-\int_{\mathbb{R}\times\mathbb{R}}\frac{\partial^2\mathcal{K}\Big(\frac{l_1-ih_1}{h_1},\frac{l_2-jh_2}{h_2}\Big)}{\partial l_1\partial l_2}dl_2(\hat{F}_n(l_1,l_2)-F(l_1,l_2))dl_1\Big\}\\
&=\int_{\mathbb{R}^2}d\mathcal{K}\Big(\frac{l_1-ih_1}{h_1},\frac{l_2-ih_2}{h_2}\Big)(\hat{F}_n(l_1,l_2)-F(l_1,l_2))
\end{aligned}$$

where $d_{l_i}\mathcal{K}(\cdot)=\dfrac{\partial\mathcal{K}(\cdot)\,dl_i}{\partial l_i}$ design the derivative of $\mathcal{K}$ with regard to l_i. Hence

$$\begin{aligned}
&|P_n^{\mathcal{K}}(z,\alpha_1,\alpha_2)-\mathbb{E}(P_n^{\mathcal{K}}(z,\alpha_1,\alpha_2))|\\
&\le\sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]}\int_{\mathbb{R}^2}\Big|d\mathcal{K}\Big(\frac{l_1-ih_1}{h_1},\frac{l_2-ih_2}{h_2}\Big)\Big|\sup_{(l_1,l_2)\in\mathbb{R}^2}|\hat{F}_n(l_1,l_2)-F(l_1,l_2)|\\
&\le\sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]}\int_{\mathbb{R}^2}dV_{-\infty}^{(\frac{l_1-ih_1}{h_1},\frac{l_2-ih_2}{h_2})}\mathcal{K}\sup_{(l_1,l_2)\in\mathbb{R}^2}|\hat{F}_n(l_1,l_2)-F(l_1,l_2)|\\
&\le\Big[\frac{z_1}{h_1}\Big]\Big[\frac{z_2}{h_2}\Big]V(\mathbb{R}\times\mathbb{R})\sup_{(l_1,l_2)\in\mathbb{R}^2}|\hat{F}_n(l_1,l_2)-F(l_1,l_2)|.
\end{aligned}$$

Noting that

$$\begin{aligned}
&|P_n^{\mathcal{K}}(z,\alpha_1,\alpha_2)-(P_(z,\alpha_1,\alpha_2))|\\
&=|P_n^{\mathcal{K}}(z,\alpha_1,\alpha_2)-\mathbb{E}(P_n^{\mathcal{K}}(z,\alpha_1,\alpha_2))+\mathbb{E}(P_n^{\mathcal{K}}(z,\alpha_1,\alpha_2))-P(z,\alpha_1,\alpha_2)|\\
&\le|P_n^{\mathcal{K}}(z,\alpha_1,\alpha_2)-\mathbb{E}(P_n^{\mathcal{K}}(z,\alpha_1,\alpha_2))|+|\mathbb{E}(P_n^{\mathcal{K}}(z,\alpha_1,\alpha_2))-P(z,\alpha_1,\alpha_2)|.
\end{aligned}$$

Because of the Lemma 1 we obtain

$$|\mathbb{E}(P_n^{\mathcal{K}}(z,\alpha_1,\alpha_2)) - P(z,\alpha_1,\alpha_2)| \to 0 \qquad n \to +\infty$$

and according to the preceding inequality we get

$$|P_n^{\mathcal{K}}(z,\alpha_1,\alpha_2) - \mathbb{E}(P_n^{\mathcal{K}}(z,\alpha_1,\alpha_2))| \to 0 \qquad n \to +\infty.$$

5.1.4. Demonstration of Corollary 2

We have:

$$\begin{aligned}&|P_n^{\mathcal{K}}(z,\alpha_1,\alpha_2) - P(z,\alpha_1,\alpha_2)|\\ &= |\mathbb{E}(P_n^{\mathcal{K}}(z,\alpha_1,\alpha_2)) - P(z,\alpha_1,\alpha_2) + P_n^{\mathcal{K}}(z,\alpha_1,\alpha_2) - \mathbb{E}(P_n^{\mathcal{K}}(z,\alpha_1,\alpha_2))|.\end{aligned}$$

We obtain:

$$\begin{aligned}&|P_n^{\mathcal{K}}(z,\alpha_1,\alpha_2) - P(z,\alpha_1,\alpha_2)|\\ &\leq |\mathbb{E}(P_n^{\mathcal{K}}(z,\alpha_1,\alpha_2)) - P(z,\alpha_1,\alpha_2)| + |P_n^{\mathcal{K}}(z,\alpha_1,\alpha_2) - \mathbb{E}(P_n^{\mathcal{K}}(z,\alpha_1,\alpha_2))|.\end{aligned} \tag{15}$$

We have because of Lemma 2:

$$\begin{aligned}&\sup_{z\in\mathbb{R}^2} |\mathbb{E}(P_n^{\mathcal{K}}(z,\alpha_1,\alpha_2)) - P(z,\alpha_1,\alpha_2)| \leq\\ &h_1h_2\Big(\big(\int_{\mathbb{R}^2} |f^{'}(x,y)|\,dxdy\big)\big(\int_{\mathbb{R}^2} (|u|+1)(|v|+1)|\mathcal{K}(u,v)|\,dudv\big)\\ &+ 4(\alpha_1\alpha_2 M + Ah_1h_2)\int_{-\infty}^{+\infty} |\mathcal{K}(u,v)|\,dudv\Big)\end{aligned}$$

and moreover we have:

$$|P_n^{\mathcal{K}}(z,\alpha_1,\alpha_2) - \mathbb{E}(P_n^{\mathcal{K}}(z,\alpha_1,\alpha_2))| \leq [\frac{z_1}{h_1}][\frac{z_2}{h_2}]V(\mathbb{R}\times\mathbb{R}) \sup_{(l_1,l_2)\in\mathbb{R}^2} |\hat{F}_n(l_1,l_2) - F(l_1,l_2)|.$$

Hence the second term of the right hand-side of inequality (15) leads to zero as $n \to +\infty$. The first term leads to $h_1h_2 = O(n^{-1}\log\log n)^{1/4}$ as $n \to +\infty$. Hence the demonstration is complete.

5.2. Uniform Almost-Sure Consistency and Bias Behavior

5.2.1. Demonstration of Lemma 3

We suppose condition *C1* verified. Let $\delta = (\delta_1,\delta_2) > 0$.
Note that thesis notations θ_1^i θ_2^i $i = 1,2$ represent indices, not powers.
Denote

$$\begin{aligned}I_n(x,y) &= (h_1h_2)^{-2}\int_{\mathbb{R}^2} \Big|\mathcal{K}\Big(\frac{u_1 - x_1 + \theta_1^1h_1}{h_1}, \frac{u_2 - x_2 + \theta_1^2h_2}{h_2}\Big)\\ &\quad\times \mathcal{K}\Big(\frac{u_1 - y_1 + \theta_2^1h_1}{h_1}, \frac{u_2 - y_2 + \theta_2^2h_2}{h_2}\Big)\Big| f(u_1,u_2)\,du_1du_2\\ &= \int_{\mathbb{R}^2} \Big((h_1h_2)^{-1}\mathcal{K}\Big(\frac{v_1}{h_1},\frac{v_2}{h_2}\Big)\Big)\Big|\Big((h_1h_2)^{-1}\mathcal{K}\Big(\frac{v_1 + x_1 - \theta_1^1 - y_1 + \theta_2^1h_1}{h_1}, \frac{v_2 + x_2 - \theta_1^2 - y_2 + \theta_2^2h_2}{h_2}\Big)\Big)\Big|\\ &\quad\times f(x_1 + v_1 - \theta_1^1h_1, x_2 + v_2 - \theta_1^2h_2)\,du_1du_2\\ &= \int_{\{|v_1-\theta_1^1h_1|\leq\delta_1\}\times\{|v_2-\theta_1^2h_2|\leq\delta_2\}} + \int_{\{|v_1-\theta_1^1h_1|>\delta_1\}\times\{|v_2-\theta_1^2h_2|>\delta_2\}}\\ &\quad+ \int_{\{|v_1-\theta_1^1h_1|\leq\delta_1\}\times\{|v_2-\theta_1^2h_2|>\delta_2\}} + \int_{\{|v_1-\theta_1^1h_1|>\delta_1\}\times\{|v_2-\theta_1^2h_2|\leq\delta_2\}}.\end{aligned}$$

f is continuous hence bounded on $I_1 \times I_2 = [x_1 - \delta_1, x_1 + \delta_1] \times [x_2 - \delta_2, x_2 + \delta_2]$. We take n large enough such that

$(x_1 + v_1 \pm \theta_1^1 h_1, x_2 + v_2 \pm \theta_1^2 h_2) \in I_1 \times I_2$. Hence

$$\begin{aligned}
&\int_{\{|v_1-\theta_1^1 h_1|\leq\delta_1\}\times\{|v_1-\theta_1^1 h_1|\leq\delta_2\}} \leq \sup_{|v_1-\theta_1^1 h_1|\leq\delta_1} \sup_{|v_2-\theta_1^2 h_2|\leq\delta_2} f(x_1+v_1-\theta_1^1 h_1, x_2+v_2-\theta_1^2 h_2) \\
&\times \int_{\{-\frac{\delta_1}{h_1}+\theta_1^1\leq u_1\leq\frac{\delta_1}{h_1}+\theta_1^1\}\times\{-\frac{\delta_1}{h_1}+\theta_1^1\leq u_1\leq\frac{\delta_1}{h_1}+\theta_1^1\}} |\mathcal{K}(u_1,u_2)| \\
&\times \Big|\mathcal{K}\Big(\frac{x_1-\theta_1^1 h_1-y_1+\theta_2^1 h_1}{h_1}+u_1, \frac{x_2-\theta_1^2 h_2-y_2+\theta_2^2 h_2}{h_2}+u_2\Big)(h_1h_2)^{-1}\Big| du_1du_2 \\
&\quad = \sup_{|v_1-\theta_1^1 h_1|\leq\delta_1} \sup_{|v_2-\theta_1^2 h_2|\leq\delta_2} f(x_1+v_1-\theta_1^1 h_1, x_2+v_2-\theta_1^2 h_2) \\
&\quad \times \int_{\mathbb{R}^2} \chi_{-\frac{\delta_1}{h_1}+\theta_1^1\leq u_1\leq\frac{\delta_1}{h_1}+\theta_1^1}(u_1)\chi_{-\frac{\delta_2}{h_2}+\theta_1^2\leq u_2\leq\frac{\delta_2}{h_2}+\theta_1^2}(u_2)|\mathcal{K}(u_1,u_2)| \\
&\quad \times \Big|\mathcal{K}\Big(\frac{x_1-\theta_1^1 h_1-y_1+\theta_2^1 h_1}{h_1}+u_1, \frac{x_2-\theta_1^2 h_2-y_2+\theta_2^2 h_2}{h_2}+u_2\Big)(h_1h_2)^{-1}\Big| du_1du_2.
\end{aligned} \tag{16}$$

For every (u_1,u_2)

$$\Big|\mathcal{K}\Big(\frac{x_1-\theta_1^1 h_1-y_1+\theta_2^1 h_1}{h_1}+u_1, \frac{x_2-\theta_1^2 h_2-y_2+\theta_2^2 h_2}{h_2}+u_2\Big)(h_1h_2)^{-1}\Big| \to 0 \quad \text{as} \quad n\to+\infty.$$

Note

$$\begin{aligned}
&\Big|\mathcal{K}\Big(\frac{x_1-\theta_1^1 h_1-y_1+\theta_2^1 h_1}{h_1}+u_1, \frac{x_2-\theta_1^2 h_2-y_2+\theta_2^2 h_2}{h_2}+u_2\Big)(h_1h_2)^{-1}\Big| \\
&= \Big|\Big(\frac{x_1-\theta_1^1 h_1-y_1+\theta_2^1 h_1}{h_1}+u_1\Big)\Big(\frac{x_2-\theta_1^2 h_2-y_2+\theta_2^2 h_2}{h_2}+u_2\Big) \\
&\times \mathcal{K}\Big(\frac{x_1-\theta_1^1-y_1+\theta_2^1 h_1}{h_1}+u_1, \frac{x_2-\theta_1^2-y_2+\theta_2^2 h_2}{h_2}+u_2\Big)\Big| \\
&\times \Big|\frac{1}{x_1-\theta_1^1 h_1-y_1+\theta_2^1 h_1+h_1u_1}\times\frac{1}{x_2-\theta_1^2 h_2-y_2+\theta_2^2 h_2+h_2u_2}\Big|.
\end{aligned}$$

We get

$$\Big|\frac{1}{x_i-\theta_1^i h_i-y_i+\theta_2^i h_i+h_iu_i}\Big| = \frac{1}{|x_i-y_i||1-\frac{\theta_1^i-\theta_2^i-u_i}{x_i-y_i}h_i|} \qquad i=1,2$$

Like $|u_i| \leq \frac{\delta_i}{h_i}+\theta_1^i$ we can take δ_i small such that for $n\geq n_0$ we obtain

$$\Big|\frac{\theta_1^i-\theta_2^i-u_i}{x_i-y_i}h_i\Big| \leq \frac{3h_i+\delta_i}{|x_i-y_i|} = \eta_i < 1.$$

Therefore

$$\Big|\frac{1}{x_i-\theta_1^i h_i-y_i+\theta_2^i h_i+h_iu_i}\Big| \leq \frac{1}{|x_i-y_i|(1-\eta_i)}. \tag{17}$$

Since H_3 implies there exists B such that

$$\begin{aligned}
&|\Big(\frac{x_1-\theta_1^1 h_1-y_1+\theta_2^1 h_1}{h_1}+u_1\Big)\Big(\frac{x_2-\theta_1^2 h_2-y_2+\theta_2^2 h_2}{h_2}+u_2\Big) \\
&\times\mathcal{K}\Big(\frac{x_1-\theta_1^1 h_1-y_1+\theta_2^1 h_1}{h_1}+u_1, \frac{x_2-\theta_1^2 h_2-y_2+\theta_2^2 h_2}{h_2}+u_2\Big)| \leq B.
\end{aligned}$$

Therefore

$$\Big|\mathcal{K}\Big(\frac{x_1-\theta_1^1 h_1-y_1+\theta_2^1 h_1}{h_1}+u_1, \frac{x_2-\theta_1^2 h_2-y_2+\theta_2^2 h_2}{h_2}+u_2\Big)(h_1h_2)^{-1}\Big| \leq \frac{2B}{|x_1-y_1|(1-\eta_1)|x_2-y_2|(1-\eta_2)}.$$

$|\mathcal{K}(u_1,u_2)|$ being integrable, by dominated convergence we get:

$$\int_{\{|v_1-\theta_1^1h_1|\leq\delta_1\}\times\{|v_1-\theta_1^1h_1|\leq\delta_2\}} \to 0 \quad \text{as} \quad n\to+\infty.$$

Let us consider $\int_{\{|v_1-\theta_1^1h_1|>\delta_1\}\times\{|v_2-\theta_12h_2|>\delta_2\}}$, write it in the form:

$$\begin{aligned}\int_{\{|v_1-\theta_1^1h_1|>\delta_1\}\times\{|v_2-\theta_1^2h_2|>\delta_2\}} &= \int_{\{|v_1-\theta_1^1h_1|>\delta_1\}\times\{|v_2-\theta_1^2h_2|>\delta_2\}} \Big| v_1v_2(h_1h_2)^{-1}\mathcal{K}\Big(\frac{v_1}{h_1},\frac{v_2}{h_2}\Big)(h_1h_2)^{-1}\\ &\times\mathcal{K}\Big(\frac{v_1+x_1-\theta_1^1-y_1+\theta_2^1h_1}{h_1},\frac{v_2+x_2-\theta_1^2-y_2+\theta_2^2h_2}{h_2}\Big)\\ &\times\frac{f(x_1+v_1-\theta_1^1h_1,x_2+v_2-\theta_1^2h_2)}{v_1v_2}\Big| dv_1dv_2.\end{aligned}$$

We get

$$\begin{aligned}\int_{\{|v_1-\theta_1^1h_1|>\delta_1\}\times\{|v_2-\theta_1^2h_2|>\delta_2\}} &\leq \frac{2}{\delta_1-\theta_1^1h_1}\times\frac{2}{\delta_2-\theta_1^2h_2}\sup_{|v_1-\theta_1^1h_1|>\delta_1}\sup_{|v_2-\theta_1^2h_2|>\delta_2}\Big|\frac{v_1v_2}{h_1h_2}\mathcal{K}\Big(\frac{v_1}{h_1},\frac{v_2}{h_2}\Big)\Big|\\ &\times\int_{\{|v_1-\theta_1^1h_1|>\delta_1\}\times\{|v_1-\theta_1^1h_1|>\delta_2\}}(h_1h_2)^{-1}\\ &\times\mathcal{K}\Big(\frac{v_1+x_1-\theta_1^1-y_1+\theta_2^1h_1}{h_1},\frac{v_2+x_2-\theta_1^2-y_2+\theta_2^2h_2}{h_2}\Big)\\ &\times |f(x_1+v_1-\theta_1^1h_1,x_2+v_2-\theta_1^2h_2)|\,dv_1dv_2. \end{aligned}\tag{18}$$

Suppose the change of variables defined by:

$$v_i+x_i-\theta_1^ih_i-y_i+\theta_2^ih_i=u_i,\quad i=1,2.$$

Then

$$\begin{aligned}\int_{\{|v_1-\theta_1^1h_1|>\delta_1\}\times\{|v_2-\theta_1^2h_2|>\delta_2\}} &\leq \frac{2}{\delta_1-\theta_1^1h_1}\times\frac{2}{\delta_2-\theta_1^2h_2}\sup_{|v_1-\theta_1^1h_1|>\delta_1}\sup_{|v_2-\theta_1^2h_2|>\delta_2}\Big|\frac{v_1v_2}{h_1h_2}\mathcal{K}\Big(\frac{v_1}{h_1},\frac{v_2}{h_2}\Big)\Big|\\ &\times\int_{\mathbb{R}^2}|(h_1h_2)^{-1}\mathcal{K}\Big(\frac{u_1}{h_1},\frac{u_2}{h_2}\Big)|f(u_1+y_1-\theta_2^1h_1,u_2+y_2-\theta_2^2h_2\Big)\,du_1du_2.\end{aligned}\tag{19}$$

Lemma 4 (using $|\mathcal{K}|$ in the place of $\mathcal{K}$) and (H_3) we have:

$$\Big|\int_{\{|v_1-\theta_1^1h_1|>\delta_1\}\times\{|v_2-\theta_1^2h_2|>\delta_2\}}\Big| \to 0 \quad \text{quand} \quad n\to+\infty$$

and we obtain, this case, the convergence uniform.
Regard the case $\int_{\{|v_1-\theta_1^1h_1|\leq\delta_1\}\times\{|v_2-\theta_1^2h_2|>\delta_2\}}(\cdot)(\cdot)$ we have:

$$\int_{\{|v_1-\theta_1^1h_1|\leq\delta_1\}\times\{|v_2-\theta_1^2h_2|>\delta_2\}}(\cdot)(\cdot)\leq\int_{\mathbb{R}\times\mathbb{R}}(\cdot)(\cdot).$$

Using the condition C_1 we get

$$\int_{\{|v_1-\theta_1^1h_1|\leq\delta_1\}\times\{|v_2-\theta_1^2h_2|>\delta_2\}} \to 0 \quad \text{as} \quad n\to+\infty.$$

Regard the case $\int_{\{|v_1-\theta_1^1h_1|>\delta_1\}\times\{|v_2-\theta_1^2h_2|\leq\delta_2\}}(\cdot)(\cdot)$ similarly we have:

$$\int_{\{|v_1-\theta_1^1h_1|>\delta_1\}\times\{|v_2-\theta_1^2h_2|\leq\delta_2\}}(\cdot)(\cdot)\leq\int_{\mathbb{R}\times\mathbb{R}}(\cdot)(\cdot).$$

Hence because of assumption C_1

$$\int_{\{|v_1-\theta_1^1h_1|>\delta_1\}\times\{|v_2-\theta_1^2h_2|\leq\delta_2\}} \to 0 \quad \text{as} \quad n\to+\infty.$$

This point completes the demonstration of Lemma 3.

Remark 3. If C_2 is satisfied then, the lemma is valid for each case evaluated.
Evaluate $\int_{\{|v_1-\theta_1^1 h_1|>\delta_1\}\times\{|v_1-\theta_1^1 h_1|>\delta_2\}}$. If C_2 is satisfied then, the integral of the right hand-side of (19) gives

$$\begin{aligned}
&\int_{\mathbb{R}^2}\Big|(h_1h_2)^{-1}\mathcal{K}\Big(\frac{u_1}{h_1},\frac{u_2}{h_2}\Big)\Big|\Big|f(u_1+y_1-\theta_2^1h_1,u_2+y_2-\theta_2^2h_2)-f\Big(\frac{u_1}{h_1},\frac{u_2}{h_2}\Big)+f\Big(\frac{u_1}{h_1},\frac{u_2}{h_2}\Big)\Big|\\
&du_1du_2\\
&\leq\int_{\mathbb{R}^2}\Big|(h_1h_2)^{-1}\mathcal{K}\Big(\frac{u_1}{h_1},\frac{u_2}{h_2}\Big)\Big|\int_{\frac{u_1}{h_1}}^{u_1+y_1-\theta_2^1h_1}\int_{\frac{u_2}{h_2}}^{u_2+y_2-\theta_2^2h_2}|f^{'}(t_1,t_2)|\,dt_1dt_2du_1du_2\\
&\quad+\int_{\mathbb{R}^2}f\Big(\frac{u_1}{h_1},\frac{u_2}{h_2}\Big)\Big|(h_1h_2)^{-1}\mathcal{K}\Big(\frac{u_1}{h_1},\frac{u_2}{h_2}\Big)\Big|\,du_1du_2\\
&\leq\int_{\mathbb{R}^2}\Big|(h_1h_2)^{-1}\mathcal{K}\Big(\frac{u_1}{h_1},\frac{u_2}{h_2}\Big)\Big|\,du_1du_2\int_{\mathbb{R}^2}|f^{'}(t_1,t_2)|\,dt_1dt_2\\
&\quad+\int_{\mathbb{R}^2}|\mathcal{K}(u_1,u_2)|f(u_1,u_2)\,du_1du_2.
\end{aligned}$$

In this previous inequality, the integrals of the right hand-side are bounded. Therefore, under the assumption C_2, the theorem is valid.

5.2.2. Demonstration of Theorem 5

We regard hypothesis C_1 satisfied. Let $\Delta_1=[0,z_1]\times[0,z_1];\Delta_2=[0,z_2]\times[0,z_2]$. We obtain

$$\begin{aligned}
&\sum_{0\leq i_1\neq j_1\leq[\frac{z_1}{h_1}]}\Big(1-\frac{i_1h_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{j_1h_1}{z_1}\Big)^{\alpha_1}\sum_{0\leq i_2\neq j_2\leq\left[\frac{z_2}{h_2}\right]}\Big(1-\frac{i_2h_2}{z_2}\Big)^{\alpha_2}\Big(1-\frac{j_2h_2}{z_2}\Big)^{\alpha_2}\\
&\times\int_{\mathbb{R}^2}\Big|\mathcal{K}\Big(\frac{u_1-i_1h_1}{h_1},\frac{u_1-j_1h_1}{h_1}\Big)\mathcal{K}\Big(\frac{u_2-i_2h_2}{h_2},\frac{u_2-j_2h_2}{h_2}\Big)\Big|f(u_1,u_2)\,du_1du_2)\\
&=\int_{\{(x_1,y_1)\in\Delta_1:|x_1-y_1|>0\}}\Phi_n(x_1,y_1)\,dx_1dy_1\int_{\{(x_2,y_2)\in\Delta_2:|x_2-y_2|>0\}}\Phi_n(x_2,y_2)\,dx_2dy_2
\end{aligned}$$

where

$$\begin{aligned}
\Phi_n(x_1,y_1)=\frac{1}{h_1^2}\sum_{0\leq i_1\neq j_1\leq\left[\frac{z_1}{h_1}\right]}&\chi_{\Delta_{h_1,i_1}\times\Delta_{h_1,j_1}}(x_1,y_1)\Big(1-\frac{i_1h_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{j_1h_1}{z_1}\Big)^{\alpha_1}\\
&\times\int_{\mathbb{R}}\Big|\mathcal{K}\Big(\frac{u_1-i_1h_1}{h_1},\frac{u_1-j_1h_1}{h_1}\Big)\Big|f_1(u_1)\,du_1
\end{aligned}$$

and $\Phi_n(x_2,y_2)$ holds with changing f_1 by f_2, x_1 by x_2, y_1 by y_2, and u_1 by u_2.
Remark that $\Phi_{n_{12}}(x_1,y_1,x_2,y_2)$ can be write $\Phi_n(x_1,y_1)\Phi_n(x_2,y_2)$. Hence

$$\begin{aligned}
\Phi_{n_{12}}(x_1,y_1,x_2,y_2)=\frac{1}{h_1^2h_2^2}&\sum_{0\leq i_1\neq j_1\leq\left[\frac{z_1}{h_1}\right]}\chi_{\Delta_{h_1,i_1}\times\Delta_{h_1,j_1}}(x_1,y_1)\Big(1-\frac{i_1h_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{j_1h_1}{z_1}\Big)^{\alpha_1}\\
&\times\sum_{0\leq i_2\neq j_2\leq\left[\frac{z_2}{h_2}\right]}\chi_{\Delta_{h_2,i_2}\times\Delta_{h_2,j_2}}(x_2,y_2)\Big(1-\frac{i_2h_2}{z_2}\Big)^{\alpha_2}\Big(1-\frac{j_2h_2}{z_2}\Big)^{\alpha_2}\\
&\times\int_{\mathbb{R}^2}\Big|\mathcal{K}\Big(\frac{u_1-i_1h_1}{h_1},\frac{u_1-j_1h_1}{h_1}\Big)\mathcal{K}\Big(\frac{u_2-i_1h_2}{h_2},\frac{u_2-j_2h_2}{h_2}\Big)\Big|f(u_1,u_2)\,du_1du_2.
\end{aligned}$$

If $(x_1,y_1)\in\Delta_{h_1,i_1}\times\Delta_{h_1,j_1}$ $i_1\neq j_1$; $(x_2,y_2)\in\Delta_{h_2,i_2}\times\Delta_{h_2,j_2}$ $i_2\neq j_2$ with the representations:

$$x_1=h_1i_1+\theta_1^1h_1,\quad y_1=h_1j_1+\theta_2^1h_1\quad 0\leq\theta_i^1<1,\quad i=1,2$$

and

$$x_2=h_2i_2+\theta_1^2h_2,\quad y_2=h_2j_2+\theta_2^2h_2\quad 0\leq\theta_i^2<1,\quad i=1,2.$$

We have:

$$\begin{aligned}
&\frac{1}{h_1^2h_2^2}\Big(1-\frac{i_1h_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{j_1h_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{i_2h_2}{z_2}\Big)^{\alpha_2}\Big(1-\frac{j_2h_2}{z_2}\Big)^{\alpha_2}\\
&\qquad\times\int_{\mathbb{R}^2}\Big|\mathcal{K}\Big(\frac{u_1-x_1+\theta_1^1h_1}{h_1},\frac{u_1-y_1+\theta_2^1h_1}{h_1}\Big)\mathcal{K}\Big(\frac{u_2-x_2+\theta_1^2h_2}{h_2},\frac{u_2-y_2+\theta_2^2h_2}{h_2}\Big)\Big|\\
&\qquad\times f(u_1,u_2)\,du_1du_2\\
&\le\frac{1}{h_1^2h_2^2}\int_{\mathbb{R}^2}\Big|\mathcal{K}\Big(\frac{u_1-x_1+\theta_1^1h_1}{h_1},\frac{u_1-y_1+\theta_2^1h_1}{h_1}\Big)\mathcal{K}\Big(\frac{u_2-x_2+\theta_1^2h_2}{h_2},\frac{u_2-y_2+\theta_2^2h_2}{h_2}\Big)\Big|\\
&\qquad\times f(u_1,u_2)\,du_1du_2.
\end{aligned}\tag{20}$$

The right hand-side of (20) equals in limit to zero as $n\to+\infty$ because of the Lemma 3.
Suppose $\delta_i=\frac{z_i}{2},\quad i=1,2$. We have:

$$\begin{aligned}
&\frac{1}{h_1^2h_2^2}\int_{\mathbb{R}^2}\Big|\mathcal{K}\Big(\frac{u_1-x_1+\theta_1^1h_1}{h_1},\frac{u_1-y_1+\theta_2^1h_1}{h_1}\Big)\mathcal{K}\Big(\frac{u_2-x_2+\theta_1^2h_2}{h_2},\frac{u_2-y_2+\theta_2^2h_2}{h_2}\Big)\Big|\\
&\qquad\times f(u_1,u_2)\,du_1du_2\\
&=\int_{\{|v_1|\le\delta_1\}\times\{|v_2|\le\delta_2\}}+\int_{\{|v_1|>\delta_1\}\times\{|v_2|>\delta_2\}}+\int_{\{|v_1|\le\delta_1\}\times\{|v_2|>\delta_2\}}+\int_{\{|v_1|>\delta_1\}\times\{|v_2|\le\delta_2\}}.
\end{aligned}$$

Hence we get:

$$\begin{aligned}
&\int_{\{(x_1,y_1)\in\Delta_1:|x_1-y_1|>0\}}\Phi_n(x_1,y_1)\,dx_1dy_1\int_{\{(x_2,y_2)\in\Delta_2:|x_2-y_2|>0\}}\Phi_n(x_2,y_2)\,dx_2dy_2\\
&\le\int_{\{(x_1,y_1)\in\Delta_1:|x_1-y_1|>0\}}\sum_{0\le i_1\ne j_1\le[\frac{z_1}{h_1}]}\chi_{\Delta_{h_1,i_1}\times\Delta_{h_1,j_1}}(x_1,y_1)\\
&\quad\times\int_{\{(x_2,y_2)\in\Delta_2:|_x2-y_2|>0\}}\sum_{0\le i_2\ne j_2\le[\frac{z_2}{h_2}]}\chi_{\Delta_{h_2,i_2}\times\Delta_{h_2,j_2}}(x_2,y_2)\\
&\quad\times\Bigg(\int_{\{|v_1|\le\delta_1\}\times\{|v_2|\le\delta_2\}}+\int_{\{|v_1|>\delta_1\}\times\{|v_2|>\delta_2\}}+\int_{\{|v_1|\le\delta_1\}\times\{|v_2|>\delta_2\}}+\int_{\{|v_1|>\delta_1\}\times\{|v_2|\le\delta_2\}}\Bigg).
\end{aligned}$$

The demonstration of the remainder is established as follow:
Firstly, we consider

$$\int_{\{(x_1,y_1)\in\Delta_1:|x_1-y_1|>0\}}\int_{\{(x_2,y_2)\in\Delta_2:|x_2-y_2|>0\}}\int_{\{|v_1|\le\delta_1\}\times\{|v_2|\le\delta_2\}}.$$

Take $A=\sup\limits_{(x,y)\in[0,z_1]\times[0,z_2]}f(x,y)$. The notations are the same in the demonstration of Lemma 3 with $\delta_i=\dfrac{z_i}{2},\quad i=1,2,$ we get according to inequality (16)

$$\begin{aligned}
&\int_{\{|v_1|\le\delta_1\}\times\{|v_2|\le\delta_2\}}\le A\int_{-\infty}^{+\infty}\chi_{-\frac{\delta_1}{h_1}\le u_1\le\frac{\delta_1}{h_1}}\int_{-\infty}^{+\infty}\chi_{-\frac{\delta_2}{h_2}\le u_2\le\frac{\delta_2}{h_2}}|\mathcal{K}(u_1,u_2)|\\
&\quad\times\Big|\mathcal{K}\Big(\frac{x_1-\theta_1^1h_1-y_1+\theta_2^1h_1}{h_1}+u_1,\frac{x_2-\theta_1^2h_2-y_2+\theta_2^2h_2}{h_2}+u_2\Big)(h_1h_2)^{-1}\Big|\,du_1du_2.
\end{aligned}$$

For every (u_1,u_2)

$$\lim_{n\to+\infty}\Big|\mathcal{K}\Big(\frac{x_1-\theta_1^1h_1-y_1+\theta_2^1h_1}{h_1}+u_1,\frac{x_2-\theta_1^2h_2-y_2+\theta_2^2h_2}{h_2}+u_2\Big)(h_1h_2)^{-1}\Big|=0.$$

Write as follow

$$
\begin{aligned}
&\Big|\mathcal{K}\Big(\frac{x_1-\theta_1^1 h_1-y_1+\theta_2^1 h_1}{h_1}+u_1,\frac{x_2-\theta_1^2 h_2-y_2+\theta_2^2 h_2}{h_2}+u_2)(h_1h_2)^{-1}\Big| \\
&=\Big|\mathcal{K}\Big(\frac{x_1-\theta_1^1 h_1-y_1+\theta_2^1 h_1}{h_1}+u_1,\frac{x_2-\theta_1^2 h_2-y_2+\theta_2^2 h_2}{h_2}+u_2\Big) \\
&-\mathcal{K}\Big(\frac{2z_1+x_1-\theta_1^1-y_1+\theta_2^1 h_1}{h_1}+u_1,\frac{z_2+x_2-\theta_1^2-y_2+\theta_2^2 h_2}{h_2}+u_2\Big) \\
&+\mathcal{K}\Big(\frac{2z_1+x_1-\theta_1^1-y_1+\theta_2^1 h_1}{h_1}+u_1,\frac{z_2+x_2-\theta_1^2-y_2+\theta_2^2 h_2}{h_2}+u_2\Big)\Big|(h_1h_2)^{-1} \\
&\leq\Big(\lambda\Big(\frac{2z_1}{h_1},\frac{2z_2}{h_2}\Big)+\Big|\mathcal{K}\Big(\frac{2z_1+x_1-\theta_1^1 h_1-y_1+\theta_2^1 h_1}{h_1}+u_1,\frac{z_2+x_2-\theta_1^2-y_2+\theta_2^2 h_2}{h_2}+u_2)\Big|\Big) \\
&\times(h_1h_2)^{-1}.
\end{aligned}
$$

Moreover write:

$$
\begin{aligned}
&\Big|\mathcal{K}(\frac{2z_1+x_1-\theta_1^1-y_1+\theta_2^1 h_1}{h_1}+u_1,\frac{z_2+x_2-\theta_1^2-y_2+\theta_2^2 h_2}{h_2}+u_2)\Big|(h_1h_2)^{-1} \\
&=\Big|\frac{2z_1+x_1-y_1+h_1u_1}{h_1}\Big|\Big|\frac{2z_2+x_2-y_2+h_2u_2}{h_2}\Big| \\
&\times\Big|\mathcal{K}(\frac{2z_1+x_1-\theta_1^1-y_1+\theta_2^1 h_1}{h_1}+u_1,\frac{z_2+x_2-\theta_1^2-y_2+\theta_2^2 h_2}{h_2}+u_2)\Big| \\
&\times\frac{1}{|2z_1+x_1-y_1+h_1u_1||2z_2+x_2-y_2+h_2u_2|}.
\end{aligned}
$$

We have

$$
\begin{aligned}
TT &= \Big|\frac{2z_1+x_1-y_1+h_1u_1}{h_1}\Big|\Big|\frac{2z_2+x_2-y_2+h_2u_2}{h_2}\Big| \\
&= \Big|\frac{2z_1+x_1-y_1+h_1u_1-\theta_1^1 h_1+\theta_2^1 h_1}{h_1}+\frac{\theta_1^1 h_1-\theta_2^1 h_1}{h_1}\Big| \\
&\times \Big|\frac{2z_2+x_2-y_2+h_2u_2-\theta_1^2 h_2+\theta_2^2 h_2}{h_2}+\frac{\theta_2^2 h_2-\theta_2^2 h_2}{h_2}\Big| \\
&\leq \Big|\frac{2z_1+x_1-y_1+h_1u_1-\theta_1^1 h_1+\theta_2^1 h_1}{h_1}\Big|+\Big|\frac{\theta_1^1 h_1-\theta_2^1 h_1}{h_1}\Big| \\
&\times \Big|\frac{2z_2+x_2-y_2+h_2u_2-\theta_1^2 h_2+\theta_2^2 h_2}{h_2}\Big|+\Big|\frac{\theta_1^2 h_2-\theta_2^2 h_2}{h_2})\Big| \\
&\leq \Big|\frac{2z_1+x_1-y_1+h_1u_1-\theta_1^1 h_1+\theta_2^1 h_1}{h_1}\Big|\Big|\frac{2z_2+x_2-y_2+h_2u_2-\theta_1^2 h_2+\theta_2^2 h_2}{h_2}\Big| \\
&+ \Big|\frac{2z_1+x_1-y_1+h_1u_1-\theta_1^1 h_1+\theta_2^1 h_1}{h_1}\Big|+\Big|\frac{2z_2+x_2-y_2+h_2u_2-\theta_1^2 h_2+\theta_2^2 h_2}{h_2}\Big|+1.
\end{aligned}
$$

Assume $B=\sup_{(x,y)\in\mathbb{R}^2}|x||y||\mathcal{K}(x,y)|$ and $C=\sup_{(x,y)\in\mathbb{R}^2}|\mathcal{K}(x,y)|$, hence we obtain:

$$
\begin{aligned}
&\Big|\frac{2z_1+x_1-y_1+h_1u_1}{h_1}\Big|\Big|\frac{2z_2+x_2-y_2+h_2u_2}{h_2}\Big| \\
&\times\Big|\mathcal{K}\Big(\frac{2z_1+x_1-\theta_1^1-y_1+\theta_2^1 h_1}{h_1}+u_1,\frac{z_2+x_2-\theta_1^2-y_2+\theta_2^2 h_2}{h_2}+u_2\Big)\Big|\leq B+2h_1h_2C.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
&\Big|\mathcal{K}\Big(\frac{2z_1+x_1-\theta_1^1-y_1+\theta_2^1 h_1}{h_1}+u_1,\frac{z_2+x_2-\theta_1^2-y_2+\theta_2^2 h_2}{h_2}+u_2\Big)\Big|(h_1h_2)^{-1} \\
&\leq\frac{B^2+2h_1h_2C}{|2z_1+x_1-y_1+h_1u_1||2z_2+x_2-y_2+h_2u_2|}.
\end{aligned}
$$

Hence

$$\Big|\mathcal{K}\Big(\frac{x_1-\theta_1^1h_1-y_1+\theta_2^1h_1}{h_1}+u_1,\frac{x_2-\theta_1^2h_2-y_2+\theta_2^2h_2}{h_2}+u_2)(h_1h_2\Big)^{-1}\Big|$$
$$\leq\lambda\Big(\frac{2z_1}{h_1},\frac{2z_2}{h_2}\Big)+\frac{B+2h_1h2C}{|2z_1+x_1-y_1+h_1u_1||2z_2+x_2-y_2+h_2u_2|}.$$

Then for $h_i \quad i=1,2$ small enough, we have

$$\begin{aligned}\int_{\{|v_1|\leq\delta_1\}\times\{|v_2|\leq\delta_2\}}&\leq\frac{A}{|2z_1+x_1-y_1+h_1u_1||2z_2+x_2-y_2+h_2u_2|}\\&\times\int_{\mathbb{R}^2}|\mathcal{K}(u_1,u_2)|(B^2+2h_1h_2C)\,du_1du_2\\&<\frac{AD}{|2z_1+x_1-y_1+h_1u_1||2z_2+x_2-y_2+h_2u_2|}\\&\leq\frac{AD}{|2z_1+x_1-y_1+h_1u_1||2z_2+x_2-y_2+h_2u_2|}\\&\leq\frac{AD}{(2z_1+x_1-y_1+h_1u_1)(2z_2+x_2-y_2+h_2u_2)}\end{aligned}$$

where D represents the finite bound of $\int_{\mathbb{R}^2}|\mathcal{K}(u_1,u_2)|(B+2C)\,du_1du_2$.
Finally, we get:

$$\int_{\{|v_1|\leq\delta_1\}\times\{|v_2|\leq\delta_2\}}\leq\frac{AD}{(2z_1+x_1-y_1+h_1u_1)(2z_2+x_2-y_2+h_2u_2)}+O(h_1h_2).$$

Since $-\delta_i\leq h_1u_1\leq\delta_i$ we obtain $\frac{z_i}{2}\leq 2z_i+x_i-y_i+h_iu_i\leq\frac{7z_i}{2}, \quad i=1,2$. Therefore

$$\int_{\{|v_1|\leq\delta_1\}\times\{|v_2|\leq\delta_2\}}\leq\frac{2AD}{z_1z_2}+O(h_1h_2).$$

Hence we have by Lebesgue-dominated convergence

$$\lim_{n\to+\infty}\int_{\Delta_1\times\Delta_2}\Bigg(\int_{\{|v_1|\leq\delta_1\}\times\{|v_2|\leq\delta_2\}}\Bigg)dx_1dx_2dy_1dy_2=\int_{\Delta_1}\int_{\Delta_2}\lim_{n\to+\infty}\Bigg(\int_{\{|v_1|\leq\delta_1\}\times\{|v_2|\leq\delta_2\}}\Bigg)dx_1dx_2dy_1dy_2=0$$

Evaluate then $\int_{\{|v_1|>\delta_1\}\times\{|v_2|>\delta_2\}}$:
We utilize the second part of lemma 3 analogously

$$\begin{aligned}\int_{\{|v_1|>\delta_1\}\times\{|v_2|>\delta_2\}}&\leq\frac{2}{\delta_1}\times\frac{2}{\delta_2}\sup_{|v_1|>\delta_1}\sup_{|v_2|>\delta_2}\Big|\frac{v_1}{h_1}\frac{v_2}{h_2}\mathcal{K}(\frac{v_1}{h_1},\frac{v_2}{h_2})\Big|\\&\times\int_{\mathbb{R}^2}\Big|(h_1h_2)^{-1}\mathcal{K}(\frac{u_1}{h_1},\frac{u_2}{h_2})\Big|f(u_1+y_1-\theta_2^1h_1,u_2+y_2-\theta_2^2h_2)\,du_1du_2.\end{aligned}\tag{21}$$

We uniformly obtain

$$\int_{\{|v_1|>\delta_1\}\times\{|v_2|>\delta_2\}}\to 0,\qquad n\to+\infty.$$

Evaluate $\int_{\{|v_1|\leq\delta_1\}\times\{|v_2|>\delta_2\}}(\cdot)$ because of condition C_1 we get:

$$\int_{\{|v_1|\leq\delta_1\}\times\{|v_2|>\delta_2\}}(\cdot)\leq\int_{\mathbb{R}\times\mathbb{R}}(\cdot)(\cdot)\to 0,\quad n\to+\infty.$$

Similarly we have

$$\int_{\{|v_1|>\delta_1\}\times\{|v_2|\leq\delta_2\}}(\cdot)\leq\int_{\mathbb{R}\times\mathbb{R}}(\cdot)(\cdot)\to 0,\quad n\to+\infty.$$

Consequently

$$\lim_{n\to+\infty}\int_{\Delta_1\times\Delta_2}\int_{\mathbb{R}^2}\to 0,\quad n\to+\infty$$

Δ_1 and Δ_2 being bounded. This completes the demonstration of the lemma.

Remark 4. Under condition C_2, the theorem remains valid. In fact it suffices to apply Remark 3 to inequality (21) in $\int_{\{|v_1|>\delta_1\}\times\{|v_2|>\delta_2\}}$.

5.2.3. Demonstration of Theorem 3

We assume condition (C_1) satisfied:

$$\begin{aligned} n\mathbb{V}ar(P_n^{\mathcal{K}}(z,\alpha)) &= \mathbb{E}\Big(\sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]}\Big(1-\frac{ih_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{jh_2}{z_2}\Big)^{\alpha_2}\mathcal{K}\Big(\frac{X_{k1}-ih_1}{h_1},\frac{X_{k2}-jh_2}{h_2}\Big)\Big)^2 \\ &- \mathbb{E}^2\Big(\sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]}\Big(1-\frac{ih_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{jh_2}{z_2}\Big)^{\alpha_2}\mathcal{K}\Big(\frac{X_{k1}-ih_1}{h_1},\frac{X_{k2}-jh_2}{h_2}\Big)\Big). \end{aligned} \tag{22}$$

Since

$$\begin{aligned} &\mathbb{E}\Big(\sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]}\Big(1-\frac{ih_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{jh_2}{z_2}\Big)^{\alpha_2}\mathcal{K}\Big(\frac{X_{k1}-ih_1}{h_1},\frac{X_{k2}-jh_2}{h_2}\Big)\Big)^2 \\ &= \mathbb{E}\Big(\sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]}\Big(1-\frac{ih_1}{z_1}\Big)^{2\alpha_1}\Big(1-\frac{jh_2}{z_2}\Big)^{2\alpha_2}\mathcal{K}^2\Big(\frac{X_{k1}-ih_1}{h_1},\frac{X_{k2}-jh_2}{h_2}\Big)\Big) \\ &+ \mathbb{E}\Big(\sum_{0\le i_1\neq j_1\le[\frac{z_1}{h_1}]}\ \sum_{0\le i_2\neq j_2\le[\frac{z_2}{h_2}]}\Big(1-\frac{i_1h_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{j_1h_1}{z_1}\Big)^{\alpha_1}\times\Big(1-\frac{i_2h_2}{z_2}\Big)^{\alpha_2}\Big(1-\frac{j_2h_2}{z_2}\Big)^{\alpha_2} \\ &\times\mathcal{K}\Big(\frac{X_{k1}-i_1h_1}{h_1},\frac{X_{k1}-j_1h_1}{h_1}\Big)\mathcal{K}\Big(\frac{X_{k2}-i_2h_2}{h_2},\frac{X_{k2}-j_2h_2}{h_2}\Big)\Big). \end{aligned} \tag{23}$$

Because of the Theorem 5 we have

$$\begin{aligned} &\mathbb{E}\Big(\sum_{0\le i_1\neq j_1\le[\frac{z_1}{h_1}]}\ \sum_{0\le i_2\neq j_2\le[\frac{z_2}{h_2}]}\Big(1-\frac{i_1h_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{j_1h_1}{z_1}\Big)^{\alpha_1}\times\Big(1-\frac{i_2h_2}{z_2}\Big)^{\alpha_2}\Big(1-\frac{j_2h_2}{z_2}\Big)^{\alpha_2} \\ &\times\mathcal{K}\Big(\frac{X_{k1}-i_1h_1}{h_1},\frac{X_{k1}-j_1h_1}{h_1}\Big)\mathcal{K}\Big(\frac{X_{k2}-i_2h_2}{h_2},\frac{X_{k2}-j_2h_2}{h_2}\Big)\Big)\to 0. \end{aligned}$$

Because of Corollary 1 we get

$$\mathbb{E}\Big(\sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]}\Big(1-\frac{ih_1}{z_1}\Big)^{2\alpha_1}\Big(1-\frac{jh_2}{z_2}\Big)^{2\alpha_2}\mathcal{K}^2\Big(\frac{X_{k1}-ih_1}{h_1},\frac{X_{k2}-jh_2}{h_2}\Big)\Big)\to\int_{\mathbb{R}^2}\mathcal{K}^2(y_1,y_2)P(z,2\alpha_1,2\alpha_2).$$

On the other hand, because of the Lemma 1 we have

$$\mathbb{E}\Big(\sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]}\Big(1-\frac{ih_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{jh_2}{z_2}\Big)^{\alpha_2}\mathcal{K}\Big(\frac{X_{k1}-ih_1}{h_1},\frac{X_{k2}-jh_2}{h_2}\Big)\Big)\to P(z,\alpha)\quad n\to+\infty.$$

Therefore

$$\mathbb{E}^2\Big(\sum_{i=0}^{[\frac{z_1}{h_1}]}\sum_{j=0}^{[\frac{z_2}{h_2}]}\Big(1-\frac{ih_1}{z_1}\Big)^{\alpha_1}\Big(1-\frac{jh_2}{z_2}\Big)^{\alpha_2}\mathcal{K}\Big(\frac{X_{k1}-ih_1}{h_1},\frac{X_{k2}-jh_2}{h_2}\Big)\Big)\to P^2(z,\alpha)\quad n\to+\infty.$$

Hence

$$n\mathbb{V}ar(P_n^{\mathcal{K}}(z,\alpha)) \to \int_{\mathbb{R}^2} \mathcal{K}^2(y_1,y_2)P(z,\alpha) - P^2(z,\alpha).$$

Because of the Lemma 1 we have

$$\mathbb{E}(P_n^{\mathcal{K}}(z,\alpha)) \to P(z,\alpha) \quad \text{as} \quad n \to +\infty.$$

Therefore

$$\mathbb{E}^2(P_n^{\mathcal{K}}(z,\alpha)) \to P^2(z,\alpha) \quad \text{as} \quad n \to +\infty.$$

Define

$$biais(P_n^{\mathcal{K}}(z,\alpha)) = \mathbb{E}(P_n^{\mathcal{K}}(z,\alpha)) - P(z,\alpha).$$

We have

$$\mathbb{E}(P_n^{\mathcal{K}}(z,\alpha) - P_(z,\alpha))^2 = biais^2(P_n^{\mathcal{K}}(z,\alpha)) + \mathbb{V}ar(P_n^{\mathcal{K}}(z,\alpha))$$

and

$$\Big|\int_{\mathbb{R}^2} \mathcal{K}^2(y_1,y_2)P(z,\alpha) - P^2(z,\alpha)\Big| \leq \int_{\mathbb{R}^2} \mathcal{K}^2(y_1,y_2) + 1.$$

Hence

$$\mathbb{V}ar(P_n^{\mathcal{K}}(z,\alpha)) = O(\frac{1}{n}).$$

Because of Lemma 1 we obtain

$$|\mathbb{E}(P_n^{\mathcal{K}}(z,\alpha)) - P(z,\alpha)| \to 0, \quad \text{as} \quad n \to +\infty.$$

Therefore

$$biais^2(P_n^{\mathcal{K}}(z,\alpha)) \to 0, \quad \text{as} \quad n \to +\infty.$$

Hence

$$\mathbb{E}(P_n^{\mathcal{K}}(z,\alpha) - P(z,\alpha))^2 \to 0, \quad \text{as} \quad n \to +\infty.$$

If assumption C_2 is verified this theorem remains true. According to Corollary 1 of Lemma 2, utilizing Remark 4 of Theorem 5 and according to Theorem 2, we get Theorem 4.

6. Conclusion

This paper has introduced a bivariate radialsymmetric kernel estimator for the bidimensional FGT poverty index. The proposed estimator addresses the limitation of multiplicative kernel estimators by assigning equal weights to observations at equal Euclidean distance from the reference point. We have established complete asymptotic results, including uniform almost-sure consistency and uniform mean square consistency under appropriate regularity conditions. The simulation study demonstrates that the radialsymmetric kernel estimator performs well, particularly for small sample sizes. However, given the resources at our disposal, we encountered difficulties in performing simulations with large sample sizes. As a next step, we plan to propose, in the short term, a study employing a variable bivariate bandwidth for both the product kernel and the spherical kernel. We also propose investigating, in the future, the case of a radialsymmetric adaptive kernel.

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Abbreviations

FGT Foster, Greer and Thorbecke
pdf Probability Density Function

Author Contributions

Youssou Ciss: Conceptualization, Formal Analysis, Methodology, Project Administration, Software, Validation, Writing – original draft, Writing – review & editing
Mamadou Djitan Sinclair: Funding Acquisition
Mohamed Dinah Bangoura: Funding Acquisition
Aboubakary Diakhaly: Funding Acquisition, Supervision, Writing – original draft

Conflicts of Interest

The authors declare no conflicts of interest.

References

[1] Al Hassana Diallo, Youssou Ciss, Baradine Zakaria et al. "ON THE ADAPTIVE BI-DIMENSIONAL KERNEL DENSITY ESTIMATION OF WELL-BEING DISTRIBUTION AND POVERTY INDEX". *Far East Journal of Theoretical Statistics*, vol. 66, Aug. 2022, pp. 15-72. http://dx.doi.org/10.17654/0972086322012

[2] Alkire, S., & Foster, J. (2011). Counting and multidimensional poverty measurement. *Journal of Public Economics*, 95(7-8), 476-487, 2011. https://doi.org/10.1016/j.jpubeco.2010.11.006

[3] Atkinson, A.B., 1987. On the Measurement of Poverty. *Econometrica*, 55 (4), 749-764, 1987. https://doi.org/10.2307/1911028

[4] Biewen, M. (2002). Bootstrap inference for inequality, mobility and poverty measurement. *Journal of Economic Inequality*, 1(1), 51-69, 2002. https://doi.org/10.1016/S0304-4076(01)00138-5

[5] Bourguignon, F., & Chakravarty, S. R. (2003). The measurement of multidimensional poverty. *Journal of Economic Inequality*, 1(1), 25-49, 2003. https://doi.org/10.1023/A:1023913831342

[6] Davidson, R., & Flachaire, E. (2007). Asymptotic and bootstrap inference for inequality and poverty measures. *Journal of Econometrics*, 141(1), 141-166, 2007. https://doi.org/10.1016/j.jeconom.2007.01.009

[7] Duclos, J. Y., Sahn, D. E. and Younger, S. D. (2006). Robust multidimensional poverty comparison. *Economic Journal*, 113:943–968, 2006. https://doi.org/10.1111/j.1468-0297.2006.01118.x

[8] Duclos, J. Y., Sahn, D. E. and Younger, S. D. (2006). Robust multidimensional spatial poverty comparisons in Ghana, Madagascar, and Uganda. *World Bank Econ. Rev.*, 20(1), 91-113, 2006.

[9] Foster, J. E., Greer, J. and E. Thorbecke (1984) A class of decomposable poverty measures. *Econometrica*, 52:761–776, 1984. https://doi.org/10.2307/1913475

[10] Foster, J. E. and Shorrocks, A.F. (1988) Poverty Orderings. *Econometrica*, 56, 173-177, 1988. https://doi.org/10.2307/1911846

[11] Foster, J. E. and Shorrocks, A.F. (1988) Poverty Orderings and Welfare Dominance. Social *Choice Welfare*, 5, 179-198, 1988. https://doi.org/10.1007/BF00735760

[12] Foster, J. E. and Shorrocks, A.F. (1988) Inequality and Poverty Orderings. *European Economic Review*, 32, 654-662, 1988.

[13] Dia, G.. Nonparametric estimation of income distribution and poverty index. *C. R. Acad. Sci. Paris*, Volume 346 (2008) no. 15-16, pp.907-912. https://doi.org/10.1016/j.crma.2008.07.004

[14] Härdle, W., Müller, M. (1997). Multivariate and Semiparametric Kernel Regression. *Sonderforschungsbereich 373: Quantification and Simulation of Economic Processes.*, 1997,26.

[15] Parzen, E. (1956). On estimation of a probability density function and mode. *Annals of Mathematical Statistics*, Volume 33, Issue 3:1065–1076, Sep. 1962. https://doi.org/10.1214/aoms/1177700074

[16] Schimek, M. G. (2000)., Smoothing and Regression. *John Wiley & Sons Inc., New York*, ed. 2000

[17] Scott, D. W. (2015). Multivariate Density Estimation: Theory, Practice, and Visualization *(2nd ed.)*. Wiley, 2015. https://doi.org/10.1002/9781118575574

[18] Sen, A. (1976). Poverty: An ordinal approach to measurement. *Econometrica*, 44(2), 219-231, 1976. https://doi.org/10.2307/1912718

[19] Sheather, S. J., & Jones, M. C. (1991). A reliable data-based bandwidth selection method for kernel density estimation. Journal of the Royal Statistical Society: Series B, 53(3), 683-690, 1991. https://doi.org/10.1111/j.2517-6161.1991.tb01857.x

[20] Silverman, B. W. (1986). Density Estimation for Statistics and Data Analysis. *Chapman & Hall, London*, 1986. https://doi.org/10.1002/bimj.4710300745

[21] Simonoff, J. S. (1996). Smoothing Methods in Statistics. *Springer, New York*, 1996. https://doi.org/10.1007/978-1-4612-4026-6

[22] Wand, M. P. & Jones, M. C. (1995) Kernel Smoothing. *Chapman and Hall Ltd., London*, 1995. https://doi.org/10.1201/b14876

[23] Y. Ciss, G. Dia, A. Diakhaby. Bidimensional non-parametric estimation of well-being distribution and poverty index. *Afrika Statistika*, Vol. 9, 2014, pages 695-725. http://dx.doi.org/1016929.as.2014.695.63

[24] Youssou Ciss; Galaye Dia; Aboubakary Diakhaby. Non-parametric estimation of income distribution and poverty index in the unidimensional context with $\alpha \in [0, 1]$. *C. R.Acad.Sci.Paris*, Ser. I 353 (2015) no. 10, pp. 947-952. http://dx.doi.org/10.1016/j.crma.2015.08.007

[25] Z. Baradine and Y. Ciss and A. Diakhaby. Adaptive kernel density estimation of income distribution and poverty index. *In A Collection of Papers in Mathematics and Related Sciences, a festschrift in honour of the late Galaye Dia (Editors : Seydi H., Lo G.S. and Diakhaby A.).* Spas Editions, Euclid Series Book, pp. 103 - 128. https://doi.org/10.16929/sbs/2018.100- 02-04