Research Article

On the Motion Past a Slip-Stick Sphere and a Shear Free Sphere in a Viscous Fluid

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Abstract

The theoretical development of the complex fluid flows such as more than one obstacle with different shapes have great interest for scientists to understand flow phenomena and verify the model or approximate solution. The complex physical properties due to a uniform streaming motion past two fixed spheres is investigated having one with shear stress and another being shear stress-free. This study concerns analytical technique of a steady incompressible viscous fluid past to two fixed spheres. The Gegenbaur function and associated Legendre polynomials is used to derive the solution that simplify the process of the theoretical calculations. The mathematical expression for the flow fields are obtained in terms of stream functions by Gegenbaur function and associated Legendre polynomials. The physical properties of interest such as the Stokes stream function, the stress and its drag are calculated. It is understandable that for the uniform streaming motion around a sphere with the stress and its drag are affected owing to the presence of another stress-free are analyzed. The present result can be considered as a generalized by making other established results as a corollary of this solution. This theoretical study helps to the numerical or computational work for the verification of their approximated results of interest.

Keywords

Uniform Stream, Strokes' Flow, Incompressible Viscous Fluid, Gegenbaur Function, Legendre Polynomials

1. Introduction

Tools development considering flow over solid bodies can be different scenarios such as all boundaries is stick-slip or shear free depending upon situations. Theoretical advancement will help to analyze and validate the engineering design and simulations. Here uniform flow past two spheres are considered where one with slip-stick boundary condition and another one with shear free condition in viscous fluid flow.

The analytical solution to the Navier-Stokes equation for the viscous incompressible steady flow past a solid sphere which was first approached by Stokes in 1851 [1] and has been discussed subsequently by many researchers [2-7]. Much research has been done to study flow problems related

to viscous fluid motion in symmetrical boundaries subject to no-slip condition which states that the velocity of the fluid is everywhere zero at the solid surface. Sometimes, the fluid may slip at the surface of the sphere. This problem was first solved by Basset [8] and in this circumstance the tangential velocity of the fluid relative to the sphere at a point on its surface is proportional to the tangential stress prevailing at that point. It should be pointed out that similar condition have been used by several authors [9-17] to solve exterior and interior flow problems past a surface. Now it is intended to present a solution for flow spheres, one with general boundary condition and another one with shear free condition in viscous fluid flow.

James W. Swan and Aditya S. Khair [18] investigate the low-Reynolds-number hydrodynamics of a novel 'slip-stick' spherical particle whose surface is partitioned into slip and no-slip regions. They compute the translation velocity of such a particle due to the force density on its surface. They also compute the rotational velocity and the response to an ambient straining filled of a slip-stick particle. Stokes [19] first solved the problem of uniform streaming motion past a rigid sphere in a viscous fluid by solving the Stokes equation. Harper [20] and Happel & Brenner [21] and Rybczynski [22] and Padrino et al. [23] treated the problem of uniform

streaming motion past shear-stress free sphere. There is a wide literature about viscous and non-viscous flows past one or more rigid spheres with shear stress. Now, the Stokes problem of a uniform streaming motion past one rigid sphere with shear stress can be solved by the Collins theorem [24]. Harper's theorem [20] for treating the axi-symmetric viscous flows past a shear stress free sphere can now solve Rybczynski-Hadamard problem referred above. Harpers theorem is valid when the singularities lie on the axis of symmetry, z-axis. But Palaniappan et al. [25] established a general sphere theorem which applies when the singularities may lie on the axis of symmetry or not. This theorem has its two-dimensional counter part in the result known as the Circle theorem [26] for the flow past a shear-stress free circular sphere. However, literature of fluid flows about shear stress-free bodies is not so wide as that of the flows about the same bodies with shear stress.

In this study, the some physical properties of interest due to a uniform streaming motion past two fixed spheres are investigated, one having shear stress and another being shear stress-free. The physical properties, such as the Stokes stream function for the uniform streaming motion, around a sphere with the stress and its drag are affected owing to the presence of another stress-free are analysed.

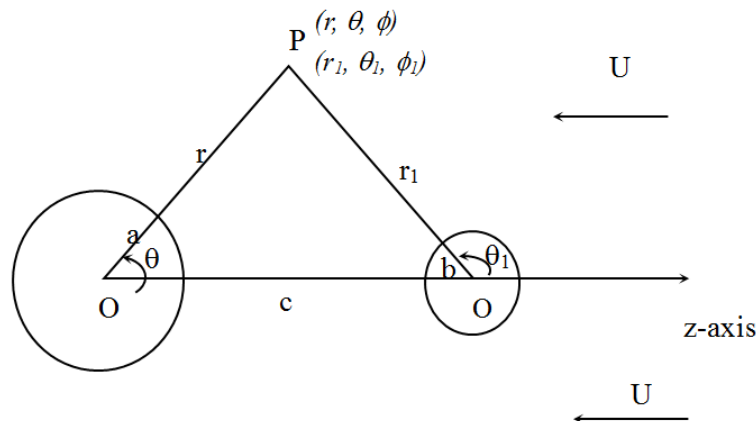## 2. Description of the Solution of the Problem

Figure 1. Sketch of the physical problem where two spheres having radii a and b are situated apart from each other with distance c under the uniform stream U.

Consider the three-dimensional axisymmetrical slow viscous fluid motion about the axis of symmetry (say, z-axis) in the case of uniform streaming motion in the negative direction of z-axis, past two rigid spheres of one is with shear stress and other shear stress-free. The physical model displays in the above figure 1 where the sphere r = a is with shear stress and the sphere r_1 = b is without shear stress.

The governing equation for the steady Stokes flow in terms of stream function ψ is given by

$$E^4\psi = 0 \qquad (1)$$

where

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\right) \qquad (2)$$

The velocity components q_r and q_θ for the axi-symmetrical flow in terms of the Stokes' stream function

$\psi = \psi(r, \theta)$ are given by

$$q_r = -\frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta} \text{ and } q_\theta = \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r} \tag{3}$$

The operator (2) is form-invariant with respect to the transformations $r\cos \theta = r_1 \cos \theta_1 + c$ and $r\sin \theta = r_1 \sin \theta_1$ where (r, θ, ϕ) and (r_1, θ_1, ϕ_1) denote the spherical coordinates of the same point P with respect to the centers O and O_1 respectively as in Figure 1.

It is the interest to find the solution of equation (1) for ψ subject to the appropriate boundary conditions at the surfaces of the spheres.

The boundary conditions

One sphere is considered as stick-slip flow and other one is considered as pure slip (shear free). Thus, the boundary conditions maybe stated as follows:

On $r = a$, $\psi = 0$ and $\beta \frac{1}{r} \frac{\partial \psi}{\partial r} = \mu r \frac{\partial}{\partial \theta} \left(\frac{1}{r^2} \frac{\partial \psi}{\partial r} \right)$ (4)

On $r_1 = b$, $\psi = 0$ and $\frac{\partial}{\partial r_1} \left(\frac{1}{r_1^2} \frac{\partial \psi}{\partial r_1} \right) = 0$ (5)

where μ and β are constants.

3. Solution Approach

The Stokes stream function for the flow in the absence of spheres is

$$\psi_0 = \frac{1}{2} U r^2 \sin \theta \tag{6}$$

If the spheres of radii 'a' and 'b' are introduced into the flow, the Stokes' stream function for the new flow field

$$\psi = \frac{1}{2} U r^2 \sin^2 \theta + \sum_{n=1}^{\infty} \left(\frac{A_n}{r^{n-2}} + \frac{B_n}{r^n} \right) P_n^1(\cos \theta) P_1^1(\cos \theta) + \sum_{m=1}^{\infty} \left(\frac{C_m}{r_1^{m-2}} + \frac{D_m}{r_1^m} \right) P_m^1(\cos \theta_1) P_1^1(\cos \theta_1) \tag{7}$$

which can also be written as

$$\psi = \frac{1}{2} U r^2 \sin^2 \theta + \sum_{n=1}^{\infty} \left(\frac{A_n}{r^{n-2}} + \frac{B_n}{r^n} \right) P_n^1(\cos \theta) P_1^1(\cos \theta) + \sum_{m=1}^{\infty} \left[C_m \frac{P_m^1(\cos \theta_1)}{r_1^{m-1}} r \sin \theta + \frac{D_m}{r_1^{m+1}} P_m^1(\cos \theta_1) \times r \sin \theta \right] \tag{8}$$

Or, in (r_1, θ_1) system as

$$\psi = \frac{1}{2} U r_1^2 \sin^2 \theta_1 + \sum_{n=1}^{\infty} \left(\frac{A_n}{r^{n-2}} + \frac{B_n}{r^n} \right) P_n^1(\cos \theta) P_1^1(\cos \theta) + \sum_{m=1}^{\infty} \left[C_m \frac{P_m^1(\cos \theta_1)}{r_1^{m-1}} r \sin \theta + \frac{D_m}{r_1^{m+1}} P_m^1(\cos \theta_1) \times r \sin \theta \right] \tag{9}$$

where, P_n^1 are the associated Legendre functions and A_n, B_n, C_m, D_m are constants to be determined.

Now the following transformations formula is required in order to solve the problem. The following relations [28] are true when r, $r_1 < c$

$$\frac{P_m^1(\cos \theta_1)}{r_1^{m-1}} = \frac{(-1)^{m-1}}{c^{m-1}} \sum_{n=1}^{\infty} \left[\frac{(2m-1)n-(m-2)}{(m+n)(2n-1)} - \frac{2m-1}{2n+3} \left(\frac{r}{c} \right)^2 \right] \times \frac{(m)_{n+1}}{(n+1)!} \left(\frac{r}{c} \right)^n P_n^1(\cos \theta), \tag{10}$$

$$\frac{P_m^1(\cos \theta_1)}{r_1^{m+1}} = \frac{(-1)^{m+1}}{c^{m+1}} \sum_{n=1}^{\infty} \frac{(m)_{n+1}}{(n+1)!} \left(\frac{r}{c} \right)^n P_n^1(\cos \theta) \tag{11}$$

where $(m)n = m(m + 1).............................(m + n - 1)$.

Multiplying both sides of (10) and (11) by $r\sin\theta$

$$\frac{P_m^1(\cos \theta_1)}{r_1^{m-1}} r \sin \theta = \frac{(-1)^{m-1}}{c^{m-1}} \sum_{n=1}^{\infty} \left[\frac{(2m-1)n-(m-2)}{(m+n)(2n-1)} - \frac{2m-1}{2n+3} \left(\frac{r}{c} \right)^2 \right] \times \frac{(m)_{n+1}}{(n+1)!} \left(\frac{r}{c} \right)^n r P_n^1(\cos \theta) \times P_1^1(\cos \theta) \tag{12}$$

$$\frac{P_m^1(\cos \theta_1)}{r_1^{m+1}} r \sin \theta = \frac{(-1)^{m+1}}{c^{m+1}} \sum_{n=1}^{\infty} \frac{(m)_{n+1}}{(n+1)!} \left(\frac{r}{c} \right)^n r P_n^1(\cos \theta) P_1^1(\cos \theta) \tag{13}$$

Substituting (12) and (13) in (7), ψ can be completely written in terms of r and θ as

$$\psi = \frac{1}{2} U r^2 \sin^2 \theta + \sum_{n=1}^{\infty} \left[\frac{A_n}{r^{n-2}} + \frac{B_n}{r^n} \right] P_n^1(\cos \theta) P_1^1(\cos \theta) +$$
$$\sum_{m=1}^{\infty} \left(C_m \frac{(-1)^{m-1}}{c^{m-1}} \sum_{n=1}^{\infty} \left[\frac{(2m-1)n-(m-2)}{(m+n)(2n-1)} - \frac{2m-1}{2n+3} \left(\frac{r}{c} \right)^2 \right] \frac{(m)_{n+1}}{(n+1)!} \left(\frac{r}{c} \right)^n r P_n^1(\cos \theta) P_1^1(\cos \theta) +$$
$$D_m \frac{(-1)^{m+1}}{c^{m+1}} \sum_{n=1}^{\infty} \frac{(m)_{n+1}}{(n+1)!} \left(\frac{r}{c} \right)^n r P_n^1(\cos \theta) P_1^1(\cos \theta) \right) \tag{14}$$

Similarly, an expression of ψ can be written in terms of r_1 and θ_1 by the following relations [27].

$$\frac{P^1{}_n(\cos\theta)}{r^{n-1}} = \frac{1}{c^{n-1}}\sum_{m=1}^{\infty}(-1)^{m-1}\left[\frac{(2n-1)m-(n-2)}{(n+m)(2m-1)}-\frac{2n-1}{2m+3}\left(\frac{r_1}{c}\right)^2\right]\times\frac{(n)_{m+1}}{(m+1)!}\left(\frac{r_1}{c}\right)^m P^1{}_m(\cos\theta_1) \tag{15}$$

$$\frac{P^1_n(\cos\theta)}{r^{n+1}} = \frac{1}{c^{n+1}}\sum_{m=1}^{\infty}(-1)^{m+1}\frac{(n)_{m+1}}{(m+1)!}\left(\frac{r_1}{c}\right)^m P^1_m(\cos\theta_1) \tag{16}$$

Multiplying both sides of (15) and (16) by $r_1\sin\theta_1$

$$\frac{P^1_n(\cos\theta)r_1\sin\theta_1}{r^{n-1}} = \frac{1}{c^{n-1}}\sum_{m=1}^{\infty}(-1)^{m-1}\left[\frac{(2n-1)m-(n-2)}{(n+m)(2m-1)}-\frac{2n-1}{2m+3}\left(\frac{r_1}{c}\right)^2\right]\times\frac{(n)_{m+1}}{(m+1)!}\left(\frac{r_1}{c}\right)^m r_1 P^1{}_m(\cos\theta_1)P^1{}_1(\cos\theta_1) \tag{17}$$

$$\frac{P^1_n(\cos\theta)r_1\sin\theta_1}{r^{n-1}} = \frac{1}{c^{n+1}}\sum_{m=1}^{\infty}(-1)^{m+1}\times\frac{(n)_{m+1}}{(m+1)!}\left(\frac{r_1}{c}\right)^m r_1 P^1{}_m(\cos\theta_1)P^1{}_1(\cos\theta_1) \tag{18}$$

Substituting (17) and (18) in (9) yields the stream function ψ as

$$\psi(r_1,\theta_1) = \frac{1}{2}Ur_1^2\sin^2\theta_1 + \sum_{n=1}^{\infty}\left(A_n\frac{1}{c^{n-1}}\sum_{m=1}^{\infty}(-1)^{m-1}\left[\frac{(2n-1)m-(n-2)}{(n+m)(2m-1)}-\frac{2n-1}{2m+3}\left(\frac{r_1}{c}\right)^2\right]\times\frac{(n)_{m+1}}{(m+1)!}\left(\frac{r_1}{c}\right)^m r_1 P^1{}_m(\cos\theta_1)P^1{}_1(\cos\theta_1)\;+ \right.$$

$$\left. B_n\frac{1}{c^{n+1}}\sum_{m=1}^{\infty}(-1)^{m-1}\frac{(n)_{m+1}}{(m+1)!}\left(\frac{r_1}{c}\right)^m r_1 P^1{}_m(\cos\theta_1)\times P^1{}_1(\cos\theta_1) + \sum_{m=1}^{\infty}\left[\frac{C_m}{r_1^{m-2}}+\frac{D_m}{r_1^m}\right]P^1{}_m(\cos\theta_1)P^1{}_1(\cos\theta_1)\right) \tag{19}$$

Now by using the boundary conditions (4) in (14) and equating the coefficients of p_n^1 on both sides of the resulting equations one obtains for n > 1

$$0 = \frac{1}{2}Ua^2\delta_1{}^n + \left(\frac{A_n}{a^{n-2}}+\frac{B_n}{a^n}\right) + \sum_{m=1}^{\infty}\left(C_m\frac{(-1)^{m-1}}{c^{m-1}}\left[\frac{(2m-1)n-(m-2)}{(m+n)(2n-1)}-\frac{2m-1}{2n+3}\left(\frac{a}{c}\right)^2\right]\frac{(m)_{n+1}}{(n+1)!}\times\left(\frac{a}{c}\right)^n a + D_m\frac{(-1)^{m+1}}{c^{m+1}}\frac{(m)_{n+1}}{(n+1)!}\left(\frac{a}{c}\right)^n a\right), \tag{20}$$

$$0 = U\left(\beta+\frac{\mu}{a}\right)\delta_1{}^n + A_n\left(-\frac{(n-2)\beta}{a^n}-\frac{(n+1)(n-2)\mu}{a^{n+1}}\right) - B_n\left(-\frac{n\beta}{a^{n+2}}+\frac{n(n+3)\mu}{a^{n+3}}\right) + \sum_{m=1}^{\infty}\left(C_m\frac{(-1)^{m-1}}{c^{m-1}}\left[\frac{(2m-1)n-(m-2)}{(m+n)(2n-1)}(n+1)\beta a^{n-1}-\frac{2m-1}{2n+3}(n+3)\beta a^{n+1}\frac{1}{c^2}-\frac{(2m-1)n-(m-2)}{(m+n)(2n-1)}(n+1)(n-2)\mu a^{n-2}+\frac{2m-1}{2n+3}n(n+3)\mu a^2\frac{1}{c^2}\right]\frac{(m)_{n+1}}{(n+1)!}\frac{1}{c^n} + D_m\frac{(-1)^{m+1}}{c^{m+1}}[(n+1)\beta a^{n-1}-(n+1)(n-2)\mu a^{n-2}]\frac{(m)_{n+1}}{(n+1)!}\frac{1}{c^n}\right) \tag{21}$$

Neglecting the terms of $\left(\frac{1}{c^k}\right)$, $k\geq 4,5,6$ in these two series one gets the following linear equations for the determination of the constants A_1, A_2, A_3 etc.

$$0 = \frac{1}{2}Ua^2 + aA_1 + \frac{B_1}{a} + \left(\frac{a^2}{c}-\frac{a^4}{5c^3}\right)C_1 - \frac{3a^2}{c^2}C_2 + \frac{6a^2}{c^3}C_3 + \frac{a^2}{c^3}D_1, \tag{22}$$

$$0 = A_2 + \frac{B_2}{a^2} + \frac{1}{3}\frac{a^3}{c^2}C_1 - \frac{2a^3}{c^3}C_2, \tag{23}$$

$$0 = \frac{A_3}{a} + \frac{B_3}{a^3} + \frac{1}{5}\frac{a^4}{c^3}C_1, \tag{24}$$

$$0 = \left(\beta+\frac{\mu}{a}\right)U + \left(\frac{\beta}{a}+\frac{2\mu}{a^2}\right)A_1 - \left(\frac{\beta}{a^3}+\frac{4\mu}{a^4}\right)B_1 + \left(\frac{2\beta}{c}-\frac{4\beta a^2}{5c^3}+\frac{2\mu}{ac}+\frac{4\mu a}{5c^3}\right)C_1 - \frac{6}{c^2}\left(\beta+\frac{\mu}{a}\right)C_2 + \frac{6}{c^3}\left(2\beta+\frac{2\mu}{a}\right)C_3 + \frac{2}{c^3}\left(\beta+\frac{\mu}{a}\right)D_1. \tag{25}$$

$$0 = -\left(\frac{2\beta}{a^4}+\frac{10\mu}{a^5}\right)B_2 + \frac{a\beta}{c^2}C_1 - \frac{6a\beta}{c^3}C_2 \tag{26}$$

$$0 = -\left(\frac{\beta}{a^3}+\frac{4\mu}{a^4}\right)A_3 - \left(\frac{3\beta}{a^5}+\frac{18\mu}{a^6}\right)B_3 + \left(\frac{4}{5}\frac{a^2}{c^3}\beta-\frac{4}{5}\frac{a}{c^3}\mu\right)C_1 \tag{27}$$

Also by using the boundary conditions (5) in (19) and equating the coefficients of p_m^1 on both sides of the resulting equations one obtains for m > 1

$$0 = \frac{1}{2}Ub^2\delta^m{}_1 + \sum_{n=1}^{\infty}\left(A_n\frac{1}{c^{n-1}}(-1)^{m-1}\left[\frac{(2n-1)m-(n-2)}{(n+m)(2m-1)} - \frac{2n-1}{2m+3}\left(\frac{b}{c}\right)^2\right] \times \frac{(n)_{m+1}}{(m+1)!} \times \left(\frac{b}{c}\right)^m b + B_n\frac{1}{c^{n+1}}(-1)^{m+1} \times \frac{(n)_{m+1}}{(m+1)!}\left(\frac{b}{c}\right)^m b\right) + \left(\frac{C_m}{b^{m-2}} + \frac{D_m}{b^m}\right) \tag{28}$$

$$0 = -\frac{U}{b^2}\delta^m{}_1 + \sum_{n=1}^{\infty}A_n\frac{1}{c^{n-1}}(-1)^{m-1}\left[\frac{(2n-1)m-(n-2)}{(n+m)(2m-1)}(m+1)(m-2)b^{m-3} - \frac{2n-1}{2m+3}m(m+3)b^{m-1}\frac{1}{c^2}\right] \times \frac{(n)_{m+1}}{(m+1)!} \times \frac{1}{c^m} + \sum_{n=1}^{\infty}B_n\frac{1}{c^{n+1}}(-1)^{m+1}\frac{(n)_{m+1}}{(m+1)!}(m+1)(m-2)b^{m-3}\frac{1}{c^m} + \left[C_m\frac{(m+1)(m-2)}{b^{m+2}} + D_m\frac{m(m+3)}{b^{m+4}}\right] \tag{29}$$

Similarly, another six linear equations arises for the same purpose of determining the constants.

$$0 = \frac{1}{2}Ub^2 + \left(\frac{b^2}{c} - \frac{b^4}{5c^3}\right)A_1 + \frac{3b^2}{c^2}A_2 + \frac{6b^2}{c^3}A_3 + \frac{b^2}{c^3}B_1 + bC_1 + \frac{D_1}{b}, \tag{30}$$

$$0 = -\frac{b^3}{3c^2}A_1 - \frac{2b^3}{c^3}A_2 + C_2 + \frac{D_2}{b^2}C \tag{31}$$

$$0 = \frac{b^4}{5c^3}A_1 + \frac{C_3}{b} + \frac{D_3}{b^3} \tag{32}$$

$$0 = -\frac{U}{b^2} - \left(\frac{2}{b^2c} + \frac{4}{5c^3}\right)A_1 - \frac{6}{b^2c^2}A_2 - \frac{12}{b^2c^3}A_3 - \frac{2}{b^2c^3}B_1 - \frac{2C_1}{b^3} + \frac{4D_1}{b^5} \tag{33}$$

$$0 = D_2 \tag{34}$$

$$0 = \frac{4}{5c^3}A_1 + \frac{4}{b^5}C_3 + \frac{18}{b^7}D_3 \tag{35}$$

Solving the equations (22) to (27) and (30) to (35) for the constants of $O(\frac{1}{c^k})$, k= 1,2,3 one gets

$$A_1 = -\frac{3}{4}\frac{a\beta+2\mu}{a\beta+3\mu}Uab\left(\frac{1}{b} - \frac{1}{c} + \frac{3a(a\beta+2\mu)}{2c^2(a\beta+3\mu)} + \left[\frac{a^3\beta}{3(a\beta+2\mu)} - \frac{3ab(a\beta+2\mu)}{2(a\beta+3\mu)}\right]\frac{1}{c^3}\right) \tag{36}$$

$$B_1 = \frac{Ua^4b\beta}{4(a\beta+3\mu)}\left(\frac{1}{b} - \frac{1}{c} + \frac{3a(a\beta+2\mu)}{2c^2(a\beta+3\mu)} + \left[\frac{a^3\beta}{3(a\beta+2\mu)} - \frac{3ab(a\beta+2\mu)}{2(a\beta+3\mu)}\right]\frac{1}{c^3}\right) + \frac{Ua^5b}{15c^3}\frac{(a\beta-3\mu)}{(a\beta+2\mu)} \tag{37}$$

$$C_1 = -\frac{Ub}{2} + \frac{3Uab}{4}\left(\frac{(a\beta+2\mu)}{(a\beta+3\mu)}\frac{1}{c} - \frac{b}{c^2}\frac{(a\beta+2\mu)}{(a\beta+3\mu)} + \frac{3ab}{2c^3}\frac{(a\beta+2\mu)^2}{(a\beta+3\mu)^2} - \frac{a^3\beta}{3c^3(a\beta+3\mu)}\right) \tag{38}$$

$$D_1 = -\frac{3Uab^5(a\beta+2\mu)}{20c^3(a\beta+3\mu)} \tag{39}$$

$$A_2 = B_2 = C_2 = D_2 = 0 \tag{40}$$

$$A_3 = \frac{7}{20}\frac{Ua^5b}{c^3}\frac{(a\beta+2\mu)}{(a\beta+7\mu)}, \tag{41}$$

$$B_3 = -\frac{Ua^8b\beta}{4c^3}\frac{1}{(a\beta+7\mu)} \tag{42}$$

$$C_3 = \frac{3}{20}\frac{Uab^5}{c^3}\frac{(\alpha\beta+2\mu)}{(\alpha\beta+3\mu)}, \tag{43}$$

$$D_3 = 0 \tag{44}$$

Other constants (for n, m > 4) are of $O(\frac{1}{c^k})$, k≥ 4; hence, they may be neglected. The expression for ψ can now be obtained in either of the coordinate system by using (36) to (44) in (14). In terms of (r, θ) the expression is

$$\psi(r,\theta) =$$

$$\frac{1}{2}U\left[r^2 + \left(\frac{1}{b} - \frac{1}{c} + \frac{3a(\alpha\beta+2\mu)}{2c^2(\alpha\beta+3\mu)} + \left[\frac{\alpha^3\beta}{3(\alpha\beta+2\mu)} - \frac{3ab(a\beta+2\mu)}{2(a\beta+3\mu)}\right]\frac{1}{c^3}\right)\left(\frac{a^4b\beta}{2(a\beta+3\mu)r} - \frac{3abr(a\beta+2\mu)}{2(a\beta+3\mu)}\right) + \frac{2a^5b(a\beta-3\mu)}{15c^3r(a\beta+2\mu)} - \frac{br^2}{c} + \frac{3ab(a\beta+2\mu)}{2(a\beta+3\mu)}\left(\frac{1}{c^2} - \frac{b}{c^3}\right)r^2 + \frac{b}{5c^3}r^4\right]sin^2\theta - \frac{Ub}{2}\left(\frac{1}{c^2} - \frac{3a(a\beta+2\mu)}{2c^3(a\beta+3\mu)}\right)r^3sin^2\theta\cos\theta + \frac{3Ub}{4c^3}\left\{\frac{7a^5(a\beta+2\mu)}{10(a\beta+7\mu)r} - \frac{a^8\beta}{2r^3(a\beta+7\mu)} - \frac{r^4}{5}\right\}sin^2\theta(5cos^2\theta - 1) \tag{45}$$

Another expression for ψ in terms of (r_1, θ_1) can be obtained by using (36) to (44) as

$$\psi(r_1,\theta_1) = \frac{U}{2}\left[r_1^2 - \frac{3ab}{2}r_1^2\frac{(a\beta+2\mu)}{(a\beta+3\mu)}\left(\frac{1}{bc} - \frac{1}{c^2} + \frac{3a(a\beta+2\mu)}{2c^3(a\beta+3\mu)}\right) + \frac{3ar_1^4(a\beta+2\mu)}{10c^3(a\beta+3\mu)} + \frac{a^4\beta r_1^2}{2c^3(a\beta+3\mu)} - br_1 + \frac{3abr_1}{2}\left(\frac{(a\beta+2\mu)}{(a\beta+3\mu)}\frac{1}{c} - \frac{b}{c^2}\frac{(a\beta+2\mu)}{(a\beta+3\mu)} + \frac{3ab(a\beta+2\mu)^2}{2c^3(a\beta+3\mu)^2} - \frac{a^3\beta}{3c^3(a\beta+3\mu)}\right) - \frac{3ab^5(a\beta+2\mu)}{10c^3r_1(a\beta+3\mu)}\right] \times sin^2\theta_1 + \frac{3Uab}{4c^2}\frac{(a\beta+2\mu)}{(a\beta+3\mu)}\left(\frac{1}{b} - \frac{1}{c}\right)r_1^3sin^2\theta_1cos\theta_1 + \left\{\frac{9}{40}\frac{Uab^5}{c^3r_1}\frac{(a\beta+2\mu)}{(a\beta+3\mu)} - \frac{9}{40}\frac{Uar_1^4}{c^3}\frac{(a\beta+2\mu)}{(a\beta+3\mu)}\right\}sin^2\theta(5cos^2\theta_1 - 1) \tag{46}$$

The pressure field p in the neighborhood of the spheres may be obtained from momentum equations.

$$\frac{\partial P}{\partial r} = -\frac{\mu}{r^2sin^2\theta}\frac{\partial}{\partial\theta}(E^2\psi) \tag{47}$$

$$\frac{\partial P}{\partial\theta} = \frac{\mu}{sin\theta}\frac{\partial}{\partial r}(E^2\psi) \tag{48}$$

Now,

$$E^2\psi = \frac{\partial^2}{\partial r^2}(\psi) + \frac{sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{sin\theta}\frac{\partial\psi}{\partial\theta}\right) = U\left\{\frac{1}{b} - \frac{1}{c} + \frac{3a(a\beta+2\mu)}{2c^2(a\beta+3\mu)} + \left[\frac{a^3\beta}{3(a\beta+2\mu)} - \frac{3ab(a\beta+2\mu)}{2(a\beta+3\mu)}\right]\frac{1}{c^3} \times \frac{3ab(a\beta+2\mu)}{2r(a\beta+3\mu)} + \frac{br^2}{c^3}\right\}sin^2\theta - \frac{21}{4}\frac{\mu Ua^5b(a\beta+2\mu)}{c^3r^3(a\beta+7\mu)} \times sin^2\theta(5cos^2\theta - 1) \tag{49}$$

$$E^2\psi(r_1,\theta_1) = U\left[\frac{3}{2}\frac{ar_1^2}{c^3}\frac{(a\beta+2\mu)}{(a\beta+3\mu)} + \frac{b}{r_1} - \frac{3ab}{2r_1}\left(\frac{(a\beta+2\mu)}{(a\beta+3\mu)}\frac{1}{c} - \frac{b}{c^2}\frac{(a\beta+2\mu)}{(a\beta+3\mu)} + \frac{3ab(a\beta+2\mu)^2}{2c^3(a\beta+3\mu)^2} - \frac{a^3\beta}{3c^3(a\beta+3\mu)}\right)sin^2\theta_1 - \frac{9}{4}\frac{Uab^5}{c^3r_1}\frac{(a\beta+2\mu)}{(a\beta+3\mu)} \times sin^2\theta_1(5cos^2\theta_1 - 1)\right] \tag{50}$$

Using (49) in (47) or (48), it is easily obtained,

$$p(r,\theta) = -\mu U\left\{\frac{2br}{c^3} - \frac{3ab}{2r^2}\frac{(a\beta+2\mu)}{(a\beta+3\mu)}\left(\frac{1}{b} - \frac{1}{c} + \frac{3a(a\beta+2\mu)}{2c^3(a\beta+3\mu)} + \left[\frac{a^3\beta}{3(a\beta+2\mu)} - \frac{3ab(a\beta+2\mu)}{2(a\beta+3\mu)}\right]\frac{1}{c^3}\right)\right]cos\theta - \frac{21}{8}\frac{\mu Ua^5b}{c^3r^4}\frac{(a\beta+2\mu)}{(a\beta+7\mu)}cos\theta(5cos2\theta - 1) \tag{51}$$

Similarly,

$$p_1(r_1,\theta_1) = -\mu U\left[\frac{3ar_1(a\beta+2\mu)}{c^3(a\beta+3\mu)} - \frac{b}{r_1^2} + \frac{3ab}{2r_1^2}\left(\frac{(a\beta+2\mu)}{(a\beta+3\mu)}\frac{1}{c} - \frac{b}{c^2}\frac{(a\beta+2\mu)}{(a\beta+3\mu)} + \frac{3ab(a\beta+2\mu)^2}{2c^3(a\beta+3\mu)^2} - \frac{a^3\beta}{3c^3(a\beta+3\mu)}\right)\right]cos\theta_1 - \frac{9\mu}{8}\frac{Uab^5}{c^3r_1^4}\frac{(a\beta+2\mu)}{(a\beta+3\mu)}cos\theta_1(5cos^2\theta_1 - 1) \tag{52}$$

4. Drag and Torque on the Spheres

Following Faxen's laws [28] the drag is found by the formula

$$\underline{D} = \int_S \left(T_{rr}\,\underline{e}_r + T_{r\theta}\,\underline{e}_\theta + T_{r\phi}\,\underline{e}_\phi\right)_{r=a} \times a^2sin\theta\, d\theta\, d\phi$$

where, S is the surface of sphere r = a and $\underline{e}_r$, $\underline{e}_\theta$, $\underline{e}_\phi$ being unit vectors in the directions of r, θ, φ increasing and where

$$T_{rr} = -p + 2\mu\frac{\partial q_r}{\partial r},\ T_{r\theta} = \mu\left(\frac{1}{r}\frac{\partial q_r}{\partial\theta} - \frac{q_\theta}{r} + \frac{\partial q_\theta}{\partial r}\right)\text{ and } T_{r\phi} = \mu\left(\frac{1}{r\,sin\theta}\frac{\partial q_r}{\partial\phi} - \frac{q_\phi}{r} + \frac{\partial q_\phi}{\partial r}\right)$$

For the case of axisymmetric motion this formula becomes

$$\underline{D} = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \left(T_{rr}\,\underline{e}_r + T_{r\theta}\,\underline{e}_\theta \right)_{r=a} \times a^2 \sin\theta \; d\theta \; d\phi \tag{53}$$

where

$$\underline{e}_r = \sin\theta\,\cos\phi\,\hat{\imath} + \sin\theta\sin\phi\,\hat{\jmath} + \cos\theta\,\hat{k}$$

$$\underline{e}_\theta = \cos\theta\cos\phi\,\hat{\imath} + \cos\theta\sin\phi\,\hat{\jmath} - \sin\theta\,\hat{k}$$

and $\theta = 0$ to π, $\phi = 0$ to 2π

Now on omitting the details of calculation, It can be shown that

$$\underline{D} = \left[-6\pi\mu U ab \frac{(\alpha\beta+2\mu)}{(\alpha\beta+3\mu)} \left(\frac{1}{b} - \frac{1}{c} + \frac{3a(\alpha\beta+2\mu)}{2c^2(\alpha\beta+3\mu)} + \left(\frac{a^3\beta}{3(\alpha\beta+3\mu)} - \frac{3ab(\alpha\beta+2\mu)}{2(\alpha\beta+3\mu)} \right) \frac{1}{c^3} \right) \right] \hat{k} \tag{54}$$

The negative sign implies that the force exerted by the fluid on the sphere r = a is in the direction of the uniform stream. To keep the sphere at rest, the force of the amount |D| must be applied to the rigid sphere.

Similarly, the drag on this sphere $r_1 = b$ is,

$$\underline{D}_1 = \int_{\theta_1=0}^{\pi} \int_{\phi_1=0}^{2\pi} \left(T_{r_1 r_1}\underline{e}_{r_1} + T_{r_1\theta_1}\underline{e}_{\theta_1} \right)_{r_1=b} b^2 \sin\theta_1 \, d\theta_1 \, d\phi_1 \tag{55}$$

$$\underline{D}_1 = \left[-4\pi\mu U b + 6\pi\mu U ab \left(\frac{(\alpha\beta+2\mu)}{(\alpha\beta+3\mu)} \frac{1}{c} - \frac{b}{c^2} \frac{(\alpha\beta+2\mu)}{(\alpha\beta+3\mu)} + \frac{3ab(\alpha\beta+2\mu)^2}{2c^3(\alpha\beta+3\mu)^2} - \frac{\alpha^3\beta}{3c^3(\alpha\beta+3\mu)} \right) \right] \underline{k} \tag{56}$$

which is negative for sufficiently large value of c. This also implies that the equal and opposite force must be applied to keep the shear-stress-free sphere at rest. Note that the magnitude of the drag on the rigid sphere is greater than that on the stress free sphere, when the spheres are particularly equal.

When the stress free sphere r_1=b is located at a very large distance relative the sphere r=a, the flow in the neighborhood of the sphere r=a will be that of a uniform streaming motion past the sphere r=a. Therefore, the Stokes stream function for this flow is the limit of the expression (45), as c approaches infinity and clearly, this is

$$\psi(r,\theta) = \frac{1}{2} U \left[r^2 - \frac{3a(\alpha\beta+2\mu)r}{2(\alpha\beta+3\mu)} + \frac{\alpha^4\beta}{2(\alpha\beta+3\mu)r} \right] \sin^2\theta \tag{57}$$

Similarly, when the rigid sphere r=a is located at a very large distance with respect to the stress free sphere r_1=b, the flow in the vicinity of the latter sphere is the limit of the expression (46) as c approaches infinity and this is obviously,

$$\psi(r_1, \theta_1) = \frac{1}{2} U (r_1{}^2 - br_1) \sin^2\theta_1 \tag{58}$$

Next the torque (T) on the sphere r = a, is calculated which is found from the formula of Faxen's laws [24].

$$T = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \left(rT_{r\theta}\underline{e}_\phi - rT_{r\phi}\underline{e}_\theta \right)_{r=a} a^2 \sin\theta \, d\theta \, d\phi = 0 \tag{59}$$

Similarly, It is shown that the torque on the sphere r_1= b is also zero.

5. Generalization of the Present Problem

The present results can be generalized by shifting the boundary condition of the referral condition. The properties of flow motion such as drag and torque are investigated as limiting case of the present development.

5.1. Transforming Boundary Condition to the Referral

When β=∞ (transforming stick-slip to no slip condition [25]) then the equations (45), (46), (50) and (51) become

$$\psi(r,\theta) = \frac{1}{2}U\left[r^2 + \left(\frac{1}{b} - \frac{1}{c} + \frac{3a}{2c^2} + \left(\frac{a^2}{3} - \frac{3ab}{2} \right)\frac{1}{c^3} \right)\left\{ \frac{a^3 b}{2r} - \frac{3abr}{2} \right\} + \frac{2a^5 b}{15c^3 r} - \frac{br^2}{c} + \frac{3ab}{2}\left(\frac{1}{c^2} - \frac{b}{c^3} \right)r^2 + \frac{br^4}{5c^3} \right]\sin^2\theta$$
$$- \frac{Ub}{2}\left(\frac{1}{c^2} - \frac{3a}{2c^3} \right)r^3 \sin^2\theta\cos\theta + \frac{3Ub}{4c^3}\left\{ \frac{7a^5}{10r} - \frac{a^7}{2r^3} - \frac{r^4}{5} \right\}\sin^2\theta(5\cos^2\theta - 1) \tag{60}$$

$$\psi(r_1, \theta_1) = \frac{1}{2}U\left\{r_1^2 - \frac{3ab}{2}r_1^2\left(\frac{1}{bc} - \frac{1}{c^2} + \frac{3a}{2c^3}\right) + \frac{3ar_1^4}{10c^3} + \frac{a^3}{2c^3}r_1^2 - br_1 + \frac{3abr_1}{2}\left(\frac{1}{c} - \frac{b}{c^2} + \frac{3ab}{2c^3} - \frac{a^2}{3c^3}\right) - \frac{3ab^5}{10c^3r_1}\right\}\sin^2\theta_1$$
$$+ \frac{3Uab}{4c^2}\left(\frac{1}{b} - \frac{1}{c}\right)r_1^3\sin^2\theta_1\cos\theta_1 + \left\{\frac{9}{40}\frac{Uab^5}{c^3r_1} - \frac{9}{40}\frac{Uar_1^4}{c^3}\right\}\sin^2\theta_1(5\cos2\theta_1 - 1) \tag{61}$$

$$p(r,\theta) = -\mu U\left\{\frac{2br}{c^3} - \frac{3ab}{2r^2}\left(\frac{1}{b} - \frac{1}{c} + \frac{3a}{2c^2} + \left(\frac{a^2}{3} - \frac{3ab}{2}\right)\frac{1}{c^3}\right)\right\}\cos\theta - \frac{21}{8}\mu\frac{Ua^5b}{c^3r^4}\cos\theta(5\cos2\theta - 1) \tag{62}$$

$$p_1(r_1,\theta_1) = -\mu U\left\{\frac{3ar_1}{c^3} - \frac{b}{r_1^2} + \frac{3ab}{2r_1^2}\left(\frac{1}{c} - \frac{b}{c^2} + \frac{3ab}{2c^3} - \frac{a^2}{3c^3}\right)\right\}\cos\theta_1 - \frac{9}{8}\frac{\mu Uab^5}{c^3r_1^4}\cos\theta_1(5\cos2\theta_1 - 1) \tag{63}$$

which agree with the results obtained by Chowdhury [29] corresponding to the no slip boundary conditions.

5.2. Drags on the Spheres

Similarly, when $\beta=\infty$, then the equations (54) and (56) give the drags for no slip boundary conditions as follows:

$$\underline{D} = \left[-6\pi\mu Uab\left(\frac{1}{b} - \frac{1}{c} + \frac{3a}{2c^2} + \left(\frac{a^2}{3} - \frac{3ab}{2}\right)\frac{1}{c^3}\right)\right]\hat{k} \tag{64}$$

$$\underline{D_1} = \left[6\pi\mu Uab\left(\frac{1}{c} - \frac{b}{c^2} + \frac{a}{c^3} + \left(\frac{3b}{2} - \frac{a}{3}\right)\right) - 4\pi\mu Ub\right]\hat{k} \tag{65}$$

which also agrees with the result obtained by Chowdhury [29].

Therefore, the present development of this kind of problem can be considered as a general formulation since other cases such no slip boundary condition can be derived from the present result.

6. Conclusions

In this study, it is examined some axi-symmetrical flows in the presence of the two fixed spheres, one having stick-slip and another being shear free. The forces have been found by employing Faxen's law [28]. The results have good agreement in the limiting case of flow field for a single sphere. Generalization of the present problem was investigated corresponding to the no slip boundary conditions. The present development of the theoretical work will help to understand the numerically approximated result for the verification.

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Conflicts of Interest

The authors declare no conflicts of interest.

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